# MATH 162A Review: Invariant Theory

## Facts to Know:

Linear algebra, the invariant theory of linear transformation.

$T: V \to V$ Let $\{v_1, \dots v_n\}$ be a basis of $V$.

$$T v_i = a_{i1} v_1 + a_{i2} v_2 + \dots + a_{in} v_n$$

$$T v_i = a_{ij} v_j \quad (\square)$$

Let $\{f_1, \dots f_n\}$ be another basis of $V$

$$T f_i = b_{ij} f_j \quad (*)$$

$A = (a_{ij})$ , $B = (b_{ij})$ .

Assume $f_i = P_{ij} v_j$ $(\triangle)$

$$P_{ij} T(v_j) = T(P_{ij} v_j) = T f_i = b_{ij} f_j = b_{ij} P_{jk} v_k$$

$$\| \quad P_{ij} a_{jk} v_k$$

$$\boxed{b_{ij} P_{jk} = P_{ij} a_{jk}} \Leftrightarrow BP = PA$$

$$B = PAP^{-1}$$

If $B$ is similar to $A$, then $\det B = \det A = \det(T)$

$$tr(T) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n b_{ii}$$

Theorem: $tr(A) = tr(B)$

$$\boxed{\begin{aligned} b_{il} &= P_{ij} a_{jk} g_{kl} \\ P_{jk} g_{kl} &= \delta_{jl} \end{aligned}}$$

$$tr(B) = \sum_{i=1}^n b_{ii} = b_{ii} = P_{ij} a_{jk} g_{ki} = g_{ki} P_{ij} a_{jk}$$

$$= \delta_{kj} a_{jk} = a_{kk} = tr(A)$$

$$\det(A - \lambda I) = \det(B - \lambda I)$$

$$= a_0 + a_1 \lambda + \dots (-1)^{n-1} a_{n-1} \lambda^{n-1} + (-1)^n \lambda^n$$

$a_0 = \det(A) = \det(T)$. $\quad a_{n-1} = + tr(A) = + tr(T)$

## Examples:

1. Let $A$ be a square matrix. Prove that
$$\limsup_{k\to\infty} |\mathrm{Tr}(A^k)|^{1/k}$$
exists.

Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of $A$.

$$tr(A^k) = \sum_{j=1}^{n} \lambda_j^k$$

$$\limsup_{k\to\infty} \left| \sum_{j=1}^{n} \lambda_j^k \right|^{1/k} = \max_{1\le j\le n} |\lambda_j|$$

2. Let $A$ be a $2 \times 2$ symmetric real-valued matrix. Prove that
$$(\mathrm{Tr}(A))^2 \geq 4\det(A).$$

$$Tr(A) = \lambda_1 + \lambda_2$$
$$det(A) = \lambda_1 \lambda_2$$
$$(\lambda_1+\lambda_2)^2 \geq 4\lambda_1\lambda_2 \iff (\lambda_1-\lambda_2)^2 \geq 0$$