MATH 13 FINAL EXAM SOLUTIONS

WINTER 2014

Problem 1 (15 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

(T)	F	For all functions $f : A \to B$ and $g : B \to C$ and all subsets $S \subseteq C$, we have $(g \circ f)^{-1}[S] = f^{-1}[g^{-1}[S]]$.
T	(F)	If A is a well-ordered set of real numbers and $x, y \in A$, then there must only be finitely many elements of A between x and y.
T	(F)	For all sets A and B, if there is a surjection from A to B and B is countable, then A is countable.
(T)	F	For every set A, there is an injection from A to $\mathcal{P}(A)$.
T	(F)	If f_1 and f_2 are functions from A to B and g is a surjection from B to C, and $g \circ f_1 = g \circ f_2$, then f_1 must equal f_2.

T	(F)	For all functions $f : A \to B$ and all subsets $S \subseteq A$, we have $f^{-1}[f[S]] = S$.
T	(F)	For all sets A and B, there is a bijection from A^B to B^A.
(T)	F	$P \implies Q$ is logically equivalent to $(\sim P) \vee Q$.
(T)	F	For every set A, there is a bijection from $\mathcal{P}(A)$ to $\{0,1\}^A$.
(T)	F	For all $a, b \in \mathbb{Z}$, if $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$, then $a \equiv b \pmod 6$.

(T)	F	If f_1 and f_2 are functions from A to B and g is an injection from B to C, and $g \circ f_1 = g \circ f_2$, then f_1 must equal f_2.
(T)	F	If A is a countably infinite set, then $\mathcal{P}(A)$ must be uncountable.
T	(F)	The relation on $\mathbb{N}$ defined by $R = \{(x, y) : x \mid y \text{ or } y \mid x\}$ is an equivalence relation.
(T)	F	$\sim(P \vee Q)$ is logically equivalent to $(\sim P) \wedge (\sim Q)$
(T)	F	If R is a transitive relation on the set A, then R^{-1} must also be a transitive relation on A.

Problem 2 (5 points). Let A and B be sets. Prove that $A \cap (\overline{A} \cup B) = (A \cup \overline{B}) \cap B$.

Problem 3 (5 points). Are the following two statements logically equivalent?

(1) $\exists x \in S, (P(x) \wedge Q(x))$

(2) $(\exists x \in S, P(x)) \wedge (\exists x \in S, Q(x))$

Justify your answer.

Problem 4 (5 points). PROVE or DISPROVE: If x, y, and z are real numbers and $x - y$, $y + z$, and $z + x$ are all rational, then x, y, and z must all be rational.

Solution. We disprove the statement with a counterexample. Let $x, y = \sqrt{2}$ and $z = -\sqrt{2}$. Then $x - y$, $y + z$, and $z + x$ are all rational (they are all zero) but x, y, and z are not all rational (in fact, they are all irrational, but we don't need this.)

Remark. A large majority of students submitted "proofs" of the problem statement. We all make mistakes sometimes, but I think it is fair to say that some of you may need to recalibrate your BS detectors.

Remark. Note that nothing in the problem says that x, y, and z are distinct real numbers. However, if we wanted a counterexample using distinct real numbers we could take $x = \sqrt{2}$, $y = 1 + \sqrt{2}$, and $z = -\sqrt{2}$, for example.

Remark. Some common mistakes appearing in false arguments:

- From the assumption that $a + b$ is rational, it does not follow that a and b are both rational.
- Every integer is rational. If $n \in \mathbb{Z}$ then we can write $n = n/1$, so n is rational.
- If you want to prove the contrapositive of the problem statement, your hypothesis will be "x, y, and z are not all rational"; that is, "at least one of x, y and z is irrational".
  - You cannot assume that *all* of x, y, and z are irrational.
  - You cannot assume that one *and only one* of x, y, and z is irrational.
  - Even if you could assume that one and only one of x, y, and z is irrational, you could not say "without loss of generality x is irrational and y and z are rational." The problem statement is not symmetric with respect to x, y, and z.

Problem 5 (5 points). The Fibonacci sequence $\{a_1, a_2, a_3, \ldots\}$ is defined by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-2} + a_{n-1}, \quad \text{if } n \geq 3. \end{aligned}$$

Prove that $a_n \geq (8/5)^{n-2}$ for all $n \in \mathbb{N}$.

Problem 6 (5 points). Define the equivalence relation E on $\mathbb{Z}$ by

$$x \, E \, y \iff x^2 \equiv y^2 \pmod 7, \quad \text{for all } x, y \in \mathbb{Z}$$

and let $\mathbb{Z}/E$ denote the set of equivalence classes of E. Is there a function $f : \mathbb{Z}/E \to \mathbb{Z}/E$ such that for all integers x we have $f([x]) = [x^3]$? Why or why not?

Solution. Yes. To show that there is such a function, it suffices (as explained below) to check that for all $x, y \in \mathbb{Z}$, if $[x] = [y]$, then $[x^3] = [y^3]$. So let $x, y \in \mathbb{Z}$ and assume that $[x] = [y]$. In other words, $x^2 \equiv y^2 \pmod 7$. Then we have

$$(x^3)^2 = x^6 = (x^2)^3 \equiv (y^2)^3 = y^6 = (y^3)^2 \pmod 7,$$

so $[x^3] = [y^3]$ as desired. This means that the relation f on $\mathbb{Z}/E$ defined by

$$f = \{([x], [x^3]) : x \in \mathbb{Z}\}$$

is a function from $\mathbb{Z}/E$ to $\mathbb{Z}/E$. It is clear that this function f has the desired property that $f([x]) = [x^3]$ for all integers x.

Remark. In the context of this problem, checking the implication "if $[x] = [y]$, then $[x^3] = [y^3]$" is sometimes called "checking whether the function f is well-defined." This terminology is confusing because it seems to presuppose that a function f is given, when actually it is not. Your task is to determine whether there is a function satisfying a desired property. In this case, there is such a function. It may be helpful to see the solutions to the yellow exam for a similar example where there is *not* a function satisfying the desired property.

Remark. Some common mistakes:

- If you want to prove that such a function exists (which is the correct solution) then you should not use the symbol "f" in a way that *presupposes* that such a function exists.
- The problem does not ask you to check that the relation E is an equivalence relation. This was given.
- The fact that $[x^3] \neq [x]$ for some integers x is not relevant. The problem asks you whether there is a function f satisfying a certain property. It does not ask you whether the *identity function* on $\mathbb{Z}/E$ satisfies that property.
- Many students repeated the proof that the operation of addition is well-defined on $\mathbb{Z}_7$. This problem is different in two ways: (1) the operation is cubing, not addition, and (2) the domain is $\mathbb{Z}/E$, not $\mathbb{Z}_7$.
- Many students rewrote the equation $y^2 - x^2 = 7k$ as $y = \sqrt{x^2 + 7k}$. I think this is meaningless because a single number can have multiple square roots modulo 7; for example, $4 = 2^2 \pmod 7$ and $4 = 5^2 \pmod 7$, and unlike in the case of the real numbers, there is no notion of "positive" in modular arithmetic that gives a canonical choice of square root. Note that the ordinary notion of square root for real numbers is irrelevant; it may result in numbers that are not integers, and it certainly isn't well-defined mod 7.

Problem 7 (5 points). Let A and B be disjoint sets. Prove that there is a bijection from $\mathcal{P}(A) \times \mathcal{P}(B)$ to $\mathcal{P}(A \cup B)$.

Solution. Define a function $f : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \cup B)$ by

$$f((C, D)) = C \cup D.$$

Here (C, D) denotes an arbitrary element of $\mathcal{P}(A) \times \mathcal{P}(B)$, so C denotes an arbitrary subset of A and D denotes an arbitrary subset of B.

The function f is injective: Let $(C, D), (C', D') \in \mathcal{P}(A) \times \mathcal{P}(B)$ and assume that $f((C, D)) = f((C', D'))$. In other words, $C \cup D = C' \cup D'$. Because $C, C' \subseteq A$ and $D, D' \subseteq B$ and $A \cap B = \emptyset$, we have

$$C = (C \cup D) \cap A = (C' \cup D') \cap A = C' \quad \text{and}$$
$$D = (C \cup D) \cap B = (C' \cup D') \cap B = D'.$$

So $(C, D) = (C', D')$ as desired.

The function f is surjective: Let $E \in \mathcal{P}(A \cup B)$. That is, $E \subseteq A \cup B$. Defining $C = E \cap A$ and $D = E \cap B$, we have $C \subseteq A$ and $D \subseteq B$ and $E = C \cup D$, so $E = f((C, D))$ as desired.

Remark. A few students argued along the following lines: Let $m = |A|$ and $n = |B|$. Then $|\mathcal{P}(A)| = 2^m$ and $|\mathcal{P}(B)| = 2^n$. We have $|A \cup B| = |A| + |B|$ because A and B are disjoint, so $|\mathcal{P}(A \cup B)| = 2^{m+n}$. Therefore $|\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(B)| = 2^m \cdot 2^n = 2^{m+n} = |\mathcal{P}(A \cup B)|$, so there is a bijection from $\mathcal{P}(A) \times \mathcal{P}(B)$ to $\mathcal{P}(A \cup B)$.

Unfortunately, this argument only works when A and B are finite sets. It is possible to define a notion of "infinite cardinal number" and do a similar argument for infinite sets. But what is the meaning of the equation

$$2^m \cdot 2^n = 2^{m+n}$$

for infinite cardinal numbers m and n? Is it a true statement? (We have seen that some properties of finite sets do not hold for infinite sets.)

In fact this equation can be given a meaning and proved to hold for infinite cardinal numbers m and n. But the proof would look a lot like the solution I gave above! This is because one *defines* the product of cardinalities as the cardinality of a product, and *defines* the sum of cardinalities as the cardinality of a disjoint union, and *defines* the equality of cardinalities as the existence of a bijection. Moral: "cardinality" is not a magic wand that you can wave to conjure up a bijection if you didn't already know how to construct one.

Problem 8 (5 points). Prove that there is a bijection from $\mathbb{R}$ to $\mathbb{R} - \{0\}$.

Solution. Define a function $f : \mathbb{R} \to \mathbb{R} - \{0\}$ by

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is a nonnegative integer} \\ x & \text{otherwise (that is, if } x \text{ is negative or } x \text{ is not an integer).} \end{cases}$$

Note that f really is a function into $\mathbb{R} - \{0\}$: the number 0 is not in the range because if x is not a nonnegative integer then neither is $f(x)$, and if x is a nonnegative integer then $f(x) > 0$.

The function f is injective: Let $x, y \in \mathbb{R}$ and assume that $x \neq y$.

- Case 1: x and y are nonnegative integers. Then $f(x) = x + 1 \neq y + 1 = f(y)$.
- Case 2: one of x and y is a nonnegative integer and the other isn't. Without loss of generality, x is and y isn't. Then $f(x)$ (which equals $x + 1$) is a nonnegative integer and $f(y)$ (which equals y) isn't, so $f(x) \neq f(y)$.
- Case 3: neither x nor y is a positive integer. Then $f(x) = x \neq y = f(y)$.

The function f is surjective: Let $y \in \mathbb{R} - \{0\}$.

- Case 1: y is a nonnegative integer. Because $y \neq 0$, the number $y - 1$ is also a nonnegative integer, and $y = f(y - 1)$.
- Case 2: otherwise. Then $y = f(y)$.

Remark. Many students either

- defined a bijection from $\mathbb{R} - \{0\}$ to $\mathbb{R} - \{0\}$,
- defined a bijection from $\mathbb{R}$ to $\mathbb{R}$,
- defined a function that was a bijection $\mathbb{Z}$ to $\mathbb{Z} - \{0\}$, but did not extend to a bijection $\mathbb{R}$ to $\mathbb{R} - \{0\}$ (in most cases either $1/2$ was not in the range, or $-1/2$ was not in the range, for example,) or
- argued that any two uncountable sets have the same cardinality (meaning that there is a bijection between them,) which is not true. For example, if X is any uncountable set then $\mathcal{P}(X)$ is also uncountable because there is an injection from X to $\mathcal{P}(X)$; however, by Cantor's theorem there can be no bijection from X to $\mathcal{P}(X)$.