

SPECTRAL NORM OF PRODUCTS OF RANDOM AND DETERMINISTIC MATRICES

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ABSTRACT. We study the spectral norm of matrices W that can be factored as $W = BA$, where A is a random matrix with independent mean zero entries and B is a fixed matrix. Under the $(4 + \varepsilon)$ -th moment assumption on the entries of A , we show that the spectral norm of such an $m \times n$ matrix W is bounded by $\sqrt{m} + \sqrt{n}$, which is sharp. In other words, in regard to the spectral norm, products of random and deterministic matrices behave similarly to random matrices with independent entries. This result along with the previous work of M. Rudelson and the author implies that the smallest singular value of a random $m \times n$ matrix with i.i.d. mean zero entries and bounded $(4 + \varepsilon)$ -th moment is bounded below by $\sqrt{m} - \sqrt{n - 1}$ with high probability.

1. INTRODUCTION

This paper grew out of an attempt to understand the class of random matrices with non-independent entries, but which can be *factorized* through random matrices with independent entries. Equivalently, we are interested in sample covariance matrices of a wide class of random vectors – the linear transformations of vectors with independent entries.

Here we study the spectral norm of such matrices. Recall that the spectral norm $\|W\|$ is defined as the largest singular value of a matrix W , which equals the largest eigenvalue of $\sqrt{WW^*}$. Equivalently, the spectral norm can be defined as the $\ell_2 \rightarrow \ell_2$ operator norm: $\|W\| = \sup_x \|Wx\|_2 / \|x\|_2$ where $\|\cdot\|_2$ denotes the Euclidean norm. The spectral norm of random matrices plays a notable role in particular in geometric functional analysis, computer science, statistical physics, and signal processing.

1.1. Matrices with independent entries. For random matrices with independent and identically distributed entries, the spectral norm is well studied. Let W be an $m \times n$ matrix whose entries are real independent and identically

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distributed random variables with mean zero, variance 1 and finite fourth moment. Estimates of the type

$$(1.1) \quad \|W\| \sim \sqrt{n} + \sqrt{m}$$

are known to hold (and are sharp) in both the limit regime for dimensions increasing to infinity, and the non-limit regime where the dimensions are fixed. The meaning of (1.1) in the limit regime is that, for a family of matrices as above whose dimensions m and n increase to infinity and whose aspect ratio m/n converges to a constant, the ratio $\|W\|/(\sqrt{n} + \sqrt{m})$ converges to 1 almost surely [32].

In the non-limit regime, i.e. for arbitrary dimensions n and m , variants of (1.1) were proved by Y. Seginer [28] and R. Latala [17]. If W is an $m \times n$ matrix whose entries are i.i.d. mean zero random variables, then denoting the rows of W by X_i and the columns by Y_j , the result of Y. Seginer [28] states that

$$\mathbb{E}\|W\| \leq C(\mathbb{E} \max_i \|X_i\|_2 + \mathbb{E} \max_j \|Y_j\|_2)$$

where C is an absolute constant. This estimate is sharp because $\|W\|$ is obviously bounded below by the Euclidean norm of any row and any column of W . Furthermore, if the entries w_{ij} of the matrix W are not necessarily identically distributed, then R. Latala's result [17] states that

$$\mathbb{E}\|W\| \leq C(\max_i \mathbb{E}\|X_i\|_2 + \max_j \mathbb{E}\|Y_j\|_2 + (\sum_{i,j} \mathbb{E}w_{ij}^4)^{1/4}).$$

In particular, if W is an $m \times n$ matrix whose entries are independent random variables with mean zero and fourth moments bounded by 1, then one can deduce from either Y. Seginer's or R. Latala's result that

$$(1.2) \quad \mathbb{E}\|W\| \leq C(\sqrt{n} + \sqrt{m}).$$

This is a variant of (1.1) in the non-limit regime.

The fourth moment hypothesis is known to be necessary. Consider again a family of matrices whose dimensions m and n increase to infinity, and whose aspect ratio m/n converges to a constant. If the entries are independent and identically distributed random variables with mean zero and infinite fourth moment, then the upper limit of the ratio $\|W\|/(\sqrt{n} + \sqrt{m})$ is infinite almost surely [32].

1.2. The main result. The main result of this paper is an extension of the optimal bound (1.2) to the class of random matrices with non-independent entries, but which can be factored through a matrix with independent entries.

Theorem 1.1. *Let $\varepsilon \in (0, 1)$ and let m, n, N be positive integers. Consider a random $m \times n$ matrix $W = BA$, where A is an $N \times n$ random matrix whose*

entries are independent random variables with mean zero and $(4+\varepsilon)$ -th moment bounded by 1, and B is an $m \times N$ non-random matrix such that $\|B\| \leq 1$. Then

$$(1.3) \quad \mathbb{E}\|W\| \leq C(\varepsilon)(\sqrt{n} + \sqrt{m})$$

where $C(\varepsilon)$ is a function that depends only on ε .

Remarks. 1. An important feature of this result is that its conclusion is independent of the dimension N .

2. The proof of Theorem 1.1 yields the stronger estimate

$$(1.4) \quad \mathbb{E}\|W\| \leq C(\varepsilon)(\|B\|\sqrt{n} + \|B\|_{\text{HS}})$$

valid for arbitrary (non-random) $m \times N$ matrix B . This result is independent of the dimensions of the matrix B , and therefore it holds for an arbitrary linear operator B acting from the N -dimensional Euclidean space ℓ_2^N to an arbitrary Hilbert space.

3. Theorem 1.1 can be interpreted in terms of *sample covariance matrices* of random vectors in \mathbb{R}^m of the form BX , where X is a random vector in \mathbb{R}^N with independent entries. Indeed, let A be the random matrix whose columns are n independent samples of the vector X . Then $W = BA$ is the matrix whose columns are n independent samples of the random vector BX . The sample covariance matrix of the random vector BX is defined as $\Sigma = \frac{1}{n}WW^*$. Theorem 1.1 states that the largest eigenvalue of Σ is bounded by $C_1(\varepsilon)(1 + m/n)$, which is further bounded by $C_2(\varepsilon)$ for the number of samples $n \gtrsim m$ (and independently of the dimension N). This problem was previously studied in [4], [5] in the limit regime for $m = N$, where the result must of course depend on N .

4. Under the stronger subgaussian moment assumption (1.6) on the entries, Theorem 1.1 is easy to prove using standard concentration and an ε -net argument. In contrast, if only some finite moment is assumed, we do not know any simple proof.

1.3. The smallest singular value. Our main motivation for Theorem 1.1 was to complete the analysis of the *smallest singular value* of random rectangular matrices carried out by M. Rudelson and the author in [27]. The smallest singular value $s_{\min}(W)$ of a matrix W can be equivalently described as $s_{\min}(W) = \inf_x \|Wx\|_2 / \|x\|_2$.

Analyzing the smallest singular value is generally harder than analyzing the largest one (the spectral norm). The analogue of (1.1) for the smallest singular value of random $m \times n$ matrices W (for $m > n$) is

$$(1.5) \quad s_{\min}(W) \sim \sqrt{m} - \sqrt{n}.$$

The optimal limit version of this result proved in [7] holds under exactly the same hypotheses as (1.1) – for i.i.d. entries with mean zero, variance 1 and finite fourth moment.

Many papers addressed (1.5) for fixed dimensions n, m . Sufficiently tall matrices ($m \geq Cn$ for sufficiently large C) were studied in [8]; extensions to genuinely rectangular matrices ($m > (1 + \varepsilon)n$ for some $\varepsilon > 0$) were studied in [20, 2, 23], with gradually improving dependence on ε . An optimal version of (1.5) for all dimensions was obtained in [27]. All these works put somewhat stronger moment assumptions than the fourth moment of the entries w_{ij} of the matrix W . A convenient assumption is that the entries w_{ij} are *subgaussian* random variables. This means that all their moments are bounded by the corresponding moments of the standard normal random variable, i.e.

$$(1.6) \quad (\mathbb{E}|w_{ij}|^p)^{1/p} \leq M\sqrt{p} \quad \text{for all } p \geq 1$$

where M is called the subgaussian moment. It was proved in [27] that if the entries of W are i.i.d. mean zero subgaussian random variables with unit variance, then for every $t > 0$ one has

$$(1.7) \quad \mathbb{P}(s_{\min}(W) \leq t(\sqrt{m} - \sqrt{n-1})) \leq (Ct)^{m-n+1} + e^{-cm}$$

where $C, c > 0$ depend only on the subgaussian moment M . In particular, for such matrices we have

$$(1.8) \quad s_{\min}(W) \geq c_1(\sqrt{m} - \sqrt{n-1}) \quad \text{with high probability}$$

where $c_1 > 0$ depends only on the desired probability and the subgaussian moment. This result encompasses the case of square matrices where $m = n$ and hence (1.8) yields $s_{\min}(W) \geq c_2/\sqrt{n}$. For Gaussian square matrices this optimal bound was obtained in [11] and [29]; for general square matrices a weaker bound $n^{-3/2}$ was obtained in [24] and the best bound as above in [25]; the estimate is shown to be optimal in [26].

Whether (1.8) holds under weaker moment assumptions was only known in the case of square matrices. It was proved in [25] using (1.2) that (1.8) holds under the fourth moment assumption for square matrices, i.e. for $m = n$. Whether the same is true for arbitrary rectangular matrices under the fourth moment assumption was left open in [27]. The bottleneck of the argument occurred in Proposition 7.3 on [27] where we needed a correct bound on the spectral norm of a product of a random matrix and a fixed orthogonal projection. Such a bound was easy to get only under the subgaussian hypothesis. Theorem 1.1 of the present paper extends the argument of [27] for random matrices with bounded $(4 + \varepsilon)$ -th moment. It follows directly from the argument of [27] and Theorem 1.1.

Corollary 1.2 (Smallest singular value). *Let $\varepsilon \in (0, 1)$ and $m \geq n$ be positive integers. Let A be a random $m \times n$ matrix whose entries are i.i.d. random*

variables with mean zero, unit variance and $(4 + \varepsilon)$ -th moment bounded by M . Then, for every $\delta > 0$ there exist $t > 0$ and n_0 which depend only on ε , δ and M , and such that

$$\mathbb{P}(s_{\min}(A) \leq t(\sqrt{m} - \sqrt{n-1})) \leq \delta \quad \text{for all } n \geq n_0.$$

This result follows by the argument in [27], where one considers probability estimates conditional on the event that the norm of a product W of a random matrix and a non-random orthogonal projection is small (see [27, Proposition 7.3]).

After this paper was written, two important related results appeared on the universality of the smallest singular value in two extreme regimes – for almost square matrices and for genuinely rectangular matrices. One of these results, by T. Tao and V. Vu [31] works for square and almost square matrices where the defect $m - n$ is constant. It is valid for matrices with i.i.d. entries with mean zero, unit variance and bounded C -th moment where C is a sufficiently large absolute constant. The result states that the smallest singular value of such $m \times n$ matrices A is asymptotically the same as of the Gaussian matrix G of the same dimensions and with i.i.d. standard normal entries. Specifically,

$$(1.9) \quad \mathbb{P}(m s_{\min}(G)^2 \leq t - m^{-c}) - m^{-c} \leq \mathbb{P}(m s_{\min}(A)^2 \leq t) \\ \leq \mathbb{P}(m s_{\min}(G)^2 \leq t + m^{-c}) + m^{-c}.$$

This universality result, combined with the known asymptotic estimates of the smallest singular value of Gaussian matrices $s_{\min}(G)$ allows one to obtain bounds sharper than in Corollary 1.2. However, the universality result of [31] is only known in the almost square regime $m - n = O(1)$ (and under stronger moment assumptions), while Corollary 1.2 is valid for all dimensions $m \geq n$.

Another recent universality result was obtained by O. Feldheim and S. Sodin [12] for genuinely rectangular matrices, i.e. with aspect ratio m/n separated from 1 by a constant, and with subgaussian i.i.d. entries. In particular they proved the inequality

$$(1.10) \quad \mathbb{P}(s_{\min}(A) \leq (\sqrt{m} - \sqrt{n})^2 - tm) \leq \frac{C}{1 - \sqrt{m/n}} \exp(-cnt^{3/2}).$$

Deviation inequalities (1.7) and (1.10) complement each other – the former is multiplicative (and is valid for arbitrary dimensions) while the latter is additive (and is applicable for genuinely rectangular matrices). Each of these two inequalities clearly has the regime where it is stronger.

1.4. Outline of the argument. Let us sketch the proof of Theorem 1.1. We can assume that $m = n$ by adding an appropriate number of zero columns to A or rows to B . Since the columns of A are independent, the columns

X_1, \dots, X_n of the matrix W are independent random vectors in \mathbb{R}^n . We would like to bound the spectral norm of $WW^* = \sum_j X_j \otimes X_j$, which is a sum of independent random operators. For random vectors X_j uniformly distributed in convex bodies, deviation inequalities for sums $\sum_j X_j \otimes X_j$ were studied in [15, 10, 22, 14, 21, 3, 1]. For general distributions, a sharp estimate for such sums has been proved by M. Rudelson [22]. This approach, which we develop in Section 3, leads us to the bound

$$(1.11) \quad \mathbb{E}\|W\| \leq C\sqrt{n \log n}.$$

This bound is already independent of the dimension N , but is off by $\sqrt{\log n}$ from being optimal. The logarithmic term is unfortunately a limitation of this method. This term comes from M. Rudelson's result, Theorem 3.1 below, where it is needed in full generality. It would be useful to understand the situations where the logarithmic term can be removed from M. Rudelson's theorem. So far, only one such situation is known from [1] where the independent random vectors X_j are uniformly distributed in a convex body.

In absence of a suitable variant of M. Rudelson's theorem without the logarithmic term, the rest of our argument will proceed to remove this term from (1.11) using the rich independence structure, which is inherited by the vectors X_j from the random matrix A . However, the independence structure is encoded nontrivially via the linear transformation B , which makes the entries of X_j dependent). A more delicate application of M. Rudelson's theorem allows one to transfer the logarithmic term from the conclusion to the assumption. Namely, Theorem 3.9 establishes the optimal bound $\mathbb{E}\|W\| \leq C\sqrt{n}$ in the case when all columns of B are logarithmically small, i.e. their Euclidean norm is at most $\log^{-O(1)} n$. While some columns of a general matrix B may be large, the boundedness of B implies that most columns are always logarithmically small – all but all but $n \log^{O(1)} n$ of them. So, we can remove from B the already controlled small columns, which will make B an almost square matrix. In other words, we can assume hereafter that $N = n \log^{O(1)} n$.

The advantage of almost square matrices is that the magnitude of their entries is easy to control. A simple consequence of the $(4 + \varepsilon)$ -th moment hypothesis and Markov's inequality yields that the entries of $A = (a_{ij})$ satisfy $\max_{i,j} |a_{ij}| \leq \sqrt{n}$ with high probability. Note that the same estimate holds for square matrices ($N = n$) under the fourth moment assumption. So, in regard to the magnitude of entries, almost square matrices are similar to exactly square matrices, for which the desired bound follows from R. Latała's result (1.2).

This prompts us to construct the proof of Theorem 1.1 for almost square matrices similarly to R. Latała's argument in [17], i.e. using fairly standard concentration of measure results in the Gauss space, coupled with delicate

constructions of nets. We first decompose A into a sum of matrices which contain entries of similar magnitude. As the magnitude increases, these matrices become sparser. This quickly reduces the problem to random sparse matrices, whose entries are i.i.d. random variables valued in $\{-1, 0, 1\}$. The spectral norm of random sparse matrices was studied in [16] as a development of the work of Z. Füredi and J. Komlos [13]. However, we need to bound the spectral norm of the matrix $W = BA$ rather than A . Independence of entries is not available for W , which makes it difficult to use the known combinatorial methods based on the bounding trace of high powers of W .

To summarize, at this point we have an almost square random sparse matrix A , and we need to bound the spectral norm of $W = BA$, which is $\|W\| = \sup_x \|Wx\|_2$, where the supremum is over all unit vectors $x \in \mathbb{R}^n$. The well known method is to first fix x and bound $\|Wx\|_2$ with high probability; then take a union bound over all x in a sufficiently fine net of the unit sphere of \mathbb{R}^n . However, a probability bound for every fixed vector x , which follows from standard concentration inequalities, is not strong enough to make this method work. *Sparse vectors* – those which have few but large nonzero coordinates – produce worse concentration bounds than *spread vectors*, which have many but small nonzero coordinates. What helps us is that there are fewer sparse vectors on the sphere than there are spread vectors. This leads to a tradeoff between concentration and entropy, i.e. between the probability with which $\|Wx\|_2$ is nicely bounded, and the size of a net for the vectors x which achieve this probability bound. One then divides the unit Euclidean sphere in \mathbb{R}^n into classes of vectors according to their “sparsity”, and uses the entropy-concentration tradeoff for each class separately. This general line is already present in Latala’s argument [17], and it was developed extensively in the recent years, see e.g. [20, 24, 25, 27]. This argument is presented in Section 4, where it leads to a useful estimate for norms of sparse matrices, Corollary 4.9. With this in hand, one can quickly finish the proof of Theorem 1.1.

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2. PRELIMINARIES

2.1. Notation. Throughout the paper, the results are stated and proved over the field of real numbers. They are easy to generalize to complex numbers.

We denote by $C, C_1, c, c_1 \dots$ positive absolute constants, and by $C(\varepsilon), C_1(\varepsilon), \dots$ positive quantities that may depend only on the parameter ε . Their values can change from line to line.

The standard inner product in \mathbb{R}^n is denoted $\langle x, y \rangle$. For a vector $x \in \mathbb{R}^n$, we denote the cardinality of its support by $\|x\|_0 = |\{j : x_j \neq 0\}|$, the Euclidean

norm by $\|x\|_2 = (\sum_j x_j^2)^{1/2}$, and the sup-norm by $\|x\|_\infty = \max_j |x_j|$. The unit Euclidean ball in \mathbb{R}^n is denoted by $B_2^n = \{x : \|x\|_2 \leq 1\}$, and the unit Euclidean sphere in \mathbb{R}^n is denoted by $S^{n-1} = \{x : \|x\|_2 = 1\}$.

The tensor product of vectors $x, y \in \mathbb{R}^n$ is the linear operator $x \otimes y$ on \mathbb{R}^n defined as $(x \otimes y)(z) = \langle x, z \rangle y$ for $z \in \mathbb{R}^n$.

2.2. Concentration of measure. The method that we carry out in Section 4 uses concentration in the Gauss space in combination with constructions of ε -nets. Here we recall some basic facts we need.

The standard Gaussian random vector $g \in \mathbb{R}^m$ is a random vector whose coordinates are independent standard normal random variables. The following concentration inequality can be found e.g. in [19, inequality (1.5)].

Theorem 2.1 (Gaussian concentration). *Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a Lipschitz function. Let g be a standard Gaussian random vector in \mathbb{R}^m . Then for every $t > 0$ one has*

$$\mathbb{P}(f(g) - \mathbb{E}f(g) > t) \leq \exp(-c_0 t^2 / \|f\|_{\text{Lip}}^2)$$

where $c_0 \in (0, 1)$ is an absolute constant.

As a very restrictive but useful example, Theorem 2.1 implies the following deviation inequality for sums of independent exponential random variables g_i^2 (which can also be derived by the more standard approach via moment generating functions).

Corollary 2.2 (Sums of exponential random variables). *Let $d = (d_1, \dots, d_m)$ be a vector of real numbers, and let g_1, \dots, g_m be independent standard normal random variables. Then, for every $t > 0$ we have*

$$\mathbb{P}\left\{\left(\sum_{i=1}^m d_i^2 g_i^2\right)^{1/2} > \|d\|_2 + t\right\} \leq \exp(-c_0 t^2 / \|d\|_\infty^2).$$

Proof. The function $f(y) = (\sum_{i=1}^m d_i^2 y_i^2)^{1/2}$ is a Lipschitz function on \mathbb{R}^m with $\|f\|_{\text{Lip}} = \|d\|_\infty$. Moreover, Hölder's inequality implies that

$$\mathbb{E}f(g) = \mathbb{E}\left(\sum_{i=1}^m d_i^2 g_i^2\right)^{1/2} \leq \left(\mathbb{E}\sum_{i=1}^m d_i^2 g_i^2\right)^{1/2} = \|d\|_2.$$

Theorem 2.1 completes the proof. □

Another classical deviation inequality we will need is Bennett's inequality, see e.g. [9, Theorem 2]:

Theorem 2.3 (Bennett’s inequality). *Let X_1, \dots, X_N be independent mean zero random variables such that $|X_i| \leq 1$ for all i . Consider the sum $S = X_1 + \dots + X_N$ and let $\sigma^2 := \text{Var}(S)$. Then, for every $t > 0$ we have*

$$\mathbb{P}(S > t) \leq \exp(-\sigma^2 h(t/\sigma^2))$$

where $h(u) = (1 + u) \log(1 + u) - u$.

We will also need M. Talagrand’s concentration inequality for convex Lipschitz functions from [30, Theorem 6.6]; see also [18, Corollary 4.10] and the discussion below it.

Theorem 2.4 (Concentration of Lipschitz convex functions). *Let X_1, \dots, X_m be independent random variables such that $|X_i| \leq K$ for all i . Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a convex and 1-Lipschitz function. Then for every $t > 0$ one has*

$$\mathbb{P}(|f(X_1, \dots, X_m) - \mathbb{E}f(X_1, \dots, X_m)| > Kt) \leq 4 \exp(-t^2/4).$$

2.3. Nets. Consider a subset U of a normed space X , and let $\varepsilon > 0$. Recall that an ε -net of U is a subset \mathcal{N} of U such that the distance from any point of U to \mathcal{N} is at most ε . In other words, for every $x \in U$ there exists $y \in \mathcal{N}$ such that $\|x - y\|_X \leq \varepsilon$.

The following estimate follows by a volumetric argument, see e.g. the proof of Lemma 9.5 in [19].

Lemma 2.5 (Cardinality of ε -nets). *Let $\varepsilon \in (0, 1)$. The unit Euclidean ball B_2^n and the unit Euclidean sphere S^{n-1} in \mathbb{R}^n both have ε -nets of cardinality at most $(1 + 2/\varepsilon)^n$.*

When computing norms of linear operators, ε -nets provide a convenient discretization of the problem. We formalize it in the next proposition.

Proposition 2.6 (Computing norms on nets). *Let $A : X \rightarrow Y$ be a linear operator between normed spaces X and Y , and let \mathcal{N} be an ε -net of either the unit sphere $S(X)$ or the unit ball $B(X)$ of X for some $\varepsilon \in (0, 1)$. Then*

$$\|A\| \leq \frac{1}{1 - \varepsilon} \sup_{x \in \mathcal{N}} \|Ax\|_Y.$$

Proof. We give the proof for an ε -net of the unit sphere; the case of the unit ball is similar. Every $z \in S(X)$ has the form $z = x + h$, where $x \in \mathcal{N}$ and $\|h\|_X \leq \varepsilon$. Since $\|A\| = \sup_{z \in S(X)} \|Az\|_Y$, the triangle inequality yields

$$\|A\| \leq \sup_{x \in \mathcal{N}} \|Ax\|_Y + \sup_{\|h\|_X \leq \varepsilon} \|Ah\|_Y.$$

The last term in the right hand side is bounded by $\varepsilon\|A\|$. Thus we have shown that

$$(1 - \varepsilon)\|A\| \leq \sup_{x \in \mathcal{N}} \|Ax\|_Y.$$

This completes the proof. \square

2.4. Symmetrization. We will use the standard symmetrization technique as was done in [17]; see more general inequalities in e.g. [19, Section 6.1]. To this end, let the matrices $A = (a_{ij})$ and B be as in Theorem 1.1. Let $A' = (a'_{ij})$ be an independent copy of A , and let ε_{ij} be independent symmetric Bernoulli random variables. Then, by Jensen's inequality,

$$\begin{aligned} \mathbb{E}\|BA\| &= \mathbb{E}\|B(A - \mathbb{E}A')\| \leq \mathbb{E}\|B(A - A')\| \\ &= \mathbb{E}\|B(\varepsilon_{ij}(a_{ij} - a'_{ij}))\| \leq 2\mathbb{E}\|B(\varepsilon_{ij}a_{ij})\|. \end{aligned}$$

Therefore, we can assume without loss of generality in Theorem 1.1 that a_{ij} are symmetric random variables. Furthermore, let g_{ij} be independent standard normal random variables. Then, again by Jensen's inequality,

$$\begin{aligned} \mathbb{E}\|B(g_{ij}a_{ij})\| &= \mathbb{E}\|B(\varepsilon_{ij}|g_{ij}|a_{ij})\| \geq \mathbb{E}\|B(\varepsilon_{ij}\mathbb{E}(|g_{ij}|)a_{ij})\| \\ &= (2/\pi)^{1/2}\mathbb{E}\|B(\varepsilon_{ij}a_{ij})\|. \end{aligned}$$

Therefore

$$(2.1) \quad \mathbb{E}\|BA\| \leq (2\pi)^{1/2}\mathbb{E}\|B(g_{ij}a_{ij})\|.$$

Conditioning on a_{ij} , we thus reduce the problem to random *gaussian* matrices.

We will use a similar symmetrization technique several times in our argument. In particular, in the proof of Lemma 3.8 we apply the following observation, which can be deduced from standard symmetrization lemma ([19] Lemma 6.3) and the contraction principle ([19] Theorem 4.4). For the reader's convenience we include a direct proof.

Lemma 2.7 (Symmetrization). *Consider independent mean zero random variables Z_{ij} such that $|Z_{ij}| \leq 1$, independent symmetric Bernoulli random variables ε_{ij} , and vectors x_{ij} in some Banach space, where both i and j range in some finite index sets. Then*

$$\mathbb{E} \max_j \left\| \sum_i Z_{ij} x_{ij} \right\| \leq 2 \mathbb{E} \max_j \left\| \sum_i \varepsilon_{ij} x_{ij} \right\|.$$

Proof. To be specific, we can assume that both indices i and j range in the interval $\{1, \dots, n\}$ for some integer n . Let (Z'_{ij}) denote an independent copy of the sequence of random variables (Z_{ij}) . Then $Z_{ij} - Z'_{ij}$ are symmetric random

variables. We have

$$\begin{aligned}
\mathbb{E} \max_j \left\| \sum_i Z_{ij} x_{ij} \right\| &\leq \mathbb{E} \max_j \left\| \sum_i (Z_{ij} - \mathbb{E} Z'_{ij}) x_{ij} \right\| \quad (\text{since } \mathbb{E} Z'_{ij} = 0) \\
&\leq \mathbb{E} \max_j \left\| \sum_i (Z_{ij} - Z'_{ij}) x_{ij} \right\| \quad (\text{by Jensen's inequality}) \\
&= \mathbb{E} \max_j \left\| \sum_i \varepsilon_{ij} (Z_{ij} - Z'_{ij}) x_{ij} \right\| \quad (\text{by symmetry}) \\
&\leq 2 \max_{|a_{ij}| \leq 1} \mathbb{E} \max_j \left\| \sum_i \varepsilon_{ij} a_{ij} x_{ij} \right\|
\end{aligned}$$

where the last line follows because $|Z_{ij} - Z'_{ij}| \leq |Z_{ij}| + |Z'_{ij}| \leq 2$. The function on \mathbb{R}^{n^2}

$$(a_{ij})_{i,j=1}^n \mapsto \mathbb{E} \max_j \left\| \sum_i \varepsilon_{ij} a_{ij} x_{ij} \right\|$$

is a convex function. Therefore, on the compact convex set $[-1, 1]^{n^2}$ it attains its maximum on the extreme points, where all $a_{ij} = \pm 1$. By symmetry, the function takes the same value at each extreme point, which equals

$$\mathbb{E} \max_j \left\| \sum_i \varepsilon_{ij} x_{ij} \right\|.$$

This completes the proof. \square

2.5. Truncation and conditioning. We will need some elementary observations related to truncation and conditioning of random variables.

Lemma 2.8 (Truncation). *Let X be a non-negative random variable, and let $M > 0$, $p \geq 1$. Then*

$$\mathbb{E} X \mathbf{1}_{\{X \geq M\}} \leq \frac{\mathbb{E} X^p}{M^{p-1}}.$$

Proof. Indeed,

$$\mathbb{E} X \mathbf{1}_{\{X \geq M\}} \leq \mathbb{E} X (X/M)^{p-1} \mathbf{1}_{\{X \geq M\}} \leq \mathbb{E} X^p / M^{p-1}.$$

The Lemma is proved. \square

We will also need two elementary conditioning lemmas. In Section 4, we will need to control the maximal magnitude of the entries $M_0 = \max_{ij} |a_{ij}|$ of the random matrix A . Conditioning on M_0 will unfortunately destroy the independence of the entries. So, we will instead condition on an event $\{M_0 \leq t\}$ for fixed t , which will clearly preserve the independence. This conditional argument used in the proof of Corollary 4.11 relies on the following two elementary lemmas.

Lemma 2.9. *Let X be a random variable and K be a real number. Then*

$$\mathbb{E}(X | X \leq K) \leq \mathbb{E}X.$$

Proof. By the law of total probability,

$$\mathbb{E}X = \mathbb{E}(X | X \leq K) \mathbb{P}(X \leq K) + \mathbb{E}(X | X > K) \mathbb{P}(X > K).$$

Thus $\mathbb{E}X$ is a convex combination of the numbers $a = \mathbb{E}(X | X \leq K)$ and $b = \mathbb{E}(X | X > K)$. Since clearly $a \leq K \leq b$, we must have $a \leq \mathbb{E}X \leq b$. \square

Lemma 2.10. *Let X, Y be non-negative random variables. Assume there exists $K, L > 0$ such that one has for every $t \geq 1$:*

$$(2.2) \quad \mathbb{E}(X^2 | Y \leq t) \leq K^2 t, \quad \mathbb{P}(Y > Lt) \leq \frac{1}{t^2}.$$

Then $\mathbb{E}X \leq CK\sqrt{L}$.

Proof. Without loss of generality we can assume that $K = 1$ by rescaling X to X/K . Thus we have for every $t \geq 1$:

$$(2.3) \quad \mathbb{E}X^2 \mathbf{1}_{\{Y \leq t\}} \leq \mathbb{E}(X^2 | Y \leq t) \leq t.$$

We consider the decomposition

$$\mathbb{E}X = \mathbb{E}X \mathbf{1}_{\{Y \leq L\}} + \sum_{k=1}^{\infty} \mathbb{E}X \mathbf{1}_{\{2^{k-1}L < Y \leq 2^k L\}}.$$

By (2.3) and Hölder's inequality, the first term is bounded as

$$\mathbb{E}X \mathbf{1}_{\{Y \leq L\}} \leq (\mathbb{E}X^2 \mathbf{1}_{\{Y \leq L\}})^{1/2} \leq \sqrt{L}.$$

Further terms can be estimated by Cauchy-Schwarz inequality and using (2.3) and the second inequality in (2.2). Indeed,

$$\begin{aligned} \mathbb{E}X \mathbf{1}_{\{2^{k-1}L < Y \leq 2^k L\}} &= \mathbb{E}X \mathbf{1}_{\{Y \leq 2^k L\}} \mathbf{1}_{\{Y > 2^{k-1}L\}} \\ &\leq (\mathbb{E}X^2 \mathbf{1}_{\{Y \leq 2^k L\}})^{1/2} (\mathbb{P}\{Y > 2^{k-1}L\})^{1/2} \\ &\leq (2^k L)^{1/2} \cdot \frac{1}{2^{k-1}} = \sqrt{L} 2^{1-k/2}. \end{aligned}$$

Therefore

$$\mathbb{E}X \leq \sqrt{L} + \sum_{k=1}^{\infty} \sqrt{L} 2^{1-k/2} \leq C\sqrt{L}.$$

This completes the proof. \square

2.6. On the deterministic matrix B in Theorem 1.1. We start with two initial observations that will make our proof of Theorem 1.1 more transparent. By adding an appropriate number of zero rows to B or zero columns to A we can assume without loss of generality that $n = m$, thus B is an $n \times N$ matrix.

Throughout the proof of Theorem 1.1, we shall denote the columns of such a matrix B by B_1, \dots, B_N . They are non-random vectors in \mathbb{R}^n , which satisfy

$$(2.4) \quad \max_i \|B_i\|_2 \leq \|B\| \leq 1; \quad \sum_{i=1}^N \|B_i\|_2^2 = \|B\|_{\text{HS}}^2 \leq n\|B\| \leq n$$

where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm. Throughout the argument, we will only have access to the matrix B through inequalities (2.4). This explains Remark 2 following Theorem 1.1, which states that the range space of B is irrelevant as long as we control the spectral and Hilbert-Schmidt norms of B .

3. APPROACH VIA M. RUDELSON'S THEOREM

3.1. M. Rudelson's theorem. Our first approach, which will yield Theorem 1.1 up to a logarithmic factor, rests on the following result. Here and thereafter, by $\varepsilon_1, \varepsilon_2, \dots$ we denote independent symmetric Bernoulli random variables, i.e. independent random variables such that $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$.

Theorem 3.1 (M. Rudelson [22]). *Let u_1, \dots, u_M be vectors in \mathbb{R}^m . Then, for every $p \geq 1$, one has*

$$\left(\mathbb{E} \left\| \sum_{i=1}^M \varepsilon_i u_i \otimes u_i \right\|^p \right)^{1/p} \leq C(\sqrt{p} + \sqrt{\log m}) \cdot \max_i \|u_i\|_2 \cdot \left\| \sum_{i=1}^M u_i \otimes u_i \right\|^{1/2}.$$

In particular, for every $t > 0$, with probability at least $1 - 2me^{-ct^2}$ one has

$$\left\| \sum_{i=1}^M \varepsilon_i u_i \otimes u_i \right\| \leq t \cdot \max_i \|u_i\|_2 \cdot \left\| \sum_{i=1}^M u_i \otimes u_i \right\|^{1/2}.$$

The first estimate is taken from [22, inequality (3.4)]. The second estimate can be easily derived from it using the following elementary lemma:

Lemma 3.2 (Moments and tails). *Suppose a non-negative random variable X satisfies for some $m \geq 1$ that*

$$(\mathbb{E}X^p)^{1/p} \leq \sqrt{p} + \sqrt{\log m} \quad \text{for every } p \geq 1.$$

Then

$$\mathbb{P}(X \geq t) \leq 2me^{-ct^2} \quad \text{for every } t > 0.$$

Proof. Suppose first that $t \geq \max(1, \sqrt{\log m})$. Let $p := t^2$. Then $\sqrt{p} \geq \sqrt{\log m}$, so the hypothesis gives $(\mathbb{E}X^p)^{1/p} \leq 2\sqrt{p}$. By Markov's inequality,

$$\mathbb{P}(X \geq 2et) = \mathbb{P}(X^p \geq (2et)^p) \leq \frac{(2\sqrt{p})^p}{(2et)^p} = e^{-t^2}.$$

Next, if $t < \max(1, \sqrt{\log m})$ then by choosing the absolute constant $c > 0$ sufficiently small right hand side of (3.1) is larger than 1 for a sufficiently small absolute constant c . Therefore, for every $t > 0$ one has

$$(3.1) \quad \mathbb{P}(X \geq 2et) \leq 2me^{-t^2/2}$$

because if $t < \max(1, \sqrt{\log m})$ then the right hand side of (3.1) is larger than one, which makes the inequality trivial. This completes the proof. \square

The next lemma is a consequence of M. Rudelson's Theorem 3.1 and a standard symmetrization argument.

Lemma 3.3. *Let X_1, \dots, X_n be independent random vectors in \mathbb{R}^m such that*

$$(3.2) \quad \|\mathbb{E}X_j \otimes X_j\| \leq 1 \quad \text{for every } j.$$

Then

$$\mathbb{E} \left\| \sum_{j=1}^n X_j \otimes X_j \right\| \leq Cn + C \log(2m) \mathbb{E} \max_j \|X_j\|_2^2.$$

Proof. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent symmetric Bernoulli random variables. By the triangle inequality, the standard symmetrization argument (see e.g. [19, Lemma 6.3]), and the assumption, we have

$$\begin{aligned} E := \mathbb{E} \left\| \sum_{j=1}^n X_j \otimes X_j \right\| &\leq \mathbb{E} \left\| \sum_{j=1}^n (X_j \otimes X_j - \mathbb{E}X_j \otimes X_j) \right\| + \left\| \sum_{j=1}^n \mathbb{E}X_j \otimes X_j \right\| \\ &\leq 2\mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j X_j \otimes X_j \right\| + n. \end{aligned}$$

Condition on the random variables X_1, \dots, X_n , and apply Theorem 3.1. Writing \mathbb{E}_ε to denote the conditional expectation (i.e. the expectation with respect to the random variables $\varepsilon_1, \dots, \varepsilon_n$), we have

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j X_j \otimes X_j \right\| \leq C\sqrt{\log(2m)} \cdot \max_j \|X_j\|_2 \cdot \left\| \sum_{j=1}^n X_j \otimes X_j \right\|^{1/2}.$$

Now we take expectation with respect to X_1, \dots, X_n and use Cauchy-Schwarz inequality to get

$$E \leq C\sqrt{\log(2m)} \cdot (\mathbb{E} \max_j \|X_j\|_2^2)^{1/2} \cdot E^{1/2} + n.$$

The conclusion of the lemma follows. \square

3.2. Theorem 1.1 up to a logarithmic term. We now state a version of Theorem 1.1 with a logarithmic factor.

Proposition 3.4. *Let N, n be positive integers. Consider an $N \times n$ random matrix A whose entries are independent random variables with mean zero and 4-th moment bounded by 1. Let B be an $n \times N$ matrix such that $\|B\| \leq 1$. Then*

$$\mathbb{E}\|BA\| \leq C\sqrt{n \log(2n)}.$$

The proof will need two auxiliary lemmas. Recall that B_1, \dots, B_N denote the columns of the matrix B .

Lemma 3.5. *Let a_1, \dots, a_N be independent random variables with mean zero and 4-th moment bounded by 1. Consider the random vector X in \mathbb{R}^n defined as*

$$X = \sum_{i=1}^N a_i B_i.$$

Then

$$\mathbb{E}\|X\|_2^2 \leq n, \quad \text{Var}(\|X\|_2^2) \leq 3n.$$

Proof. The estimate on the expectation follows easily from (2.4):

$$(3.3) \quad \mathbb{E}\|X\|_2^2 = \sum_{i=1}^N \mathbb{E}(a_i^2) \|B_i\|_2^2 \leq \sum_{i=1}^N \|B_i\|_2^2 \leq n.$$

To estimate the variance, we need to compute

$$\mathbb{E}\|X\|_2^4 = \mathbb{E}\langle X, X \rangle^2 = \sum_{i,j,k,l=1}^N \mathbb{E}(a_i a_j a_k a_l) \langle B_i, B_j \rangle \langle B_k, B_l \rangle.$$

By independence and the mean zero assumption, the only nonzero terms in this sum are those for which $i = j; k = l$ or $i = k; j = l$ or $i = l; j = k$. Therefore

$$\begin{aligned} \mathbb{E}\|X\|_2^4 &= \sum_{i,j=1}^N \mathbb{E}(a_i^2 a_j^2) \|B_i\|_2^2 \|B_j\|_2^2 + 2 \sum_{i,j=1}^N \mathbb{E}(a_i^2 a_j^2) \langle B_i, B_j \rangle^2 \\ &= \sum_{i=1}^N \mathbb{E}(a_i^4) \|B_i\|_2^4 + \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathbb{E}(a_i^2) \mathbb{E}(a_j^2) \|B_i\|_2^2 \|B_j\|_2^2 + 2 \sum_{i,j=1}^N \mathbb{E}(a_i^2 a_j^2) \langle B_i, B_j \rangle^2 \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By the fourth moment assumption and using (2.4) we have

$$I_1 \leq \sum_{i=1}^N \|B_i\|_2^4 \leq \max_i (\|B_i\|_2^2) \sum_{i=1}^N \|B_i\|_2^2 \leq n$$

Squaring the sum in (3.3), we see that

$$I_2 \leq (\mathbb{E}\|X\|_2^2)^2.$$

Finally, since by Cauchy-Schwarz inequality $\mathbb{E}(a_i^2 a_j^2) \leq \sqrt{\mathbb{E}(a_i^4)\mathbb{E}(a_j^4)} \leq 1$, and using (2.4) again, we obtain

$$I_3 \leq 2 \sum_{i,j=1}^N \langle B_i, B_j \rangle^2 = 2\|B^*B\|_{\text{HS}}^2 \leq 2\|B^*\|^2\|B\|_{\text{HS}}^2 = 2\|B\|^2\|B\|_{\text{HS}}^2 \leq 2n.$$

Putting all this together, we obtain

$$\text{Var}(\|X\|_2^2) = \mathbb{E}\|X\|_2^4 - (\mathbb{E}\|X\|_2^2)^2 \leq I_1 + I_3 \leq 3n.$$

This completes the proof. \square

Lemma 3.6. *Let A and B be matrices as in Proposition 3.4. Let $X_1, \dots, X_n \in \mathbb{R}^n$ denote the columns of the matrix BA . Then*

$$\mathbb{E} \max_{j=1, \dots, n} \|X_j\|_2^2 \leq Cn.$$

Remark. This result says that all columns of the matrix BA have norm $O(\sqrt{n})$ with high probability. Since the spectral norm of a matrix is bounded below by the norm of any column, this result is a necessary step in proving our desired estimate $\|BA\| = O(\sqrt{n})$.

Proof. Let, as usual, $B_1, \dots, B_N \in \mathbb{R}^n$ denote the columns of the matrix B , and let a_{ij} denote the entries of the matrix A . Then

$$(3.4) \quad X_j = \sum_{i=1}^N a_{ij} B_i, \quad j = 1, \dots, n.$$

Let us fix $j \in \{1, \dots, n\}$ and use Lemma 3.5. This gives

$$(3.5) \quad \mathbb{E}\|X_j\|_2^2 \leq n, \quad \text{Var}(\|X_j\|_2^2) \leq 3n.$$

Now we use Chebychev's inequality, which states that for a random variable Z with $\sigma^2 = \text{Var}(Z)$ and for an arbitrary $k > 0$, one has

$$\mathbb{P}(|Z - \mathbb{E}Z| > k\sigma) \leq \frac{1}{k^2}.$$

Let $t > 0$ be arbitrary. Using Chebychev's inequality along with (3.5) for $Z = \|X_j\|_2^2$, $k = t\sqrt{n}$, we obtain

$$\mathbb{P}(\|X_j\|_2^2 > (1 + \sqrt{3}t)n) \leq \frac{1}{t^2 n}.$$

Taking the union bound over all $j = 1, \dots, n$, we conclude that

$$\mathbb{P}\left(\max_{j=1, \dots, n} \|X_j\|_2^2 > (1 + \sqrt{3}t)n\right) \leq n \cdot \frac{1}{t^2 n} = \frac{1}{t^2}.$$

Integration completes the proof. \square

Proof of Proposition 3.4. Let $X_1, \dots, X_n \in \mathbb{R}^n$ denote the columns of the matrix BA . We are going to apply Lemma 3.3. In order to check that condition (3.2) holds, we consider an arbitrary vector $x \in S^{n-1}$ and use representation (3.4) to compute

$$\begin{aligned} \mathbb{E}\langle X_j, x \rangle^2 &= \mathbb{E}\left(\sum_{i=1}^N a_{ij}\langle B_i, x \rangle\right)^2 = \sum_{i=1}^N \mathbb{E}(a_{ij}^2)\langle B_i, x \rangle^2 \leq \sum_{i=1}^N \langle B_i, x \rangle^2 \\ &= \|B^*x\|_2^2 \leq \|B^*\|^2 = \|B\|^2 \leq 1. \end{aligned}$$

This shows that condition (3.2) holds. Lemma 3.3 then gives

$$\mathbb{E}\|BA\|^2 = \mathbb{E}\left\|\sum_{j=1}^n X_j \otimes X_j\right\|^2 \leq Cn + C \log(2n) \mathbb{E} \max_{j=1, \dots, n} \|X_j\|_2^2.$$

Estimating the maximum in the right hand side using Lemma 3.6, we conclude that

$$\mathbb{E}\|BA\|^2 \leq C_1 n \log(2n).$$

This completes the proof. \square

3.3. Tradeoff between the matrix norm and the magnitude of entries.

We would like now to gain more control over the logarithmic factor than we have in Proposition 3.4. Our next result establishes a tradeoff between the logarithmic factor and the magnitude of the matrices A, B . It will be used in the proof of Theorem 3.9.

Proposition 3.7. *Let $a, b \geq 0$ and N, n be positive integers. Let A be an $N \times n$ matrix whose entries are random independent variables a_{ij} with mean zero and such that*

$$\mathbb{E}a_{ij}^2 \leq 1, \quad |a_{ij}| \leq a \quad \text{for every } i, j.$$

Let B be an $n \times N$ matrix such that $\|B\| \leq 1$, and whose columns satisfy

$$\|B_i\|_2 \leq b \quad \text{for every } i.$$

Then

$$\mathbb{E}\|BA\| \leq C(1 + ab^{1/2} \log^{1/4}(2n))\sqrt{n}.$$

The proof will again be based on M. Rudelson's Theorem 3.1, although this time we use Rudelson's theorem in a more delicate way:

Lemma 3.8. *Under the assumptions of Proposition 3.7, we have*

$$\mathbb{E} \max_{j=1, \dots, n} \left\| \sum_{i=1}^N a_{ij}^2 B_i \otimes B_i \right\| \leq C(1 + a^2 b \sqrt{\log(2n)}).$$

Proof. Fix $j \in \{1, \dots, n\}$. Let $\mu_{ij}^2 := \mathbb{E}a_{ij}^2$. By the triangle inequality,

$$(3.6) \quad \left\| \sum_{i=1}^N a_{ij}^2 B_i \otimes B_i \right\| \leq \left\| \sum_{i=1}^N (a_{ij}^2 - \mu_{ij}^2) B_i \otimes B_i \right\| + \left\| \sum_{i=1}^N \mu_{ij}^2 B_i \otimes B_i \right\|.$$

Since $0 \leq \mu_{ij}^2 \leq 1$ and

$$(3.7) \quad \left\| \sum_{i=1}^N B_i \otimes B_i \right\| \leq \|B\|^2 \leq 1,$$

we have

$$(3.8) \quad \left\| \sum_{i=1}^N \mu_{ij}^2 B_i \otimes B_i \right\| \leq \left\| \sum_{i=1}^N B_i \otimes B_i \right\| \leq 1.$$

Next, clearly $\mu_{ij}^2 \leq a^2$, so

$$\mathbb{E}(a_{ij}^2 - \mu_{ij}^2) = 0, \quad |a_{ij}^2 - \mu_{ij}^2| \leq 2a^2.$$

Symmetrization Lemma 2.7 yields

$$(3.9) \quad \mathbb{E} \max_{j=1, \dots, n} \left\| \sum_{i=1}^N (a_{ij}^2 - \mu_{ij}^2) B_i \otimes B_i \right\| \leq 2a^2 \mathbb{E} \max_{j=1, \dots, n} \left\| \sum_{i=1}^N \varepsilon_{ij} B_i \otimes B_i \right\|$$

where ε_{ij} denote independent symmetric Bernoulli random variables.

Let $t > 0$. By the second part of M. Rudelson's Theorem 3.1 and taking the union bound over n random variables, we conclude that, with probability at least $1 - 2n^2 e^{-ct^2}$, we have

$$\max_{j=1, \dots, n} \left\| \sum_{i=1}^N \varepsilon_{ij} B_i \otimes B_i \right\| \leq t \cdot \max_{i=1, \dots, N} \|B_i\|_2 \cdot \left\| \sum_{i=1}^N B_i \otimes B_i \right\|^{1/2} \leq tb$$

The second estimate follows from (3.7) and since $\max_i \|B_i\|_2 \leq b$ by the hypothesis.

Let $s > 0$ be arbitrary. We apply the above estimate for t chosen so that $2n^2 e^{-ct^2} = e^{-s^2}$. This shows that, with probability at least $1 - e^{-s^2}$, one has

$$\max_{j=1, \dots, n} \left\| \sum_{i=1}^N \varepsilon_{ij} B_i \otimes B_i \right\| \leq tb \leq C_1 b (\sqrt{\log(2n)} + s).$$

Integration implies that

$$\mathbb{E} \max_{j=1, \dots, n} \left\| \sum_{i=1}^N \varepsilon_{ij} B_i \otimes B_i \right\| \leq C_2 b \sqrt{\log(2n)}.$$

Putting this into (3.9) and, together with (3.8), back into (3.6), we complete the proof. \square

Proof of Proposition 3.7. By the symmetrization argument (see (2.1)), we can assume that the entries of the matrix A are $g_{ij}a_{ij}$, where a_{ij} are random variables satisfying the assumptions of the proposition, and g_{ij} are independent standard normal random variables. We will write $\mathbb{E}_g, \mathbb{P}_g$ when we take expectations and probability estimates with respect to (g_{ij}) (i.e. conditioned on (a_{ij})), and we write \mathbb{E}_a to denote the expectation with respect to (a_{ij}) .

By Lemma 3.8, the random variable

$$K^2 := \max_{j=1, \dots, n} \left\| \sum_{i=1}^N a_{ij}^2 B_i \otimes B_i \right\|$$

which does not depend on the random variables (g_{ij}) , has expectation

$$(3.10) \quad \mathbb{E}_a(K^2) \leq C(1 + a^2 b \sqrt{\log(2n)}).$$

We condition on the random variables (a_{ij}) ; this fixes a value of K .

Let $X_1, \dots, X_n \in \mathbb{R}^n$ denote the columns of the matrix BA ; then

$$X_j = \sum_{i=1}^N g_{ij} a_{ij} B_i, \quad j = 1, \dots, n.$$

Consider a $(1/2)$ -net \mathcal{N} of the unit Euclidean sphere S^{n-1} of cardinality $|\mathcal{N}| \leq 5^n$, which exists by Lemma 2.5. Using Proposition 2.6, we have

$$(3.11) \quad \|BA\|^2 = \|(BA)^*\|^2 \leq 4 \max_{x \in \mathcal{N}} \|(BA)^*x\|_2^2 = 4 \max_{x \in \mathcal{N}} \sum_{j=1}^n \langle X_j, x \rangle^2.$$

Fix $x \in \mathcal{N}$. For every $j = 1, \dots, n$, the random variable

$$\langle X_j, x \rangle = \sum_{i=1}^N g_{ij} \langle a_{ij} B_i, x \rangle$$

is a Gaussian random variable with mean zero and variance

$$\sum_{i=1}^N \langle a_{ij} B_i, x \rangle^2 \leq \left\| \sum_{i=1}^N a_{ij}^2 B_i \otimes B_i \right\| \leq K^2.$$

(To obtain the first inequality, take the supremum over $x \in S^{n-1}$). Therefore, by Corollary 2.2 with $d_i = (\text{Var} \langle X_i, x \rangle)^{1/2} \leq K$, we have for every $t > 0$:

$$\mathbb{P}_g \left\{ \left(\sum_{j=1}^n \langle X_j, x \rangle^2 \right)^{1/2} > K\sqrt{n} + t \right\} \leq e^{-c_0 t^2 / K^2}.$$

Let $s > 0$ be arbitrary. The previous estimate for $t = sK\sqrt{n}$ gives

$$\mathbb{P}_g \left\{ \left(\sum_{j=1}^n \langle X_j, x \rangle^2 \right)^{1/2} > (1+s)K\sqrt{n} \right\} \leq e^{-c_0 s^2 n}.$$

Taking the union bound over $x \in \mathcal{N}$ and using (3.11), we obtain

$$\mathbb{P}_g\{\|BA\| > 2(1+s)K\sqrt{n}\} \leq |\mathcal{N}|e^{-c_0s^2n} = 5^n e^{-c_0s^2n} \leq e^{(2-c_0s^2)n}.$$

Integration yields

$$\mathbb{E}_g\|BA\| \leq CK\sqrt{n}.$$

Finally, we take expectation with respect to the random variables (a_{ij}) and use (3.10) to conclude that

$$\mathbb{E}\|BA\| \leq C\mathbb{E}_a(K)\sqrt{n} \leq C_1(1+a^2b\sqrt{\log(2n)})^{1/2}\sqrt{n}.$$

This completes the proof. \square

3.4. Theorem 1.1 for logarithmically small columns. Our next step is to combine Propositions 3.4 and 3.7 and obtain a weaker version of the main Theorem 1.1 – this time with the correct bound $O(\sqrt{n})$ on the norm, but under the additional assumption that the columns of the matrix B are logarithmically small.

Theorem 3.9. *Let $\varepsilon \in (0, 1)$ and let N, n be positive integers. Consider an $N \times n$ random matrix A whose entries are independent random variables with mean zero and $(4 + \varepsilon)$ -th moment bounded by 1. Let B be an $n \times N$ matrix such that $\|B\| \leq 1$, and whose columns satisfy for some $M \geq 1$ that*

$$\|B_i\|_2 \leq M \log^{-\frac{1}{2}-\frac{1}{\varepsilon}}(2n) \quad \text{for every } i.$$

Then

$$\mathbb{E}\|BA\| \leq CM^{1/2}\sqrt{n}.$$

Proof. By the symmetrization argument described in Section 2, we can assume without loss of generality that all entries a_{ij} of the matrix $A = (a_{ij})$ are symmetric random variables. Let

$$a := \log^{\frac{1}{2\varepsilon}}(2n).$$

We decompose every entry of the matrix A according to its absolute value as

$$\bar{a}_{ij} := a_{ij}\mathbf{1}_{\{|a_{ij}| \leq a\}}, \quad \tilde{a}_{ij} := a_{ij}\mathbf{1}_{\{|a_{ij}| > a\}}.$$

Then all random variables \bar{a}_{ij} and \tilde{a}_{ij} have mean zero, and we have the following decomposition of matrices:

$$BA = B\bar{A} + B\tilde{A}, \quad \text{where } \bar{A} = (\bar{a}_{ij}), \quad \tilde{A} = (\tilde{a}_{ij}).$$

The norm of $B\tilde{A}$ can be bounded using Proposition 3.4. Indeed, by the Truncation Lemma 2.8 with $p = 1 + \varepsilon/4$, we have

$$\mathbb{E}\tilde{a}_{ij}^4 = \mathbb{E}a_{ij}^4\mathbf{1}_{\{a_{ij}^4 > a^4\}} \leq \frac{\mathbb{E}a_{ij}^{4+\varepsilon}}{a^\varepsilon} \leq a^{-\varepsilon},$$

where the last inequality follows from the moment hypothesis. Therefore, the matrix $a^\varepsilon \tilde{A}$ satisfies the hypothesis of Proposition 3.4, which then yields

$$\mathbb{E}\|B\tilde{A}\| \leq Ca^{-\varepsilon} \sqrt{n \log(2n)} = C\sqrt{n}.$$

The norm of $B\bar{A}$ can be bounded using Proposition 3.7, which we can apply with a as above and $b = M \log^{-\frac{1}{2}-\frac{1}{\varepsilon}}(2n)$. This gives

$$\mathbb{E}\|B\bar{A}\| \leq C(1 + ab^{1/2} \log^{1/4}(2n))\sqrt{n} \leq 2CM^{1/2}\sqrt{n},$$

where the last inequality follows by our choice of a and b .

Putting the two estimates together, we conclude by the triangle inequality that

$$\mathbb{E}\|BA\| \leq \mathbb{E}\|B\bar{A}\| + \mathbb{E}\|B\tilde{A}\| \leq C'M^{1/2}\sqrt{n}.$$

This completes the proof. \square

Remark. The factor $M^{1/2}$ in the conclusion of Theorem 3.9 can easily be improved to about $M^{\varepsilon/2}$ by choosing $a = t \log^{\frac{1}{2\varepsilon}}(2n)$ in the proof and optimizing in t . We will not need this improvement in our argument.

4. APPROACH VIA CONCENTRATION

In this section, we develop an alternative way to bound the norm of BA , which rests on Gaussian concentration inequalities and elaborate choice of ε -nets. The main technical result of this section is the following theorem, which, like Theorem 3.9, gives the correct bound $O(\sqrt{n})$ under some boundedness assumptions on the entries of A .

Theorem 4.1. *Let $\varepsilon \in (0, 1)$, $M \geq 1$ and let $N \geq n$ be positive integers such that $\log(2N) \leq Mn$. Consider an $N \times n$ random matrix A whose entries are independent random variables a_{ij} with mean zero and such that*

$$\mathbb{E}|a_{ij}|^{2+\varepsilon} \leq 1, \quad |a_{ij}| \leq \left(\frac{Mn}{\log(2N)}\right)^{\frac{1}{2+\varepsilon}} \quad \text{for every } i, j.$$

Let B be an $n \times N$ matrix such that $\|B\| \leq 1$. Then

$$\mathbb{E}\|BA\| \leq C(\varepsilon)\sqrt{Mn}$$

where $C(\varepsilon)$ depends only on ε .

Remarks. 1. If the entries a_{ij} have bounded $(4 + \varepsilon)$ -th moment, it is easy to check that $\max_{ij} a_{ij} \sim (nN)^{\frac{1}{4+\varepsilon}}$ holds with high probability. Therefore, under the $(4 + \varepsilon)$ -th moment assumption, the hypotheses of Theorem 4.1 are satisfied for almost square matrices, i.e. those for which $N \leq n^{1+c\varepsilon}$. This will quickly yield the main Theorem 1.1 for almost square matrices, see Corollary 4.11 below.

2. The hypotheses of Theorem 4.1 are almost sharp when $N \sim n$. Indeed, let us assume for simplicity that the random variables a_{ij} are identically distributed and B is the identity matrix. The $(2 + \varepsilon)$ -th moment hypothesis is almost sharp: if $\mathbb{E}a_{ij}^2 \gg 1$ then $(\mathbb{E}\|A\|^2)^{1/2} \geq (\frac{1}{n}\|A\|_{\text{HS}}^2)^{1/2} \gg \sqrt{n}$. Also, the boundedness hypothesis is almost sharp, since $\|A\| \geq \max_{i,j} |a_{ij}|$.

3. Using M. Talagrand's concentration result, Theorem 2.4, one can also obtain tail bounds for the norm $\|BA\|$:

Corollary 4.2. *Under the assumptions of Theorem 4.1, one has for every $t > 0$:*

$$\mathbb{P}(\|BA\| > (C(\varepsilon) + t)\sqrt{Mn}) \leq 4e^{-t^2/4}.$$

In particular, one has for every $q \geq 1$:

$$(\mathbb{E}\|BA\|^q)^{1/q} \leq C_0(\varepsilon)\sqrt{qMn}.$$

Proof. We can consider the $N \times n$ matrix A as a vector in \mathbb{R}^{Nn} . The Euclidean norm of such a vector equals the Hilbert-Schmidt norm $\|A\|_{\text{HS}}$. Since $\|BA\| \leq \|B\|\|A\| \leq 1 \cdot \|A\|_{\text{HS}}$, the function $f : \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ defined by $f(A) = \|BA\|$ is 1-Lipschitz and convex. Since we have $|a_{ij}| \leq \sqrt{Mn}$ for all i, j by the assumptions, M. Talagrand's Theorem 2.4 gives

$$\mathbb{P}(\|BA\| - \mathbb{E}\|BA\| > t\sqrt{Mn}) \leq 4e^{-t^2/4}, \quad t > 0.$$

The estimate for $\mathbb{E}\|BA\|$ in Theorem 4.1 completes the proof. \square

4.1. Sparse matrices: rows and columns. Theorem 4.1 will follow from our analysis of sparse matrices. We will decompose the entries a_{ij} according to their magnitude. As the magnitude increases, the moment assumptions will ensure that there will be fewer such entries, i.e. the resulting matrix becomes sparser.

We start with an elementary lemma, which will help us analyze the magnitude of the rows and columns of the matrix BA when A is a sparse matrix.

Lemma 4.3. *Let N, n be positive integers. Consider independent random variables a_{ij} , $i = 1, \dots, N$, $j = 1, \dots, n$. Let $p \in (0, 1]$, and suppose that*

$$\mathbb{E}a_{ij}^2 \leq p, \quad |a_{ij}| \leq 1 \quad \text{for every } i, j.$$

Let B be an $n \times N$ matrix such that $\|B\| \leq 1$, whose columns are denoted B_i . Then

$$(4.1) \quad \mathbb{E} \max_{i=1, \dots, N} \sum_{j=1}^n a_{ij}^2 \leq C(np + \log(2N)),$$

$$(4.2) \quad \mathbb{E} \max_{j=1, \dots, n} \sum_{i=1}^N a_{ij}^2 \|B_i\|_2^2 \leq C(np + \log(2n)).$$

Remark. The test case for this lemma, as well as for most of the results that follow, is the random variables a_{ij} with values in $\{-1, 0, 1\}$ and such that $\mathbb{P}(a_{ij} \neq 0) = p$. The $N \times n$ random matrix $A = (a_{ij})$ will then become sparser as we decrease p ; it will have on average np nonzero entries per row. Estimate (4.1) gives a bound on the Euclidean norm of all rows of A .

Proof. We will only prove inequality (4.2); the proof of inequality (4.1) is similar. By the assumptions, we have

$$\text{Var}(a_{ij}^2) \leq \mathbb{E}a_{ij}^4 \leq \mathbb{E}a_{ij}^2 \leq p \quad \text{for every } i, j.$$

Also, recall that (2.4) gives

$$\sum_{i=1}^N \|B_i\|_2^2 \leq n, \quad \sum_{i=1}^N \|B_i\|_2^4 \leq \max_i \|B_i\|_2^2 \cdot \sum_{i=1}^N \|B_i\|_2^2 \leq n.$$

Consider the sums of independent random variables

$$S_j := \sum_{i=1}^N a_{ij}^2 \|B_i\|_2^2, \quad j = 1, \dots, n.$$

The above estimates show that for every j we have

$$\mathbb{E}S_j = \sum_{i=1}^N \mathbb{E}(a_{ij}^2) \|B_i\|_2^2 \leq np, \quad \text{Var}(S_j) = \sum_{i=1}^N \text{Var}(a_{ij}^2) \|B_i\|_2^4 \leq np.$$

We apply Bennett's inequality, Theorem 2.3, for $X_i = \frac{1}{2}(a_{ij}^2 - \mathbb{E}a_{ij}^2) \|B_i\|_2^2$, which clearly satisfy $|X_j| \leq 1$ because $|a_{ij}| \leq 1$ and $\|B_i\|_2 \leq 1$ by (2.4). We obtain

$$(4.3) \quad \mathbb{P}\left\{\frac{1}{2}(S_j - \mathbb{E}S_j) > t\right\} \leq \exp(-\sigma^2 h(t/\sigma^2))$$

where $\mathbb{E}(\frac{1}{2}S_j) \leq np$ and $\sigma^2 = \text{Var}(\frac{1}{2}S_j) \leq np$. Note that $h(x) \geq cx$ for $x \geq 1$, where c is some positive absolute constant. Therefore, if $t \geq np$, then $\sigma^2 h(t/\sigma^2) \geq ct$, so (4.3) yields

$$\mathbb{P}\{S_j > 2t\} \leq e^{-ct} \quad \text{for } t \geq np.$$

Taking the union bound over all j , we conclude that

$$\mathbb{P}\left\{\max_{j=1, \dots, n} S_j > 2t\right\} \leq ne^{-ct} \quad \text{for } t \geq np.$$

Now let $s \geq 1$ be arbitrary, and use the last inequality for $t = (np + \log(2n))s$. We obtain

$$\mathbb{P}\left\{\max_{j=1, \dots, n} S_j > 2(np + \log(2n))s\right\} \leq ne^{-c \log(2n)s} = 2^{-cs} n^{1-cs}.$$

Integration yields

$$\mathbb{E} \max_{j=1, \dots, n} S_j \leq C(np + \log(2n)).$$

This completes the proof of (4.2). \square

The estimates in Lemma 4.3 motivate us to consider the class of $N \times n$ matrices $A = (a_{ij})$ whose entries satisfy the following inequalities for some parameters $p \in (0, 1]$ and $K \geq 1$:

$$(4.4) \quad \begin{aligned} \max_{i,j} |a_{ij}| &\leq 1; \\ \max_{i=1,\dots,N} \left(\sum_{j=1}^n a_{ij}^2 \right)^{1/2} &\leq K \sqrt{np + \log(2N)}; \\ \max_{j=1,\dots,n} \left(\sum_{i=1}^N a_{ij}^2 \|B_i\|_2^2 \right)^{1/2} &\leq K \sqrt{np + \log(2n)}. \end{aligned}$$

We have proved that for random matrices whose entries satisfy $|a_{ij}| \leq 1$ and $Ea_{ij}^2 \leq p$, conditions (4.4) hold with a random parameter K that satisfies $\mathbb{E}K \leq C$.

4.2. Concentration for a fixed vector. Our goal will be to estimate the magnitude of $\|BA\|$ for matrices of the form $A = (g_{ij}a_{ij})$, where g_{ij} are independent standard normal random variables, and a_{ij} are fixed numbers that satisfy conditions (4.4). Such an estimate will be established in Proposition 4.8 below. By the standard symmetrization, the same estimate will hold true if $A = (a_{ij})$ is a random matrix with entries as in Lemma 4.3. This will be done in Corollary 4.9. Finally, Theorem 4.1 will be deduced from this by decomposing the entries of a random matrix according to their magnitude.

Our first step toward this goal is to check the magnitude of $\|BAx\|_2$ for a fixed vector x .

Lemma 4.4. *Let N, n be positive integers. Consider an $N \times n$ random matrix $A = (g_{ij}a_{ij})$ where g_{ij} are independent standard normal random variables and a_{ij} are numbers that satisfy conditions (4.4). Let B be an $n \times N$ matrix such that $\|B\| \leq 1$. Then, for every vector $x \in B_2^n$ we have*

$$\mathbb{E}\|BAx\|_2 \leq K \sqrt{np + \log(2n)}.$$

Proof. Denoting as usual the columns of B by B_i , we have

$$BAx = \sum_{i=1}^N \left(\sum_{j=1}^n g_{ij} a_{ij} x_j \right) B_i.$$

Since $\|x\|_2 \leq 1$ and using the last condition in (4.4), we have

$$\begin{aligned} \mathbb{E}\|BAx\|_2^2 &= \sum_{i=1}^N \sum_{j=1}^n a_{ij}^2 x_j^2 \|B_i\|_2^2 \\ &= \sum_{j=1}^n \left(\sum_{i=1}^N a_{ij}^2 \|B_i\|_2^2 \right) x_j^2 \\ &\leq \max_{j=1, \dots, n} \sum_{i=1}^N a_{ij}^2 \|B_i\|_2^2 \leq K^2(np + \log(2n)). \end{aligned}$$

This completes the proof. \square

We will now strengthen Lemma 4.4 into a deviation inequality for $\|BAx\|_2$. This is a simple consequence of the Gaussian concentration, Theorem 2.1. This deviation inequality is universal in that it holds for any vector x ; in the sequel we will need more delicate inequalities that depend on the distribution of the coordinates in x .

Lemma 4.5 (Universal deviation). *Let A and B be matrices as in Lemma 4.4. Then, for every vector $x \in B_2^n$ and every $t > 0$ we have*

$$(4.5) \quad \mathbb{P}\{\|BAx\|_2 > K\sqrt{np + \log(2n)} + t\} \leq e^{-c_0 t^2}.$$

Proof. As in the proof of Lemma 4.4, we write

$$BAx = \sum_{i=1}^N \left(\sum_{j=1}^n g_{ij} a_{ij} x_j \right) B_i$$

where B_i are the columns of the matrix B . Therefore, the random vector BAx is distributed identically with the random vector

$$\sum_{i=1}^N g_i \lambda_i B_i, \quad \text{where } \lambda_i = \left(\sum_{j=1}^n a_{ij}^2 x_j^2 \right)^{1/2}$$

and where g_i are independent standard normal random variables. Since all $|a_{ij}| \leq 1$ by conditions (4.4), and $\|x\|_2 \leq 1$ by the assumptions, we have

$$0 \leq \lambda_i \leq 1, \quad i = 1, \dots, N.$$

Consider the map $f : \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$f(y) = \left\| \sum_{i=1}^N y_i \lambda_i B_i \right\|_2.$$

Its Lipschitz norm equals

$$\|f\|_{\text{Lip}} = \left\| \sum_{i=1}^N \lambda_i^2 B_i \otimes B_i \right\|^{1/2} \leq \max_i |\lambda_i| \cdot \left\| \sum_{i=1}^N B_i \otimes B_i \right\|^{1/2} \leq 1 \cdot \|B\| \leq 1.$$

Then the Gaussian concentration, Theorem 2.1, gives for every $t > 0$:

$$\mathbb{P}(f(g) - \mathbb{E}f(g) > t) \leq \exp(-c_0 t^2),$$

where $g = (g_1, \dots, g_N)$. Since as we noted above, $f(g)$ is distributed identically with $\|BAx\|_2$, Lemma 4.4 completes the proof. \square

4.3. Control of sparse vectors. Since the spectral norm of BA is the supremum of $\|BAx\|_2$ over all $x \in S^{n-1}$, the result of Lemma 4.5 suggests that $\mathbb{E}\|BA\| \lesssim \sqrt{np + \log N}$ should be true. However, the deviation inequality in Lemma 4.5 is not strong enough to prove this bound. This is because the metric entropy of the sphere, measured e.g. as the cardinality of its $\frac{1}{2}$ -net, is e^{cn} . If we are to make the bound on $\|BAx\|_2$ uniform over the net, we would need the probability estimate in (4.5) at most e^{-cn} (to allow a room for the union bound over e^{cn} points x in the net). This however would force us to make $t \sim \sqrt{n}$ or larger, so the best bound we can get this way is $\mathbb{E}\|BA\|_2 \lesssim \sqrt{n}$. This bound is too weak as it ignores the last two assumptions in (4.4).

Nevertheless, the bound in Lemma 4.5 can be made uniform over a set of sparse vectors, whose metric entropy is smaller than that of the whole sphere:

Proposition 4.6 (Sparse vectors). *Let A and B be matrices as in Lemma 4.4. There exists an absolute constant $c > 0$ such that the following holds. Consider the set of vectors*

$$B_{2,0} := \left\{ x \in \mathbb{R}^n, \|x\|_2 \leq 1, \|x\|_0 \leq cnp / \log(e/p) \right\}.$$

Then

$$\mathbb{E} \sup_{x \in B_{2,0}} \|BAx\|_2 \leq 3K \sqrt{np + \log(2n)}.$$

Proof. Let $c > 0$ be a constant to be determined later, and let $\lambda := cp / \log(e/p)$. Then

$$B_{2,0} = \bigcup_{|J|=\lfloor \lambda n \rfloor} B_2^J,$$

where the union is over all subsets $J \subset \{1, \dots, n\}$ of cardinality $\lfloor \lambda n \rfloor$, and where $B_2^J = \{x \in \mathbb{R}^J : \|x\|_2 \leq 1\}$ denotes the unit Euclidean ball in \mathbb{R}^J . By Lemma 2.5, B_2^J has a $\frac{1}{2}$ -net \mathcal{N}_J of cardinality at most $e^{2\lambda n}$. Let $t \geq 1$. For a fixed $x \in \mathcal{N}_J$, Lemma 4.5 gives

$$\mathbb{P}\{\|BAx\|_2 > (K+1)\sqrt{np + \log(2n)} + t\} \leq \exp(-c_0(np + t^2)).$$

Using Proposition 2.6 and taking the union bound over all $x \in \mathcal{N}_J$, we obtain

$$\begin{aligned} & \mathbb{P}\left\{\frac{1}{2} \sup_{x \in B_2^J} \|BAx\|_2 > (K+1)\sqrt{np + \log(2n)} + t\right\} \\ & \leq \mathbb{P}\left\{\sup_{x \in \mathcal{N}_J} \|BAx\|_2 > (K+1)\sqrt{np + \log(2n)} + t\right\} \\ & \leq |\mathcal{N}_J| \exp(-c_0(np + t^2)) \leq \exp(2\lambda n - c_0(np + t^2)). \end{aligned}$$

Since there are $\binom{n}{\lfloor \lambda n \rfloor} \leq (e/\lambda)^{\lambda n}$ ways to choose the subset J , by taking the union bound over all J we conclude that

$$(4.6) \quad \mathbb{P}\left\{\frac{1}{2} \sup_{x \in B_{2,0}} \|BAx\|_2 > 2(K+1)\sqrt{np + \log(2n)} + t\right\} \leq \exp(\lambda \log(e/\lambda)n + 2\lambda n - c_0(np + t^2)).$$

Finally, if the absolute constant $c > 0$ in the definition of λ is chosen sufficiently small, we have $\lambda \log(e/\lambda)n + 2\lambda n \leq c_0 np$. Thus the right hand side of (4.6) is at most

$$\exp(-c_0 t^2).$$

Integration completes the proof. \square

4.4. Control of spread vectors. Although we now have a good control of sparse vectors, they unfortunately comprise a small part of the unit ball B_2^n . More common but harder to deal with are “spread vectors” – those having many coordinates that are not close to zero. The next result gains control of the spread vectors.

Proposition 4.7 (Spread vectors). *Let A and B be matrices as in Lemma 4.4 with $N \geq n$. Let $M \geq 2$. Consider the set of vectors*

$$B_{2,\infty} := \left\{x \in \mathbb{R}^n, \|x\|_2 \leq 1, \|x\|_\infty \leq \frac{M}{\sqrt{n}}\right\}.$$

Then

$$\mathbb{E} \sup_{x \in B_{2,\infty}} \|BAx\|_2 \leq C \log^{3/2}(M) \cdot K \sqrt{np + \log(2N)}.$$

Proof. This time we will need to work with multiple nets to account for different possible distributions of the magnitude of the coordinates of vectors $x \in B_{2,\infty}$. Since $\|x\|_\infty \leq \|x\|_2$, without loss of generality we can assume that $M \leq \sqrt{n}$.

Step 1: construction of nets. Let

$$h_k := \frac{2^k}{\sqrt{n}}, \quad k = -2, -1, 0, 1, 2, \dots, \log_2 M$$

and let

$$\mathcal{N} := \{x \in B_{2,\infty} : \forall j \exists k \text{ such that } |x_j| = h_k\}.$$

A standard calculation shows that \mathcal{N} is an $\frac{1}{2}$ -net of $B_{2,\infty}$ in the $B_{2,\infty}$ -norm, i.e. for every $x \in B_{2,\infty}$ there exists $y \in \mathcal{N}$ such that $x - y \in \frac{1}{2}B_{2,\infty}$. Therefore, by Proposition 2.6,

$$\sup_{x \in B_{2,\infty}} \|BAx\|_2 \leq 2 \sup_{x \in \mathcal{N}} \|BAx\|_2.$$

Fix $x \in \mathcal{N}$. Since $\|x\|_2 \leq 1$, the number of coordinates of x that satisfy $|x_j| = h_k$ is at most $\lfloor h_k^{-2} \rfloor$, for every k . Decomposing x according to the coordinates whose absolute value is h_k , we have by the triangle inequality that

$$(4.7) \quad \sup_{x \in B_{2,\infty}} \|BAx\|_2 \leq 2 \sum_{k=-2}^{\log_2 M} \sup_{x \in \mathcal{N}_k} \|BAx\|_2,$$

where

$$\mathcal{N}_k = \{x \in B_2^n : \|x\|_0 \leq \lfloor h_k^{-2} \rfloor; \text{ all nonzero coordinates of } x \text{ satisfy } |x_j| = h_k\}.$$

Fix k and assume that $\mathcal{N}_k \neq \emptyset$. Since $h_k \leq M/\sqrt{n}$, we have

$$(4.8) \quad m := \lfloor h_k^{-2} \rfloor \geq \lfloor n/M^2 \rfloor \geq 1.$$

To estimate the cardinality of \mathcal{N}_k , note that there are at most $\min(m, n)$ ways to choose $\|x\|_0 := l$; there are $\binom{n}{l}$ ways to choose the support of x ; and there are 2^l ways to choose the (signs of) nonzero coordinates of x . Hence by Stirling's approximation and using (4.8), we have

$$(4.9) \quad |\mathcal{N}_k| \leq \sum_{l=1}^{\min(m,n)} \binom{n}{l} 2^l \leq \min \left\{ \left(\frac{2en}{m} \right)^m, 4^n \right\} \leq (4eM^2)^m \leq \exp(Cm \log M)$$

where $C \geq 1$ is an absolute constant.

Step 2: control of a fixed vector. Fix m and fix $x \in \mathcal{N}_k$. As we saw in the proof of Lemma 4.5,

$$\|BAx\|_2 \text{ is distributed identically with } \left\| \sum_{i=1}^N g_i \lambda_i B_i \right\|_2$$

where

$$\lambda_i = \left(\sum_{j=1}^n a_{ij}^2 x_j^2 \right)^{1/2}$$

and where g_i are independent standard normal random variables. Since $x \in \mathcal{N}_k$, we have $\|x\|_\infty = h_k \leq \frac{1}{\sqrt{m}}$. This and the second condition in (4.4) yield

$$\lambda_i \leq \left(\frac{1}{m} \sum_{j=1}^n a_{ij}^2 \right)^{1/2} \leq K \sqrt{\frac{np + \log(2N)}{m}}.$$

We consider the map $f : \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$f(y) = \left\| \sum_{i=1}^N y_i \lambda_i B_i \right\|_2.$$

Repeating the estimate in the proof of Lemma 4.5, we bound the Lipschitz norm as

$$\|f\|_{\text{Lip}} \leq \max_i |\lambda_i| \leq K \sqrt{\frac{np + \log(2N)}{m}}.$$

Then the Gaussian concentration, Theorem 2.1, gives for every $t > 0$:

$$\mathbb{P}(f(g) - \mathbb{E}f(g) > t) \leq \exp\left(-\frac{c_0 t^2 m}{K^2(np + \log(2N))}\right),$$

where $g = (g_1, \dots, g_N)$. Since as we noted above, $f(g)$ is distributed identically with $\|BAx\|_2$, Lemma 4.4 yields that

$$\mathbb{P}(\|BAx\|_2 > K\sqrt{np + \log(2n)} + t) \leq \exp\left(-\frac{c_0 t^2 m}{K^2(np + \log(2N))}\right),$$

Let $u > 0$ be arbitrary. Applying the above estimate for $t = uK\sqrt{np + \log(2N)}$ and using $N \geq n$ we conclude that

$$(4.10) \quad \mathbb{P}(\|BAx\|_2 > (1+u)K\sqrt{np + \log(2N)}) \leq \exp(-c_0 u^2 m).$$

Step 3: union bound. Taking the union bound in (4.10) over all $x \in \mathcal{N}_k$ and using estimate (4.9) on the cardinality of \mathcal{N}_k , we have for all $u > 0$:

$$\begin{aligned} \mathbb{P}\left(\sup_{x \in \mathcal{N}_k} \|BAx\|_2 > (1+u)K\sqrt{np + \log(2N)}\right) &\leq |\mathcal{N}_k| \exp(-c_0 u^2 m) \\ &\leq \exp(Cm \log M - c_0 u^2 m). \end{aligned}$$

Let $s \geq 1$. We choose $u = C_1 s \sqrt{\log M}$, where $C_1 := \sqrt{C/c_0}$. Since $u \geq 1$ and $m \geq 1$, $M \geq 2$, we obtain from the above estimate that

$$\begin{aligned} \mathbb{P}\left(\sup_{x \in \mathcal{N}_k} \|BAx\|_2 > 2C_1 s K \sqrt{\log(M)(np + \log(2N))}\right) &\leq \exp(C(1-s^2)m \log M) \\ &\leq \exp(c(1-s^2)). \end{aligned}$$

Integrating yields that

$$\mathbb{E} \sup_{x \in \mathcal{N}_k} \|BAx\|_2 \leq C_2 K \sqrt{\log(M)(np + \log(2N))}.$$

Putting this back in (4.7), we conclude that

$$\mathbb{E} \sup_{x \in B_{2,\infty}} \|BAx\|_2 \leq 2(3 + \log M) \cdot C_2 K \sqrt{\log(M)(np + \log(2N))}.$$

This completes the proof. \square

4.5. Norms of sparse matrices, and proof of Theorem 4.1. Propositions 4.6 and 4.7 together handle all vectors in the unit ball, and yield the following norm estimate:

Proposition 4.8. *Let A and B be matrices as in Lemma 4.4 with $N \geq n$. Then*

$$\mathbb{E}\|BA\| \leq C \log^{3/2} \left(\frac{e}{p} \right) \cdot K \sqrt{np + \log(2N)}.$$

Proof. Let c be the absolute constant as in Proposition 4.6; we can clearly assume that $c \leq 1/4$. We define

$$M = \sqrt{\frac{1}{cp} \log \frac{e}{p}}.$$

Note that $M \geq 2$ as required in Proposition 4.6.

Fix a vector $x \in B_2^n$. We decompose it according to the magnitude of the coordinates, as follows:

$$x = y + z, \quad y := x \mathbf{1}_{\{|x_j| > M/\sqrt{n}\}}, \quad z := x \mathbf{1}_{\{|x_j| \leq M/\sqrt{n}\}}.$$

Clearly, $\|y\|_2 \leq \|x\|_2 \leq 1$, $\|z\|_2 \leq \|x\|_2 \leq 1$. By Markov's inequality, we have

$$\|y\|_0 = |\{j : |x_j| > M/\sqrt{n}\}| \leq \frac{n}{M^2} = \frac{cnp}{\log(e/p)}.$$

Then $y \in B_{2,0}$ as in Proposition 4.6. On the other hand, $\|z\|_\infty \leq M/\sqrt{n}$ by definition, so $z \in B_{2,\infty}$ as in Proposition 4.7. Therefore, by Propositions 4.6 and 4.7 we have

$$\begin{aligned} \mathbb{E}\|BA\| &= \mathbb{E} \sup_{x \in B_2^n} \|BAx\|_2 \leq \mathbb{E} \sup_{y \in B_{2,0}} \|BAy\|_2 + \mathbb{E} \sup_{z \in B_{2,\infty}} \|BAz\|_2 \\ &\leq 3K \sqrt{np + \log(2n)} + C \log^{3/2}(M) \cdot K \sqrt{np + \log(2N)}. \end{aligned}$$

Our choice of M and the assumption $N \geq n$ completes the proof. \square

Finally, a standard symmetrization argument yields the following norm estimate, which we shall use for sparse random matrices.

Corollary 4.9. *Let $p \in (0, 1]$ and let $N \geq n$ be positive integers. Consider an $N \times n$ random matrix A whose entries are independent random variables a_{ij} with mean zero and such that*

$$\mathbb{E}|a_{ij}|^2 \leq p, \quad |a_{ij}| \leq 1 \quad \text{for every } i, j.$$

Let B be an $n \times N$ matrix such that $\|B\| \leq 1$. Then

$$\mathbb{E}\|BA\| \leq C \log^{3/2} \left(\frac{e}{p} \right) \sqrt{np + \log(2N)}.$$

Remark. It would be interesting to remove the logarithmic term from this estimate.

Proof. Let g_{ij} be independent standard normal random variables. Consider the random matrix $\tilde{A} = (g_{ij}a_{ij})$. By (2.1), we have

$$(4.11) \quad \mathbb{E}\|BA\| \leq (2\pi)^{1/2} \mathbb{E}\|B\tilde{A}\|.$$

By Lemma 4.3, conditions (4.4) hold with some random parameter $K \geq 1$ which only depends on the random variables (a_{ij}) and not on (g_{ij}) , and which satisfies

$$(4.12) \quad \mathbb{E}_a K \leq C_1$$

where C_1 is an absolute constant. Here and below we write \mathbb{E}_a when the expectation is with respect to (a_{ij}) , and \mathbb{E}_g if the expectation is with respect to (g_{ij}) .

Condition on the random variables (a_{ij}) . Proposition 4.8 then yields

$$\mathbb{E}_g\|B\tilde{A}\| \leq C \log^{3/2} \left(\frac{e}{p} \right) \cdot K \sqrt{np + \log(2N)}.$$

Therefore, when we remove the conditioning, we obtain by (4.12) that

$$\mathbb{E}\|B\tilde{A}\| = \mathbb{E}_a \mathbb{E}_g\|B\tilde{A}\| \leq C \log^{3/2} \left(\frac{e}{p} \right) \cdot C_1 \sqrt{np + \log(2N)}.$$

This and (4.11) complete the proof. \square

Proof of Theorem 4.1. By the standard symmetrization technique described in Section 2, we can assume without loss of generality that all a_{ij} are symmetric random variables. We decompose the matrix A according to the magnitude of its entries as follows. Given a subset $I \subset \mathbb{R}$, we define the truncated matrix

$$\text{trunc}(A, I) = (a_{ij} \mathbf{1}_{\{|a_{ij}| \in I\}}).$$

Consider

$$\begin{aligned} A^{(0)} &= \text{trunc}(A, [0, 1]); \\ A^{(k)} &= 2^{-k} \text{trunc}(A, (2^{k-1}, 2^k]), \quad k = 1, 2, \dots \end{aligned}$$

Then we have a decomposition $A = \sum_{k=0}^{\infty} 2^k A^{(k)}$. This sum is actually finite because of the boundedness assumption on a_{ij} . Indeed, we have

$$(4.13) \quad A = A^{(0)} + \sum_{k=1}^{k_0} 2^k A^{(k)}$$

where k_0 is the maximal integer such that

$$(4.14) \quad 2^{k_0-1} \leq \left(\frac{Mn}{\log(2N)} \right)^{\frac{1}{2+\varepsilon}}.$$

Because a_{ij} are symmetric random variables, all entries $a_{ij}^{(k)}$ of the matrices $A^{(k)}$ satisfy $\mathbb{E}a_{ij}^{(k)} = 0$ and $|a_{ij}^{(k)}| \leq 1$.

Using Corollary 4.9 for the matrix $A^{(0)}$ and $p = 1$, we obtain

$$(4.15) \quad \mathbb{E}\|BA^{(0)}\| \leq C_1\sqrt{n + \log(2N)} \leq 2C_1\sqrt{Mn},$$

where the last line follows because $\log(2N) \leq Mn$ and $M \geq 1$ by the hypothesis.

Now we fix $1 \leq k \leq k_0$. Using the $(2 + \varepsilon)$ -th moment assumption, we have by Markov's inequality that

$$\mathbb{P}(a_{ij}^{(k)} \neq 0) \leq \mathbb{P}(a_{ij} > 2^{k-1}) \leq 2^{-(2+\varepsilon)(k-1)} =: p_k.$$

This and the bound $|a_{ij}^{(k)}| \leq 1$ yield $\mathbb{E}(a_{ij}^{(k)})^2 \leq p_k$. With this, we apply Corollary 4.9 for the matrix $A^{(k)}$ and obtain

$$\mathbb{E}\|BA^{(k)}\| \leq C \log^{3/2} \left(\frac{e}{p_k} \right) \sqrt{np_k + \log(2N)}.$$

By the definition of p_k and by (4.14), we have

$$p_k \geq p_{k_0} \geq \frac{\log(2N)}{Mn}.$$

Therefore, $np_k + \log(2N) \leq (1 + M)np_k \leq 2Mnp_k$, so

$$(4.16) \quad \begin{aligned} \mathbb{E}\|BA^{(k)}\| &\leq C \log^{3/2} \left(\frac{e}{p_k} \right) \sqrt{2Mnp_k} \\ &\leq C_2 [1 + (2 + \varepsilon)(k - 1)]^{3/2} 2^{-(1+\varepsilon/2)(k-1)} \cdot \sqrt{Mn}. \end{aligned}$$

Using (4.13) and the triangle inequality, then using (4.15) and (4.16), we conclude that

$$\begin{aligned} \mathbb{E}\|BA\| &\leq \mathbb{E}\|BA^{(0)}\| + \sum_{k=1}^{k_0} 2^k \mathbb{E}\|BA^{(k)}\| \\ &\leq 2C_1\sqrt{Mn} + \sum_{k=1}^{k_0} C_2 [1 + (2 + \varepsilon)(k - 1)]^{3/2} 2^{k-(1+\varepsilon/2)(k-1)} \cdot \sqrt{Mn} \\ &\leq C_2\sqrt{Mn} \cdot \sum_{k=1}^{\infty} k^{3/2} 2^{-(\varepsilon/2)k} \\ &= C(\varepsilon)\sqrt{Mn}. \end{aligned}$$

This completes the proof of Theorem 4.1. \square

4.6. Almost square matrices. The main application of Theorem 4.1 is for almost square matrices – those for which $N = n^{1+o(1)}$. The next lemma verifies the hypotheses of Theorem 4.1 for such matrices.

Lemma 4.10. *Let $\varepsilon \in (0, 1)$ and let N, n be positive integers satisfying $N \leq n^{1+\varepsilon/10}$. Let A be an $N \times n$ random matrix whose entries are independent random variables with $(4 + \varepsilon)$ -th moment bounded by 1. Define the random variable M by the equation*

$$(4.17) \quad \max_{i,j} |a_{ij}| = \left(\frac{Mn}{\log(2N)} \right)^{\frac{1}{2+\varepsilon/4}}.$$

Then, for every $t \geq 1$, one has

$$\mathbb{P}(M > C(\varepsilon)t) \leq \frac{1}{t^2}.$$

In particular, one has $\mathbb{E}M \leq C_1(\varepsilon)$.

Proof. By Markov's inequality, we have for every i, j that

$$\mathbb{P}(|a_{ij}| > s) \leq \frac{1}{s^{4+\varepsilon}}, \quad s > 0.$$

Let $t \geq 1$. We then have

$$\mathbb{P}(|a_{ij}| > (t^2 n N)^{\frac{1}{4+\varepsilon}}) \leq \frac{1}{t^2 n N}.$$

Taking the union bound over all nN random variables a_{ij} , we obtain

$$(4.18) \quad \mathbb{P}\left(\max_{i,j} |a_{ij}| > (t^2 n N)^{\frac{1}{4+\varepsilon}}\right) \leq \frac{1}{t^2}.$$

The assumption $N \leq n^{1+\varepsilon/10}$ yields that

$$nN \leq \left(\frac{C(\varepsilon)n}{\log(2N)} \right)^{2+\varepsilon/8}.$$

Therefore, since $\frac{2+\varepsilon/8}{4+\varepsilon} \leq \frac{1}{2+\varepsilon/4}$ and $t \geq 1$, we have

$$(t^2 n N)^{\frac{1}{4+\varepsilon}} \leq \left(\frac{C(\varepsilon)tn}{\log(2N)} \right)^{\frac{1}{2+\varepsilon/4}}.$$

Using this in (4.18), we obtain

$$\mathbb{P}(M > C(\varepsilon)t) \leq \mathbb{P}\left(\max_{i,j} |a_{ij}| > \left(\frac{C(\varepsilon)tn}{\log(2N)} \right)^{\frac{1}{2+\varepsilon/4}}\right) \leq \frac{1}{t^2}.$$

Integration completes the proof. \square

We are now ready to state and prove a partial case of Theorem 1.1 for almost square matrices.

Corollary 4.11. *Let $\varepsilon \in (0, 1)$ and let N, n be positive integers satisfying $N \leq n^{1+\varepsilon/10}$. Let A be an $N \times n$ random matrix whose entries are independent random variables with mean zero and $(4 + \varepsilon)$ -th moment bounded by 1. Let B be an $n \times N$ matrix such that $\|B\| \leq 1$. Then*

$$\mathbb{E}\|BA\| \leq C(\varepsilon)\sqrt{n}.$$

Proof. Without loss of generality we may assume that $N \geq n$ by adding an appropriate number of zero rows to A and zero columns to B . Also, using the standard symmetrization, we can assume that the random variables a_{ij} are symmetric. Let M be the random variable as in Lemma 4.10, and let $t \geq 1$. By the definition, $\{M \leq t\}$ is the product event. Therefore, conditioning on this event (i) preserves the independence of the entries of A ; (ii) makes all these entries bounded as in (4.17); (iii) can only reduce their moments by Lemma 2.9, thus for all i, j we have

$$\mathbb{E}(|a_{ij}|^{2+\varepsilon/4} | M \leq t) \leq \mathbb{E}|a_{ij}|^{2+\varepsilon/4} \leq 1.$$

Therefore, we can apply Corollary 4.2 conditionally, with $\varepsilon/4$ and with M replaced by $\max(M, 10)$, which gives

$$[\mathbb{E}(\|BA\|^2 | M \leq t)]^{1/2} \leq C_0(\varepsilon)\sqrt{tn} \quad \text{for } t \geq 1.$$

Additionally, by Lemma 4.10 we have

$$\mathbb{P}(M > C(\varepsilon)t) \leq \frac{1}{t^2} \quad \text{for } t \geq 1.$$

By Lemma 2.10, this yields

$$\mathbb{E}\|BA\| \leq (\mathbb{E}\|BA\|^2)^{1/2} \leq C_1(\varepsilon)\sqrt{n}$$

as claimed. □

5. COMPLETION OF THE PROOF OF THEOREM 1.1

Proof of Theorem 1.1. By adding an appropriate number of zero rows to B or zero columns to A we can assume that $m = n$, thus B is an $n \times N$ matrix. Consider the exponent

$$K = K(\varepsilon) = \frac{1}{2} + \frac{1}{\varepsilon}.$$

As usual, let B_1, \dots, B_N be the columns of the matrix B . Consider the subset $I \subset \{1, \dots, N\}$ of large columns defined as

$$I := \{i : \|B_i\|_2 > C_0(\varepsilon) \log^{-K}(2n)\}.$$

Here we choose $C_0(\varepsilon)$ sufficiently large so that, by (2.4) and Markov's inequality, we have

$$N_0 := |I| < C_0(\varepsilon)^{-2} n \log^{2K}(2n) \leq n^{1+\varepsilon/10}.$$

Denote by A_I the $N_0 \times n$ submatrix of A whose rows are in I , by B_I the $n \times N_0$ submatrix of B whose columns are in I (and similarly for I^c). The decomposition $BA = B_I A_I + B_{I^c} A_{I^c}$ implies by the triangle inequality that

$$(5.1) \quad \|BA\| \leq \|B_I A_I\| + \|B_{I^c} A_{I^c}\|.$$

This splits our problem into two subproblems, one for I and one for I^c . Of course, if I or I^c is empty then the corresponding matrix is zero and we can skip its estimation.

The matrices A_I , B_I are almost square, so Corollary 4.11 applies for them, giving

$$(5.2) \quad \mathbb{E}\|B_I A_I\| \leq C(\varepsilon)\sqrt{n}.$$

On the other hand, the columns of the matrix B_{I^c} are small by the definition of I :

$$\|B_i\|_2 \leq C_0(\varepsilon) \log^{-K}(2n) \quad \text{for every } i \in I^c.$$

Therefore, Theorem 3.9 applies to the matrices A_{I^c} , B_{I^c} , which gives

$$(5.3) \quad \mathbb{E}\|B_{I^c} A_{I^c}\| \leq C_1(\varepsilon)\sqrt{n}.$$

Putting estimates (5.2) and (5.3) into (5.1), we conclude that

$$\mathbb{E}\|BA\| \leq C_2(\varepsilon)\sqrt{n}.$$

Theorem 1.1 is proved. □

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