# Math 147 — Complex Analysis

Neil Donaldson

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## 1 Complex Numbers

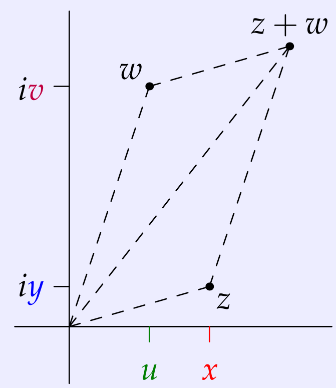
### 1.1 Definition and Basic Algebraic Properties

In the 1500s, Italian mathematician Rafael Bombelli posited a solution to the seemingly absurd equation $x^2 = -1$. By supposing that this solution behaved according to the usual rules of algebra, Bombelli and others were able to describe the solutions of any quadratic equation. To some extent, this was math for its own sake; Bombelli always considered his solutions to be entirely 'fictitious.'

For a modern definition, we start with the Cartesian plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

**Definition 1.1.** Given real numbers $x, y$, the *complex number* $z = x + iy$ is the point with co-ordinates $(x, y) \in \mathbb{R}^2$. Its *real* and *imaginary parts* are the individual co-ordinates

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$




The *complex numbers* $\mathbb{C}$ comprise the *real* vector space $\mathbb{R}^2$ with the extra operation of *complex multiplication*: if $z = x + iy$ and $w = u + iv$, define

(Vector) Addition: $z + w := (x + u) + i(y + v)$

Complex multiplication: $zw := (xu - yv) + i(xv + yu)$

When drawn with axes, the complex plane is known as the *Argand diagram* and we refer, respectively, to the *real* and *imaginary axes*.

Since $\mathbb{C} = \mathbb{R}^2$ is a real vector space under addition, several properties are immediate:

**Lemma 1.2 (Basic properties of complex addition).**

*Associativity:* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

*Commutativity:* $z + w = w + z$ *(this is the parallelogram law illustrated in the picture)*

*(Real) scalar multiplication:* $\forall \lambda \in \mathbb{R},\ \lambda(x + iy) = \lambda x + i\lambda y$

*Additive inverse:* $-z = -(x + iy) = (-x) + i(-y) = -x - iy$

**Example 1.3.** If $z = 3 + 4i$ and $w = 2 - 7i$, then

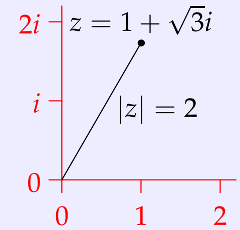
$$z - w = z + (-w) = \big(3 - 2\big) + \big(4 - (-7)\big)i = 1 + 11i$$

The natural distance measure from $\mathbb{R}^2$ transfers to $\mathbb{C}$ (the complex numbers are a real metric space).

**Definition 1.4.** The *modulus* of a complex number $z = x + iy$ is the Euclidean distance of the point $(x, y)$ from the origin:

$$|z| := \sqrt{x^2 + y^2}$$

In the picture, $z = 1 + \sqrt{3}i$ has modulus $|z| = \sqrt{1+3} = 2$.




Some natural inequalities follow straight from the picture in Definition 1.1.

**Lemma 1.5 (Triangle inequalities).** *For all* $z, w \in \mathbb{C}$,

$$|z + w| \le |z| + |w| \quad \textit{and} \quad |z + w| \ge \big||z| - |w|\big|$$

In the second ("extended") inequality, we take the *absolute value* of the difference of the moduli. Unlike in $\mathbb{R}$, these inequalities relate to honest triangles! We can easily generalize the first by induction,

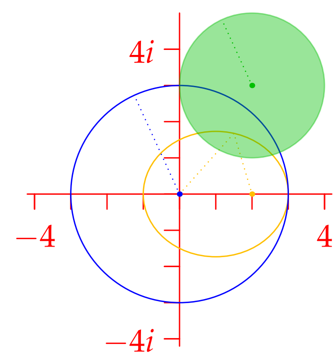
$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$$

The modulus may be used to describe various curves and regions in the plane.

**Examples 1.6.**

1. $|z| = 3$ describes the circle with radius 3 centered at the origin. In Cartesian co-ordinates, this is $x^2 + y^2 = 9$.
2. $|z - 2 - 3i| \le 2$ describes the *disk* with radius 2 centered at $2 + 3i$.
3. $|z| + |z - 2| = 4$ describes an *ellipse* with foci 0 and 2. This is more familiar after multiplying out (try it!):

$$\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4 \implies (x-2)^2 + y^2 = \cdots$$
$$\implies \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$$




**Complex multiplication, division and the complex conjugate**

Multiplication is what really distinguishes the complex numbers from $\mathbb{R}^2$, and lead to all the interesting structure in this course. For starters, we instantly see that $i$ is a solution to Bombelli's absurd equation:

$$i^2 = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1$$

The upshot is that we can treat complex addition, subtraction and multiplication as if we are working with *linear polynomials*[1] in the abstract variable $i$; simply replace $i^2$ with $-1$ when needed.

[1] This is precisely the definition encountered in a Rings & Fields course: $\mathbb{C}$ is the *factor ring* of real polynomials modulo the *ideal* $\langle x^2 + 1 \rangle$.

**Example 1.7.** If $z = 3 + 4i$ and $w = 2 - 7i$, then

$$\begin{aligned} zw &= (3+4i)(2-7i) = 3\cdot 2 + 4i\cdot 2 - 3\cdot 7i - 4i\cdot 7i = 6 + 8i - 21i - 28i^2 \\ &= 6 + 8i - 21i + 28 = 34 - 13i \end{aligned}$$

The basic algebraic properties of complex multiplication are straightforward, if tedious, to verify:

**Lemma 1.8 (Basic properties of multiplication).** *For any complex numbers $z_1, z_2, z_3$,*

*Associativity:* $z_1(z_2z_3) = (z_1z_2)z_3$

*Commutativity:* $z_1z_2 = z_2z_1$

*Distributivity:* $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

To develop division, it is is helpful to introduce a new concept.

**Definition 1.9.** The *(complex) conjugate* of $z = x + iy$ is $\bar{z} := x - iy$ ("$z$-bar"). Geometrically, $\bar{z}$ is obtained by reflection in the real axis.

Observe that $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$, from which:

**Lemma 1.10.** *Every non-zero complex number $z = x + iy$ has a unique multiplicative inverse*

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \quad \textit{satisfies} \quad zz^{-1} = 1$$

*Proof.* That $zz^{-1} = 1$ is trivial. For uniqueness, suppose we also have $zw = 1$; now use associativity and commutativity to conclude that

$$w = (z^{-1}z)w = z^{-1}(zw) = z^{-1}$$

■

Division is simply multiplication by an inverse.

**Example 1.11.** Given $z = 3 + 4i$ and $w = 2 - 7i$, compute

$$\begin{aligned} \frac{w}{z} &= \frac{2-7i}{3+4i} = wz^{-1} = \frac{w\bar{z}}{|z|^2} = \frac{(2-7i)(3-4i)}{|3+4i|^2} = \frac{2\cdot 3 - 2\cdot 4i - 7i\cdot 3 + 7i\cdot 4i}{3^2 + 4^2} \\ &= \frac{6 - 8i - 21i + 28i^2}{25} = \frac{-22 - 29i}{25} \end{aligned}$$

Alternatively, we could multiply numerator and denominator by the conjugate[2] of the denominator:

$$\frac{2-7i}{3+4i} = \frac{2-7i}{3+4i}\cdot\frac{3-4i}{3-4i} = \cdots$$

[2] Compare this approach to elementary algebra where, for instance, we use the *conjugate* $5 + \sqrt{3}$ of $5 - \sqrt{3}$ to evaluate the reciprocal $\frac{1}{5-\sqrt{3}} = \frac{5+\sqrt{3}}{(5-\sqrt{3})(5+\sqrt{3})} = \frac{5+\sqrt{3}}{22}$.

This initial section *should* be revision, though some unfamiliarity with terminology is expected. The goal at this stage is to be fluent when *computing* with complex numbers. The exercises should feel straightforward, particularly 1–6; if not, ask for help!

**Exercises 1.1.** 1. The real number 2 is also a complex number. What is it in co-ordinate form $(x, y)$? What about the complex number $i$?

2. For any $z \in \mathbb{C}$, prove that $\operatorname{Re}(iz) = -\operatorname{Im} z$ and that $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. (a) Check that both $z = 2 + 3i$ and its conjugate $\overline{z} = 2 - 3i$ solve the quadratic equation $z^2 - 4z + 13 = 0$.

   (b) More generally, suppose $a, b, c \in \mathbb{R}$ where $\omega := 4ac - b^2 > 0$. Verify that $z = \frac{-b+i\sqrt{\omega}}{2a}$ and its conjugate $\overline{z}$ both solve[3] the quadratic equation $az^2 + bz + c = 0$.

4. Prove the commutativity of complex multiplication (Lemma 1.8) using Definition 1.1.

5. Evaluate the following in the form $x + iy$.

   (a) $\dfrac{2-i}{3-5i}$ (b) $(1+i)^4$ (c) $(2+3i)^{-2} - (2-3i)^{-2}$

6. Prove the following. Write $z = x + iy$ rather than using the vector definition.

   (a) $\overline{\overline{z}} = z$ (b) $(z^{-1})^{-1} = z$ (c) $\overline{zw} = \overline{z} \cdot \overline{w}$

7. (a) Use the first version of the triangle inequality (Lemma 1.5) to prove the second: for any $z, w \in \mathbb{C}$, we have $|z + w| \geq \big||z| - |w|\big|$.

   (b) What relationship between $z, w$ corresponds to *equality* here? Draw a picture!

8. Suppose that $|z| \geq 2$ and consider the polynomial $P(z) = z^3 + 3z - 1$.

   (a) Use the triangle inequality to prove that $\left|\frac{3z-1}{z^3}\right| \leq \frac{7}{8}$

   (b) Write $|P(z)| = \left|z^3 + 3z - 1\right| = \left|z^3\right| \left|1 + \frac{3z-1}{z^3}\right|$. Now use the extended triangle inequality to prove that $|P(z)| \geq 1$.
   (*This shows that all zeros of $P(z)$ lie inside the circle $|z| < 2$*)

9. By considering the inequality $(|x| - |y|)^2 \geq 0$, prove that

$$\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z| \quad \text{for any} \quad z \in \mathbb{C}$$

10. Prove that the hyperbola $x^2 - y^2 = 1$ can be written in the form $z^2 + \overline{z}^2 = 2$.

11. Draw a picture of the ellipse satisfying the equation $|z| + |z - 4i| = 6$. Find the equation of the curve in Cartesian coordinates: $\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1$ where $(c, d)$ is the center of the ellipse and $a, b$ are its semi-axes.

    (*Hint: write $|z - 4i| = 6 - |z|$, square both sides, cancel $x^2, y^2$ terms and repeat...*)

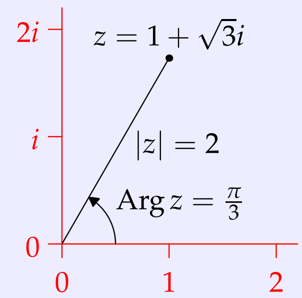
[3] Since $i^2 = -1$, we may write $i\sqrt{\omega} = \sqrt{-\omega}$, whence the quadratic formula applies to all real quadratic equations!

## 1.2 The Exponential/Polar Form of a Complex Number

**Definition 1.12.** A complex number can be written in polar co-ordinates:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Plainly $r = |z|$ is the modulus. We call the angle $\theta = \arg z$ the *argument* of $z$. The argument is *multi-valued* in the sense that $\arg z = \theta + 2\pi n$ for any integer $n$. For clarity, we distinguish the *principal argument* $\operatorname{Arg} z$ by insisting that $-\pi < \operatorname{Arg} z \leq \pi$.




Note that 0 has no argument; it is the only such complex number.

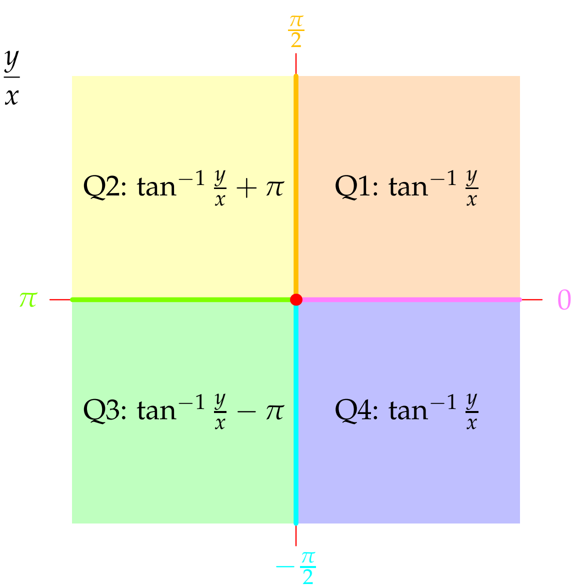
**Example 1.13.** In the above picture, $z = 1 + \sqrt{3}i$ has principal argument $\operatorname{Arg} z = \frac{\pi}{3}$. You can write the argument either as many different values, or as a set:[4] the following are all legitimate

$$\arg z = \{\tfrac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\} \quad \text{or} \quad \arg z = \tfrac{\pi}{3} \quad \text{or} \quad \arg z = \tfrac{7\pi}{3}$$

If $x \neq 0$, is it almost trivial to compute the argument:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x} \overset{???}{\implies} \operatorname{Arg} z = \tan^{-1}\frac{y}{x}$$

This isn't quite right. Since arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, an addition or subtraction of $\pi$ is required to find the correct value whenever $z$ lies in the second or third quadrants.




**Example 1.14.** Since $z = -3 - 3i$ lies in the third quadrant, we have

$$\operatorname{Arg} z = \tan^{-1}\frac{-3}{-3} - \pi = -\frac{3\pi}{4}$$

If we wanted the argument to be positive, we could choose a non-principal argument $\arg z = \frac{5\pi}{4}$.

The polar form allows us to define a crucial new function.

**Definition 1.15.** Given $\theta \in \mathbb{R}$, the *complex exponential* $e^{i\theta}$ is defined by *Euler's formula*

$$e^{i\theta} := \cos\theta + i\sin\theta$$

The *polar form* of a complex number is $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$ (*any* argument! why?). If $w = u + iv$ is complex, then its exponential is defined by $e^w := e^u e^{iv} = e^u(\cos v + i\sin v)$.

[4] This is merely the common mathematical fudge of replacing an equivalence class $\{\frac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}$ by any of its representatives, e.g. $\frac{\pi}{3}$ or $\frac{7\pi}{3}$.

Euler's formula provides a sensible definition of $e^{i\theta}$ since it fits with two common definitions of the exponential in real analysis.

1. If $k \in \mathbb{R}$, then $e^{k\theta}$ is the solution to the initial value problem $y' = ky$ with $y(0) = 1$. Assuming that differentiation works when $k = i$, Euler's formula also satisfies this criterion:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}e^{i\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$$

2. The real and imaginary parts of the Maclaurin series $\exp z = \sum \frac{z^n}{n!}$ evaluated at $z = i\theta$ are, respectively, the Maclaurin series of $\cos\theta$ and $\sin\theta$.

A third reason is that the definition satisfies the usual exponential laws.

**Lemma 1.16 (Exponential laws).** *Let $z = re^{i\theta}$ and $w = se^{i\psi}$ be written in polar form. Then:*

1. $zw = rse^{i(\theta+\psi)}$. *In particular,* $|zw| = |z|\,|w|$ *and* $\arg zw = \arg z + \arg w$
2. $\frac{z}{w} = \frac{r}{s}e^{i(\theta-\psi)}$
3. $z^n = r^n e^{in\theta}$ *for all* $n \in \mathbb{Z}$

Note that the *principal argument* might not behave so nicely for products; the best we can say is that

$$\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi n \text{ for some } n = 0, \pm 1$$

*Proof.* Part 1 follows from the multiple-angle formulæ for sine and cosine:

$$\begin{aligned} e^{i(\theta+\psi)} &= \cos(\theta+\psi) + i\sin(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi + i(\sin\theta\cos\psi + \cos\theta\sin\psi) \\ &= (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) = e^{i\theta}e^{i\psi} \end{aligned}$$

Parts 2 and 3 are now straightforward. ∎

**Examples 1.17.** 1. Given $z = -7 + i$ and $w = 3 + 4i$, we compute the modulus and argument of $zw$ in two ways:

(a) Find the polar forms of $z, w$ before applying the Lemma:

$$\begin{aligned} z = |z|\,e^{i\arg z} = 5\sqrt{2}\exp\left(i(\pi - \tan^{-1}\tfrac{1}{7})\right), \quad w = |w|\,e^{i\arg w} = 5\exp\left(i\tan^{-1}\tfrac{4}{3}\right) \\ \implies |zw| = |z|\,|w| = 25\sqrt{2}, \quad \arg zw = \arg z + \arg w = \pi - \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{4}{3} \end{aligned}$$

(b) Find $zw = (-7+i)(3+4i) = -25 - 25i$ then compute its polar form:

$$zw = 25\sqrt{2}e^{-\frac{3\pi i}{4}} \implies |zw| = 25\sqrt{2}, \qquad \operatorname{Arg} zw = -\frac{3\pi}{4}$$

The first approach is certainly uglier! It is usually better to use the second approach unless the arguments of $z, w$ are easily computable. In Exercise 7, we check that these values correspond.

2. We compute $z^{10}$ when $z=\sqrt{3}-i$. First observe that $z=2e^{-\frac{\pi i}{6}}$, from which

$$z^{10}=2^{10}e^{-\frac{5\pi i}{3}}=1024e^{\frac{\pi i}{3}}=512(1+\sqrt{3}i)$$

3. The identity $(e^{i\theta})^n=e^{in\theta}$ ($n\in\mathbb{Z}$) is known as *de Moivre's formula*. It is usually written

$$\boxed{(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta}$$

Many trigonometric identities follow by taking real/imaginary parts. For instance, when $n=3$,

$$\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$$
$$\implies \cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta$$

As before, this shouldn't be difficult. This section is merely polar co-ordinates in a different language!

**Exercises 1.2.** 1. Use induction to prove that for any $n\in\mathbb{N}_{\geq 2}$,

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n}=e^{i(\theta_1+\theta_2+\cdots+\theta_n)}$$

2. Find the principal argument of $(1+i)^{2024}$.

3. Prove that $\left|e^{i\theta}\right|=1$ and that $\overline{e^{i\theta}}=e^{-i\theta}$.

4. (a) Show that if $\operatorname{Re} z>0$ and $\operatorname{Re} w>0$, then $\operatorname{Arg} zw=\operatorname{Arg} z+\operatorname{Arg} w$.
   (b) If $z$ and $w$ both lie in quadrant 2, explain why $\operatorname{Arg} zw=\operatorname{Arg} z+\operatorname{Arg} w-2\pi$.

5. Prove that non-zero $z,w\in\mathbb{C}$ have the same modulus if and only if $\exists p,q\in\mathbb{C}$ such that $z=pq$ and $w=p\overline{q}$.

6. Use de Moivre's formula to establish the identity

$$\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$$

7. (a) Let $\alpha=\tan^{-1}\frac{4}{3}$ and $\beta=\tan^{-1}\frac{1}{7}$. Use right-triangles to show that

$$\cos\alpha=\frac{3}{5},\qquad \sin\alpha=\frac{4}{5},\qquad \cos\beta=\frac{7}{\sqrt{50}},\qquad \sin\beta=\frac{1}{\sqrt{50}}$$

   Now use the cosine multiple-angle formula to check that $\alpha-\beta=\frac{\pi}{4}$.
   (*This shows that* $\arg zw=\frac{5\pi}{4}$ *in Example 1.17(a)*)

   (b) Generalize the approach in part (a): if $0<\beta<\alpha<\frac{\pi}{2}$, prove the multiple-angle formula

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

8. The polar form of a complex number is well-suited to describing *circles.* For instance the circle centered at $i$ with radius 3 may be described by $z=i+3e^{i\theta}$ where $-\pi<\theta\leq\pi$.

   (a) Describe the circle centered at $z_0=3+4i$ with radius 2.
   (b) Show that the points $z=re^{i\theta}$ for which $r=2a\cos\theta$ describe a circle.
   (*Hint: Multiply by* $r$)

### 1.3 Roots of Complex Numbers

A naïve approach to taking roots in $\mathbb{C}$ is very messy.

**Example 1.18.** To find $c$ such that $c^2 = -5 + 12i$, we need to solve an equation:

$$-5 + 12i = c^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \iff \begin{cases} x^2 - y^2 = -5 \\ xy = 6 \end{cases}$$

Substituting $y = 6x^{-1}$ into the first equation yields a quadratic in $x^2$:

$$x^4 + 5x^2 - 36 = (x^2 - 4)(x^2 + 9)$$

We conclude that $x = \pm 2$, and obtain the square roots $\pm c = \pm(2 + 3i)$.

The example is reassuring in that we obtain precisely two square roots. Unfortunately, extending the method to cube, or higher, roots seems doomed! Instead we use the polar form. Suppose $n \in \mathbb{N}$ and that $c, z$ satisfy $z = c^n$. In polar form

$$z = re^{i\theta}, \quad c = se^{i\psi} \implies re^{i\theta} = s^n e^{in\psi}$$

By equating moduli and arguments, we see that

$$r = s^n, \qquad n\psi = \theta + 2\pi k \tag{$*$}$$

where $k$ is some integer. We'll shortly put this together to obtain a proper definition, but we already have enough for a calculation.

**Example 1.19.** We compute the fifth roots of $z = 2e^{\frac{2\pi i}{3}} = -1 + i\sqrt{3}$.

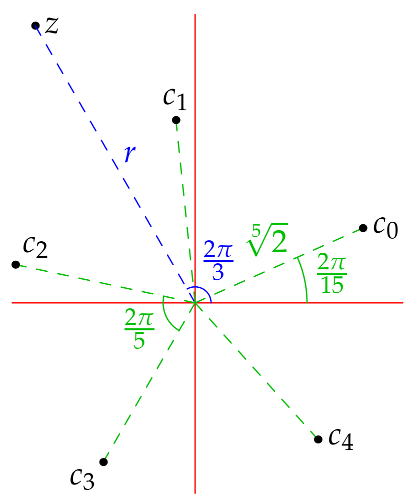
In the above language, $2 = s^5$, whence

$$s = \sqrt[5]{2}, \quad \psi = \frac{1}{5}\left(\frac{2\pi}{3} + 2\pi k\right) = \frac{2\pi}{15}(1 + 3k)$$

which results in the fifth roots

$$c_0 = \sqrt[5]{2}e^{\frac{2\pi i}{15}} \qquad c_1 = \sqrt[5]{2}e^{\frac{8\pi i}{15}} \qquad c_2 = \sqrt[5]{2}e^{\frac{14\pi i}{15}}$$

$$c_3 = \sqrt[5]{2}e^{\frac{20\pi i}{15}} = \sqrt[5]{2}e^{-\frac{10\pi i}{15}} \qquad c_4 = \sqrt[5]{2}e^{\frac{26\pi i}{15}} = \sqrt[5]{2}e^{-\frac{4\pi i}{15}}$$




There are precisely *five* fifth roots: once $k \geq 5$, the roots start repeating! The last two roots were written both with positive and principal arguments; both have their advantages.

Observe also how the fifth roots form the vertices of a regular pentagon, equally spaced around the circle of radius $\sqrt[5]{2}$. Is it obvious to you why this is so?

Finally, note how essential the polar form was to this calculation. We could convert back to rectangular form, but since $\cos\frac{2\pi}{15}$ and $\sin\frac{2\pi}{15}$ are unfriendly values, there is little benefit.

**Definition 1.20.** Given a non zero complex number $z = re^{i\theta}$ and a positive integer $n$, the $n^{th}$ *roots* of $z$ are the $n$ distinct complex numbers

$$c_k = \sqrt[n]{r} \exp \frac{(\theta + 2k\pi)i}{n} \qquad k = 0, 1, \ldots, n-1$$

where $\sqrt[n]{r}$ is the real (positive!) $n^{\text{th}}$ root of $r$.

There are some conventions to follow. Let $\theta = \operatorname{Arg} z$ be the *principal argument* of $z$:

(a) The *principal $n^{th}$ root* is written $\sqrt[n]{z} := \sqrt[n]{r} e^{\frac{i\theta}{n}}$.

(b) The *set of $n^{th}$ roots* is denoted $z^{\frac{1}{n}} := \{c_0, \ldots, c_{n-1}\}$.

Denote by $\omega_n = e^{\frac{2\pi i}{n}}$ a *primitive $n^{th}$ root of unity* ($n^{\text{th}}$ root of 1). Then the full set of $n^{th}$ *roots of unity* is

$$1^{\frac{1}{n}} = \{\omega_n^k : k = 0, \ldots, n-1\} = \{e^{\frac{2\pi ki}{n}} : k = 0, \ldots, n-1\}$$

The $n^{\text{th}}$ roots of $z$ may be written in terms of the principal root and the $n^{\text{th}}$ roots of unity

$$z^{\frac{1}{n}} = \sqrt[n]{z}\, 1^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$$

By the exponential laws (Lemma 1.16), multiplication by $\omega_n^k = e^{\frac{2\pi ki}{n}}$ has the geometric effect of rotating counter-clockwise by $\frac{2\pi k}{n}$ radians:

$$\arg \sqrt[n]{z}\omega_n^k = \arg \sqrt[n]{z} + \arg \omega_n^k = \arg \sqrt[n]{z} + \frac{2\pi k}{n}$$

It follows that the $n^{\text{th}}$ roots of $z = re^{i\theta}$ form the vertices of a regular $n$-gon spaced equally round the circle of radius $\sqrt[n]{r}$. Compare this with the previous example.

**Examples 1.21.** 1. As a sanity check, we compare what happens when $n = 4$ and $z = r = 16$ is *real* and *positive.*

- The principal fourth root $\sqrt[4]{16} = 2$ is the usual positive real fourth root.
- Within the real numbers, we have *two* fourth roots: $16^{\frac{1}{4}} = \pm 2$.
- Within $\mathbb{C}$, there are *four* fourth roots: $16^{\frac{1}{4}} = \{2, 2i, -2, -2i\}$ where $i = \omega_4 = e^{\frac{i\pi}{2}}$ is a primitive fourth root of unity.

2. We compute the fourth roots of $z = 8\sqrt{2}(1+i)$.

First we write in polar form: $z = 16e^{\frac{\pi i}{4}}$. Since $\operatorname{Arg} z = \frac{\pi}{4}$, the principal fourth root is

$$\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$$

To find all fourth roots, simply multiply by the fourth roots of unity $1^{\frac{1}{4}} = \{1, i, -1, -i\}$:

$$\left(8\sqrt{2}(1+i)\right)^{\frac{1}{4}} = \{\pm 2e^{\frac{\pi i}{16}}, \pm 2ie^{\frac{\pi i}{16}}\} = \{2e^{\frac{\pi i}{16}}, 2e^{\frac{9\pi i}{16}}, 2e^{-\frac{15\pi i}{16}}, 2e^{-\frac{7\pi i}{16}}\}$$

Evaluating these in rectangular form is messy but doable (see Exercise 7). In practice it is better to leave such expressions in polar form.

The language of $n^{\text{th}}$ roots is likely less familiar, and will take more getting used to, than the material in the previous sections. With a little practice it is very easy, but fluency will take some investment of time.

**Exercises 1.3.** 1. Find the sixth roots of $i$ in polar co-ordinates. Which is the principal root?

2. Find the square roots of $-\sqrt{3}+i$ and express them in rectangular co-ordinates.
   (*Hint: you may find the fact that* $(\sqrt{3}-1)^2 = 4-2\sqrt{3}$ *useful*)

3. Use the fact that the cube roots of unity are $1, \omega_3 = \frac{-1+\sqrt{3}i}{2}$ and $\omega_3^2 = \frac{-1-\sqrt{3}i}{2}$ to evaluate the cube roots of $-27$ in rectangular co-ordinates.

4. We previously found the fourth roots of 16. Use these to find the fourth roots of $-16$. Hence factorize the equation $z^4+16=0$ as a product of two quadratic equations with real coefficients.

5. If $\omega$ is an $n^{\text{th}}$ root of unity *other than 1*, prove that $\sum\limits_{k=0}^{n-1} \omega^k = 0$.
   (*Hint: recall the geometric series formula*)

6. (a) Suppose that $a,b,c \in \mathbb{C}$ with $a \neq 0$ and suppose that $z$ satisfies the quadratic equation $az^2+bz+c=0$. Prove the quadratic formula:

   $$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$

   Note that $(b^2-4ac)^{1/2}$ is the *set* of square roots of $b^2-4ac$, so that this provides *two* solutions whenever $b^2-4ac \neq 0$.

   (b) Find the roots of the equation $iz^2+(1+i)z+3=0$ in rectangular form.

7. Use the half-angle formula $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ to explicitly evaluate $\cos\frac{\pi}{8}$ and then $\cos\frac{\pi}{16}$. Hence find an expression for the rectangular form of $\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$ using only square roots.

8. Recall Example 1.18. Verify that the method in Definition 1.20 gives the same value for the principal square root $\sqrt{-5+12i}$.
   (*You'll need some trig identities...*)

# 2 Holomorphic Functions

In this chapter we discuss functions of a complex variable and what it means for such to be *differentiable.* This turns out to be more subtle and restrictive than in real analysis.

## 2.1 Functions of a Complex Variable

As in real analysis, it is common to express a function via a rule/formula.

**Example 2.1.** Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^3 - z$. Evaluating is straightforward: e.g.

$$f(2+i) = (2+i)^3 - (2+i) = 2^3 + 3 \cdot 2^2 i + 3 \cdot 2i^2 + i^3 - 2 - i = 10i$$

The *implied domain* of a rule is the largest possible set $D \subseteq \mathbb{C}$ on which the rule is defined.

**Example 2.2.** $f(z) = \frac{1}{z^2+9}$ has implied domain $D = \mathbb{C} \setminus \{\pm 3i\}$. Complex functions can be represented in *polar form* by substituting $z = re^{i\theta}$:

$$f(z) = \frac{1}{r^2 e^{2i\theta} + 9}$$

It is also common to separate the *real and imaginary parts* by writing $f(z) = u(x,y) + iv(x,y)$ where $u, v : D \to \mathbb{R}$. In this case,

$$f(z) = \frac{1}{(x+iy)^2+9} = \frac{1}{x^2 - y^2 + 9 + 2ixy} = \frac{x^2 - y^2 + 9}{(x^2-y^2+9)^2 + 4x^2y^2} + i\frac{-2xy}{(x^2-y^2+9)^2 + 4x^2y^2}$$

These approaches may be combined by writing $u, v$ as functions of $r, \theta$.

The major initial challenge presented by functions of a complex variable is that of *visualization*. Since $\mathbb{C}$ has two real dimensions, graphing a function $f : \mathbb{C} \to \mathbb{C}$ requires *four real dimensions*! Since we cannot see four dimensions at once, any visualization we obtain will only be partial. To see this at work, we consider a simple function in some detail.

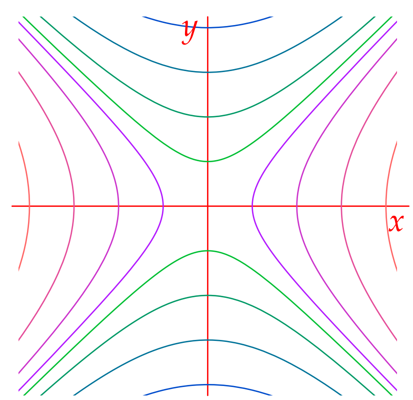
**Example 2.3.** To help understand $f(z) = z^2$, we start by computing the various forms described above:

$$\begin{aligned} f(z) &= z^2 = x^2 - y^2 + 2ixy \\ &= r^2 e^{2i\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta \end{aligned}$$

While we cannot graph the entire function (don't even think about a parabola!), we can visualize its real and imaginary parts

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy$$

as graphs of functions $\mathbb{R}^2 \to \mathbb{R}$. Both are *saddle surfaces*, which may be analyzed using the standard tools of multivariable calculus. For instance, the *level curves* $u =$ constant and $v =$ constant are *hyperbolæ.* You might find this easier using the polar forms of $u, v$.

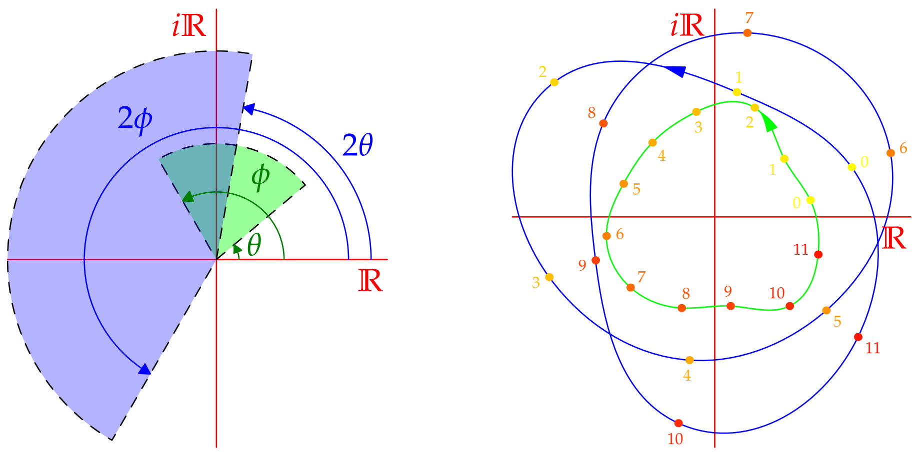


Level curves of $u = x^2 - y^2$

The polar form allows us to think about the what $f$ does to the argument:

$$f(z) = r^2 e^{2i\theta} \implies \arg f(z) = 2 \arg z$$

Given a sector between arguments $\theta$ and $\phi$, the function *doubles* this to the sector between $2\theta$ and $2\phi$. This can also be visualized pathwise. If $z$ traces a path which once encircles the origin, then the path traced by $z^2$ orbits the origin *twice*. The colored dots on the two paths correspond under $z \mapsto z^2$.




Similar behavior occurs with higher powers. For instance, $z \mapsto z^3$ maps a single loop round the origin to a *triple* loop. By contrast, the principal square root function $\sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ *halves* the principal argument: $\operatorname{Arg} \sqrt{z} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. In the fractional-exponent language of Section 1.3,

$$z \mapsto z^{1/2} = \{\sqrt{r}e^{\frac{i\theta}{2}}, -\sqrt{r}e^{\frac{i\theta}{2}}\}$$

This is sometimes called a *two-valued* function—we'll return to such objects in Chapter 3.

**2D geometry with complex numbers**

Complex functions easily describe the basic geometric transformations of the plane.

**Translation** If $w$ is constant, the function $f(z) = z + w$ translates the plane, shifting the origin to $w$.

**Scaling** If $R > 0$ is a real constant, the function $f(z) = Rz$ scales/stretches the complex plane relative to the origin.

**Rotation** Given $\phi \in \mathbb{R}$, the function $f(z) = e^{i\phi}z$ rotates the plane by $\phi$ radians counter-clockwise around the origin. This is easy to see in polar form:

$$f(z) = f(re^{i\theta}) = e^{i\phi}re^{i\theta} = re^{i(\theta+\phi)}$$

**Reflection** Complex conjugation $f(z) = \overline{z} = x - iy$ reflects the plane in the horizontal axis.

By composing such functions, we can describe arbitrary rotations, reflections and scalings relative to any points/lines.

**Examples 2.4.** 1. We construct rotation by $\frac{\pi}{3} = 60°$ counter-clockwise about $2i$ in three stages:

(a) Translate $2i$ to the origin: $z \mapsto z - 2i$

(b) Rotate by $\frac{\pi}{3}$ about the origin: $z \mapsto e^{\frac{i\pi}{3}}(z - 2i)$

(c) Translate the origin back to $2i$: $z \mapsto f(z) = e^{\frac{i\pi}{3}}(z - 2i) + 2i$

2. We may similarly produce any reflection. To reflect in the line through 0 and $w$ with $\operatorname{Arg} w = \phi$;

(a) Rotate the plane by $-\phi$: $z \mapsto e^{-i\phi}z$

(b) Reflect in the real axis: $z \mapsto \overline{e^{-i\phi}z} = e^{i\phi}\overline{z}$

(c) Rotate the plane back by $\phi$: $z \mapsto f(z) = e^{2i\phi}\overline{z}$

3. We compute the function that reflects across the line joining $\alpha = 2 + i$ and $\beta = 4 + 3i$.

Since $\beta - \alpha = 2 + 2i$ has argument $\phi = \frac{\pi}{4}$, we combine reflection across the line making $\phi$ through the origin with translation by $\alpha$ (translate by $-\alpha$, reflect, translate back by $\alpha$):

$$\begin{aligned} f(z) &= e^{2i\phi}(\overline{z - \alpha}) + \alpha = e^{\frac{i\pi}{2}}(\overline{z - 2 - i}) + 2 + i = i(\overline{z} - 2 + i) + 2 + i \\ &= i\overline{z} + 1 - i \end{aligned}$$

As a sanity check, you should verify explicitly that $f(\alpha) = \alpha$ and $f(\beta) = \beta$: why?

**Exercises 2.1.** 1. For each function, describe its implied domain (page 11).

(a) $f(z) = \dfrac{1}{4 + z^2}$ (b) $f(z) = \dfrac{z - 1}{e^z - 1}$ (c) $f(z) = \dfrac{z^2 + z + 1}{z^4 - 1}$

2. Write the function in terms of its real and imaginary parts: $f(z) = u(x, y) + iv(x, y)$.

(a) $f(z) = z^3 - 4z^2 + 2$ (b) $f(z) = \dfrac{z^2}{1 - \overline{z}}$ (c) $f(z) = e^{\overline{z}}$

3. Write the function $f(z) = \dfrac{1}{|z|^2}\overline{z}$ in polar form.

4. Find an expression for the function which reflects across the vertical line through $\alpha = -1$.

5. In Example 2.4.3, evaluate the function $g(z) = e^{2i\phi}(\overline{z - \beta}) + \beta$. Why are you not surprised by the result?

6. Let $\phi = \tan^{-1}\frac{3}{4}$. Find the result (in rectangular co-ordinates) of rotating $z = -2 + i$ counter-clockwise by $\phi$ radians around the origin.

(*Hint: consider a 3:4:5 triangle!*)

7. Prove, using the expressions on page 12, that the composition of two reflections is a rotation, and that the composition of a rotation and a reflection is a reflection.

## 2.2 Open sets, Limits and Continuity

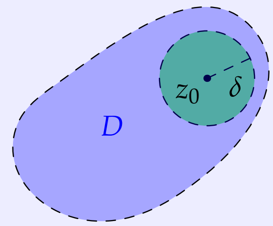
This section is mostly for reference, since limits and continuity behave similarly to real analysis. Indeed proofs are often identical, so we mostly omit them. If you're only familiar with *single-variable* real analysis, note particularly how the first definition generalizes the notion of an *open interval*.

**Definition 2.5 (Disks and Neighborhoods).** Given $\delta > 0$ and $z_0 \in \mathbb{C}$, the *open disk* (or *$\delta$-ball*) centered at $z_0 \in \mathbb{C}$ with radius $\delta$ is the set

$$B_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

Remove the central point $z_0$ for the corresponding *punctured open disk*:

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$$



An open neighborhood of $z_0$

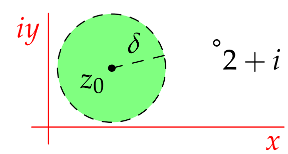
Now let $D \subseteq \mathbb{C}$ be a subset.

- $D$ is *open* if every point is *interior*: at every $z_0 \in D$ we may center some open disk $B_\delta(z_0) \subseteq D$:

$$\forall z_0 \in D,\ \exists \delta > 0 \text{ such that } |z - z_0| < \delta \implies z \in D$$

- $D$ is a *neighborhood* of $z_0$ if it contains some $B_\delta(z_0)$. A neighborhood can, but need not, be open. A *punctured neighborhood* instead contains a punctured disk.
- $D$ is *closed* if its complement $\mathbb{C} \setminus D$ is open.

**Example 2.6.** The punctured plane $D = \mathbb{C} \setminus \{2 + i\}$ is an open set. For instance, given $z_0 \in D$, let $\delta = \frac{1}{2}|z_0 - 2 - i|$, then $B_\delta(z_0) \subset D$.
(The $\frac{1}{2}$ is superfluous, since the disk $B_\delta(z_0)$ does not contain its boundary, it just makes for a clearer picture.)




**Definition 2.7 (Sequences and Limits).** Let $(z_n) = (z_1, z_2, \ldots)$ be a *sequence* of complex numbers.

1. $(z_n)$ *converges* to $z_0 \in \mathbb{C}$, and we write $\lim_{n\to\infty} z_n = z_0$ or simply $z_n \to z_0$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z_0| < \epsilon$$

2. $(z_n)$ is a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |z_m - z_n| < \epsilon$$

**Theorem 2.8 (Useful Facts about Sequences).** 1. *$(z_n)$ is convergent/Cauchy if and only if its real and imaginary parts are also, as follows from two straightforward inequalities:*

$$|z_n - z_0| \le |x_n - x_0| + |y_n - y_0| \le 2\,|z_n - z_0|$$

2. *Cauchy completeness: $(z_n)$ converges in $\mathbb{C}$ if and only if it is Cauchy*
3. *Bolzano–Weierstraß: If $(z_n)$ is bounded, then it has a convergent subsequence.*
4. *$D \subseteq \mathbb{C}$ is closed if and only if every Cauchy sequence $(z_n) \subseteq D$ has its limit in $D$.*

**Definition 2.9 (Continuity).** Let $f : D \to \mathbb{C}$ and $z_0 \in D$. We say that $f$ is *continuous at* $z_0$ if

For all sequences $(z_n) \subseteq D$ with $\lim z_n = z_0$ we have $\lim f(z_n) = f(z_0)$

$f$ is *continuous* (on $D$) if it is continuous at all points $z_0 \in D$.

Rather than using sequences, we'll typically consider continuity via limits of functions.

**Definition 2.10 (Limits of Functions).** Let $f : D \to \mathbb{C}$, where $D$ contains an open punctured neighborhood of $z_0$. We say that $w_0$ is the *limit of $f$ as $z$ approaches $z_0$*, written $\lim_{z \to z_0} f(z) = w_0$, if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

Otherwise said, given an $\epsilon$-ball centered at $w_0$, there is a $\delta$-ball $B_\delta(z_0)$ such that $f\big(B_\delta(z_0)\big) \subseteq B_\epsilon(w_0)$. Equivalently, $\forall (z_n) \subseteq D \setminus \{z_0\}$, $\lim z_n = z_0 \implies \lim f(z_n) = w_0$. This is almost continuity!

**Theorem 2.11.** *Let $z_0$ be an* interior *point of $D$. Then $f : D \to \mathbb{C}$ is continuous at $z_0$ if and only if*

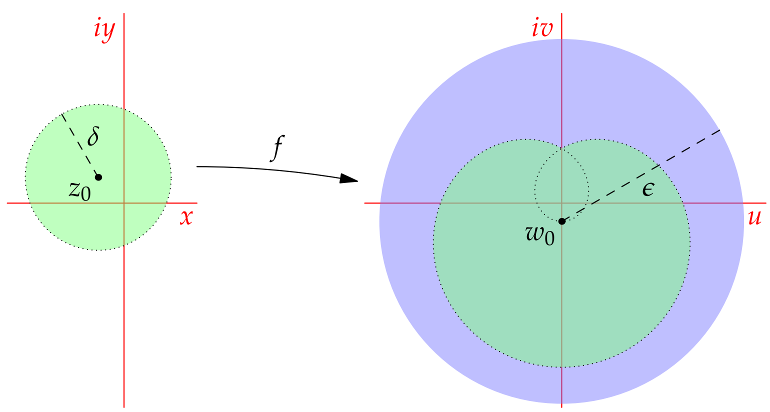
$$\lim_{z \to z_0} f(z) = f(z_0)$$

**Example 2.12.** Calculations tend to proceed similarly to real analysis. For instance, we show that $f(z) = z^2$ is continuous (on $\mathbb{C}$) by proving that $\lim_{z \to z_0} z^2 = z_0^2$ for all $z_0$.

Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$ be given. Define $\delta = \min\{1, \frac{\epsilon}{1+2|z_0|}\}$. By the triangle-inequality,

$$\begin{aligned} |z - z_0| < \delta &\implies |z + z_0| = |z - z_0 + 2z_0| \le |z - z_0| + 2|z_0| < \delta + 2|z_0| \le 1 + 2|z_0| \\ &\implies \left|z^2 - z_0^2\right| = |z - z_0|\,|z + z_0| < \delta(1 + 2|z_0|) \le \epsilon \end{aligned}$$

The picture should help: given an $\epsilon$-ball centered at $w_0 = z_0^2$, we've described how to choose $\delta > 0$ so that $f$ maps the $\delta$-ball centered at $z_0$ to a region *inside* the original $\epsilon$-ball.




The picture illustrates the case when

$$z_0 = \frac{1}{2} e^{\frac{3\pi i}{4}} = \frac{1}{2\sqrt{2}}(-1 + i), \quad w_0 = -\frac{i}{4}, \quad \epsilon = \frac{5}{2} \quad \text{and} \quad \delta = \min\left\{1, \frac{5/2}{1 + 2 \cdot \frac{1}{2}}\right\} = 1$$

**Theorem 2.13 (Basic Facts about Limits of Functions).** *Suppose $f, g : D \to \mathbb{C}$ are functions and $z_0 = x_0 + iy_0$ is a point satisfying the assumptions of Definition 2.10.*

1. *Limits are unique: If $w_0$ and $\widetilde{w_0}$ satisfy Definition 2.10, then $\widetilde{w_0} = w_0$.*

2. *If $f(z) = u(x,y) + iv(x,y)$ and $w_0 = u_0 + iv_0$, then*

$$\lim_{z\to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

   *In particular $\lim_{z\to z_0} \overline{z} = \overline{z_0}$*

3. *Suppose $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$:*

   (a) *For any $a, b \in \mathbb{C}$, $\lim_{z\to z_0} \big(af(z) + bg(z)\big) = aw_0 + bw_1$*

   (b) $\lim_{z\to z_0} \big(f(z)g(z)\big) = w_0w_1$

   (c) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$*, provided $w_1 \neq 0$ and $g(z) \neq 0$ on a punctured neighbourhood of $z_0$.*

   (d) *If $h$ is a function such that $\lim_{w\to w_0} h(w) = w_2$, then $\lim_{z\to z_0} h(f(z)) = w_2$*

The basic limit laws should feel familiar and intuitive; indeed analogues of most should have been proved in real analysis. Parts 2 & 3 have obvious corollaries for continuous functions. For instance:

**Corollary 2.14.** *$f : D \subseteq \mathbb{C} \to \mathbb{C}$ is continuous if and only if its real and imaginary parts $u, v : D \to \mathbb{R}$ are also continuous.*

Several pieces of part 3 follow from part 2 by considering real and imaginary parts and the limit laws for functions $\mathbb{R}^2 \to \mathbb{R}$. For instance, write $f(z) = u_1 + iv_1$ and $g(z) = u_2 + iv_2$, then

$$\lim_{z\to z_0} \big(f(z)g(z)\big) = \lim_{z\to z_0} \big(u_1u_2 - v_1v_2 + i(u_1v_2 + v_1u_2)\big) = w_0w_1$$

**Examples 2.15.** 1. $\lim_{z\to 1+3i} (z^2 - i\overline{z}) = (1+3i)^2 - i\big(\overline{1+3i}\big) = 1 + 6i - 9 - i(1-3i) = -11 + 5i.$

2. Every polynomial function is continuous on $\mathbb{C}$.

3. Every rational function $f(z) = \frac{p(z)}{q(z)}$ (where $p, q$ are polynomials), is continuous on its implied domain $D = \{z : q(z) \neq 0\}$.

4. The exponential function

$$\exp(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

   is continuous on $\mathbb{C}$, since the exponential, cosine and sine are continuous on $\mathbb{R}$.

### Limits and the Point at Infinity

In (single-variable) real analysis there are *two* infinities ($\pm\infty$). In complex analysis, the convention is to have only one: for instance, the sequences $z_n = n$ and $w_n = in$ both diverge to the *same* infinity.

**Definition 2.16.** The *extended complex plane* (*Riemann sphere*) is the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where $\infty$ denotes the *point at infinity*. The concepts of *neighborhood*, *openness* and *limit* extend naturally:

1. A *neighborhood* of $\infty$ is any set containing an *open disk* at $\infty$, namely a subset of the form
$$\{\infty\} \cup \{z \in \mathbb{C} : |z| > M\}$$
2. $\lim_{z \to z_0} f(z) = \infty$ means: $\forall M > 0, \exists \delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z)| > M$
3. $\lim_{z \to \infty} f(z) = w_0$ means: $\forall \epsilon > 0, \exists N > 0$ such that $|z| > N \implies |f(z) - w_0| < \epsilon$
4. $\lim_{z \to \infty} f(z) = \infty$ means: $\forall M > 0, \exists N > 0$ such that $|z| > N \implies |f(z)| > M$

These limits have the same interpretation as Definition 2.10: '$z$ close to $z_0$' implies '$f(z)$ close to $w_0$.' The difficulty is merely about the meaning of 'close to $\infty$' when $z_0$ or $w_0$ is the point at infinity.

**Example 2.17.** We verify that $\lim_{z \to -3i} \frac{z^2}{z+3i} = \infty$. Let $M > 0$ be given and define $\delta = \min\{1, \frac{4}{M}\}$. Then

$$0 < |z + 3i| < \delta \implies |z| \overset{\triangle}{\geq} |3i| - |z + 3i| > 3 - \delta \geq 2 \implies \left| \frac{z^2}{z+3i} \right| > \frac{4}{\delta} \geq M$$

The relationship between limits, zero and infinity feel like the dubious claims $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$!

**Theorem 2.18.** *Provided all limits make sense, Theorem 2.13 also applies to limits involving infinity. Moreover, we have the additional relationships:*

1. $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{\zeta \to 0} f\left(\frac{1}{\zeta}\right) = w_0$
3. $\lim_{z \to \infty} f(z) = \infty \iff \lim_{\zeta \to 0} \frac{1}{f(1/\zeta)} = 0$

*Sketch Proof.* All six results are similar; we prove only the $\Rightarrow$ direction of the second. Suppose $\epsilon > 0$ is given. Then $N$ exists according to Definition 2.16 (part 3). Let $\delta = \frac{1}{N}$ and $\zeta = \frac{1}{z}$, then

$$0 < |\zeta| < \delta \implies |z| = \frac{1}{|\zeta|} > N \implies \left| f(\tfrac{1}{\zeta}) - w_0 \right| = |f(z) - w_0| < \epsilon$$

∎

**Example 2.19.** Consider $f(z) = \frac{5iz+1}{3z-2i}$. Plainly

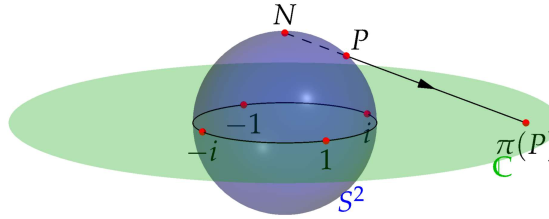
$$\lim_{z \to \frac{2}{3}i} \frac{1}{f(z)} = \lim_{z \to \frac{2}{3}i} \frac{3z - 2i}{5iz + 1} = 0 \implies \lim_{z \to \frac{2}{3}i} f(z) = \infty, \qquad \lim_{z \to \infty} f(z) = \frac{5i + \lim \frac{1}{z}}{3 - 2i \lim \frac{1}{z}} = \frac{5}{3}i$$

Because of this, it is common to view $f$ as a continuous bijection $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (exercise):

$$f(z) = \frac{5iz + 1}{3z - 2i} \quad (\text{if } z \neq \tfrac{2}{3}i, \infty), \qquad f\left(\frac{2}{3}i\right) = \infty, \qquad f(\infty) = \frac{5}{3}i$$

**Aside (non-examinable)** The Riemann sphere is merely a fun diversion for us. Unless indicated otherwise, all sets should be assumed to be subsets of the (*finite*) complex plane.

The Riemann sphere can be visualized as a sphere $S^2$, with $\infty$ playing the role of the north pole. The rest of the sphere is identified bijectively with the equatorial plane $\mathbb{C}$ via *stereographic projection* $\pi : S^2 \to \overline{\mathbb{C}}$. In the picture, the image of $P \in S^2$ is the intersection $\pi(P)$ of $\mathbb{C}$ with the line through $P$ and $N = \pi^{-1}(\infty)$.



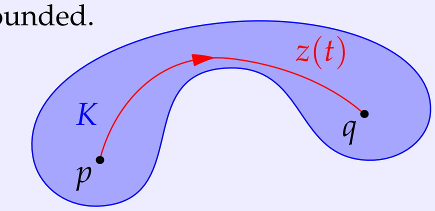

**Compactness, Path-Connectedness & Continuity**

We finish with two further desirable properties of domains.

**Definition 2.20.** 1. A subset $K \subseteq \mathbb{C}$ is *compact* if it is closed and bounded.

2. $K \subseteq \mathbb{C}$ is *(path-)connected*[5] if any two points can be joined by a path lying within $K$: more precisely, $\forall p, q \in K$,

$$\exists z : [0,1] \to K \text{ continuous with } z(0) = p \text{ and } z(1) = q$$




These concepts generalize notions of intervals from single-variable real analysis. In view of this, we translate two familiar results, essentially the *extreme* and *intermediate value theorems* of real analysis.

**Theorem 2.21.** *Suppose $f : K \to \mathbb{C}$ is continuous.*

1. *If $K$ is compact, so is the image $f(K)$.*
2. *If $K$ is path-connected, so is $f(K)$.*

*Proof.* 1. Consider the *real-valued* function $|f|$ and apply the same argument from real analysis:

- Let $M = \sup\{|f(z)| : z \in K\}$. Then $\exists (z_n) \subseteq K$ such that $\lim |f(z_n)| = M$.
- $K$ is bounded; Bolzano–Weierstraß says $(z_n)$ has a convergent subsequence $(z_{n_k})$.
- Let $z_0 = \lim z_{n_k}$. Since $K$ is closed we have $z_0 \in K$ (Theorem 2.8.4), whence $f(z_0)$ is defined.
- $f$ continuous $\implies \lim f(z_{n_k}) = f(z_0)$; thus $M = f(z_0)$ is finite and $f(K)$ is bounded.
- The closure of $f(K)$ is a short exercise.

2. This is also an exercise. ∎

The upshot of this section is that continuity and limits are essentially the same in both real and complex analysis and behave as you should expect. Differentiation, however, is another beast entirely…

[5]Loosely speaking, a connected set consists of one 'lump.' If you've studied topology: almost all domains in these notes will be open, so there is no benefit to distinguishing between connectedness and path-connectedness.

**Exercises 2.2.** 1. As in Example 2.6, prove that the doubly-punctured plane $D := \mathbb{C} \setminus \{1, -2i\}$ is an open set. State a function whose implied domain is this set.

2. Use the $\epsilon$–$\delta$ definition (2.10) to prove the following.

   (a) $\lim\limits_{z\to z_0} \overline{z} = \overline{z_0}$ (b) $\lim\limits_{z\to 0} \dfrac{\overline{z}^2}{z} = 0$ (c) $\lim\limits_{z\to 2} \dfrac{1}{z-i} = \dfrac{1}{2-i}$ (d) $\lim\limits_{z\to z_0} z^3 = z_0^3$

3. Show that $f(z) = (z/\overline{z})^2$ equals 1 at all non-zero points on the real and imaginary axes, and $-1$ at all non-zero points on the line $y = x$. Explain why $\lim\limits_{z\to 0} f(z)$ doesn't exist.

4. Prove part 3(c) of Theorem 2.13.

5. Suppose $\lim\limits_{z\to z_0} f(z) = w_0$. Prove that $\lim\limits_{z\to z_0} |f(z)| = |w_0|$.

6. Use Definition 2.16 to prove part of Theorem 2.18: $\lim\limits_{z\to z_0} f(z) = \infty \implies \lim\limits_{z\to z_0} \frac{1}{f(z)} = 0$.

7. Use Definition 2.16 to prove: (a) $\lim\limits_{z\to 2i} \frac{iz-1}{z-2i} = \infty$, (b) $\lim\limits_{z\to\infty} \frac{iz-1}{z-2i} = i$

8. (a) Show that $f(z) = \frac{5iz+1}{3z-2i}$ defines a bijection of the Riemann sphere $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.
   (*Hint: let $w = f(z)$ and solve for $z\ldots$*)

   (b) In general: Given $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, prove that $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ defines a bijection of the Riemann sphere if and only if $\alpha\delta - \beta\gamma \neq 0$. How does this relate to the matrix $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$?

9. We complete the proof of Theorem 2.21. Suppose $f : K \subseteq \mathbb{C} \to \mathbb{C}$ is continuous.

   (a) Let $K$ be compact. Suppose $(w_n) \subseteq K$ is a sequence where $\big(f(w_n)\big)$ is convergent in $\mathbb{C}$. Explain why there exists a convergent subsequence $(w_{n_k})$, and use it to show that $\lim f(w_n) \in f(K)$. Hence conclude that $f(K)$ is closed.

   (b) Suppose $K$ is path-connected. If $f(p), f(q) \in f(K)$, show that $\exists w : [0,1] \to f(K)$ continuous such that $w(0) = f(p)$ and $w(1) = f(q)$. Hence conclude that $f(K)$ is path-connected.

10. (Hard) "Every open cover has a finite subcover," is a crucial result in topology:[6]

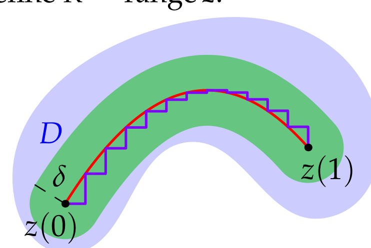
    **Theorem** Suppose a compact $K$ is a subset of a (possibly infinite) union $\bigcup U_j$ of open sets. Then there are *finitely* many $U_j$ (labelled WLOG) such that $K \subset U_1 \cup \cdots \cup U_n$.

    Suppose $z : [0,1] \to D \subseteq \mathbb{C}$ is a path in an open domain $D$ and define $K = \operatorname{range} z$.

    (a) Explain why $K$ is compact.

    (b) Prove that $K$ may be covered by *finitely* many *closed* balls $\overline{B_k}$ for which $K \subset B_0 \cup \cdots \cup B_n \subset \overline{B_0} \cup \cdots \cup \overline{B_n} \subset D$.
    (*Hint: start by centering a ball at **every** point of $K$*)

    (c) Prove that $D$ contains a zig-zag path (finitely many horizontal/vertical segments) from $z(0)$ to $z(1)$.




    (d) Show that we can fit a 'tube' around $K$: $\exists\delta > 0$ such that $\forall z \in K$, $|z - w| \le \delta \implies w \in D$.
    (*Hint: following part (b), let $V = \bigcup \overline{B_k} \setminus \bigcup B_k$ and define $\delta := \inf\{|z - v| : z \in K, v \in V\}$*)

[6]This is often taken as the definition of compactness in topology. Its equivalence to $K$ being closed and bounded in $\mathbb{C}$ (or any Euclidean space) is the famous Heine–Borel Theorem.

## 2.3 Derivatives & the Cauchy–Riemann Equations

Differentiation is where complex analysis starts to produce interesting and novel results. It won't appear so initially, since the definition of derivative is exactly as you've previously encountered.

**Definition 2.22.** Let $f : D \to \mathbb{C}$ be a complex function and $z_0$ an *interior* point of $D$. We say that $f$ is *differentiable at* $z_0$ if the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We call this limit the *derivative* of $f$ at $z_0$ and denote it $f'(z_0)$. If $f$ is differentiable everywhere[7] on $D$ then the derivative is itself a function, written $f'(z)$ or $\frac{\mathrm{d}f}{\mathrm{d}z}$.

**Example 2.23.** The function $f(z) = z^2$ is everywhere differentiable,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0$$

The second last equality follows since we only compute on the punctured disk $0 < |z - z_0| < \delta$ when taking limits (Definition 2.16).

The basic rules of differentiation are identical those in real analysis and can be proved similarly.

**Theorem 2.24.** *Suppose $f$ and $g$ are differentiable (either at a point $z_0$ or as functions).*

1. *(Linearity)* *For any constants $a, b \in \mathbb{C}$,* $\frac{\mathrm{d}}{\mathrm{d}z}\big(af(z) + bg(z)\big) = af'(z) + bg'(z)$
2. *(Power Law)* *For any $n \in \mathbb{N}_0$,* $\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}$
3. *(Product rule)* $\frac{\mathrm{d}}{\mathrm{d}z}\big(f(z)g(z)\big) = f'(z)g(z) + f(z)g'(z)$
4. *(Quotient Rule)* *If $g(z) \neq 0$, then* $\frac{\mathrm{d}}{\mathrm{d}z} \frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$
5. *(Chain Rule)* *If $h$ is differentiable at $g(z_0)$, then $h \circ g$ is differentiable at $z_0$ and*

$$(h \circ g)'(z_0) = h'\big(g(z_0)\big)g'(z_0)$$

We immediately see that all polynomials and rational functions are differentiable. Familiar examples behave the same regardless of whether we are in $\mathbb{R}$ or $\mathbb{C}$!

**Example 2.25.** $\frac{\mathrm{d}}{\mathrm{d}z} \frac{3(z^2 - 2)^5 + z^2}{z^3 + 1} = \frac{[30z(z^2 - 2)^4 + 2z)](z^3 + 1) - 3z^2[3(z^2 - 2)^5 + z^2]}{(z^3 + 1)^2}$

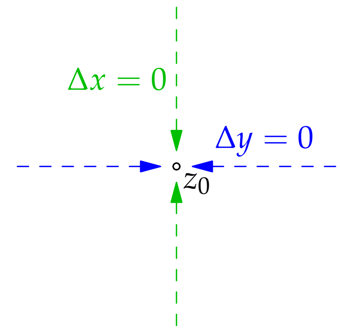
[7] Such *holomorphic* functions are the main topic of the course: we'll consider them more properly in the next section. Necessarily, $D$ must be an open set if $f$ is to be holomorphic (think about the definition of the limit!).

### The Cauchy–Riemann Equations

Differentiation thus far seems uncontroversial; here is an example that might change your mind...

**Example 2.26.** Consider $f(z) = \overline{z}$ at a generic point $z_0 = x_0 + iy_0$, and the difference quotient

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{x-iy-x_0+iy_0}{x+iy-x_0-iy_0} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \qquad \text{(where } \Delta x = x - x_0 \text{ and } \Delta y = y - y_0\text{)}$$




For $f'(z)$ to exist, we must obtain the same limit *regardless of how* $(\Delta x, \Delta y) \to (0,0)$ (equivalent to $z \to z_0$). There are two obvious ways to take the limit:

*Horizontally* $\quad \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{\Delta x}{\Delta x} = 1$

*Vertically* $\quad \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{-i\Delta y}{i\Delta y} = -1$

The quotient plainly takes different values depending on how $(\Delta x, \Delta y) \to (0,0)$. We conclude that $f'(z_0)$ *does not exist*, and that the function $f(z) = \overline{z}$ is *nowhere differentiable*!

The approach extends to general functions. We attempt to differentiate $f(z) = u(x,y) + iv(x,y)$ (written in real and imaginary parts) at a generic point $z_0 = x_0 + iy_0$:

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x,y)-u(x_0,y_0)+i\big(v(x,y)-v(x_0,y_0)\big)}{\Delta x + i\Delta y}$$

*If this limit exists*, the same result must be obtained when evaluating along paths approaching $z_0$ horizontally and vertically. This results in a key relationship between the partial derivatives of $u, v$.

*Horizontally* We have $\Delta y = 0$ ($y = y_0$), so the limit becomes

$$\lim_{\Delta x\to 0} \frac{u(x,y_0)-u(x_0,y_0)+i\big(v(x,y_0)-v(x_0,y_0)\big)}{\Delta x} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x_0,y_0)$$

*Vertically* We have $\Delta x = 0$ ($x = x_0$), from which

$$\lim_{\Delta y\to 0} \frac{u(x_0,y)-u(x_0,y_0)+i\big(v(x_0,y)-v(x_0,y_0)\big)}{i\Delta y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)(x_0,y_0)$$

**Theorem 2.27 (Cauchy–Riemann equations).** *Write $f(z) = u + iv$ in its real and imaginary parts.*

1. *If $f$ is complex-differentiable at $z_0$, then $u, v$ satisfy the* Cauchy–Riemann equations *at $z_0$:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(equivalently } u_x = v_y \text{ and } u_y = -v_x\text{)}$$

2. *Conversely, if $u, v$ have partial derivatives on a neighborhood of $z_0$, which are continuous and satisfy the Cauchy–Riemann equations at $z_0$, then $f$ is complex-differentiable at $z_0$.*

*In either situation, we may write $f'(z) = u_x + iv_x = v_y - iu_y$.*

We proved part 1 above. For a sketch of the almost-converse (part 2), see Exercise 11.

**Examples 2.28.** 1. (Example 2.26) $f(z) = \overline{z} = x - iy$ has $u(x,y) = x$ and $v(x,y) = -y$, whence

$$u_x = 1 \neq -1 = v_y \quad \text{and} \quad u_y = 0 = v_x$$

Since $u, v$ do not satisfy the Cauchy–Riemann equations, $f$ fails to be differentiable anywhere.

2. $f(z) = |z| = \sqrt{x^2+y^2}$ has $u = \sqrt{x^2+y^2}$ and $v = 0$. Away from $z_0 = 0$, the Cauchy–Riemann equations are

$$u_x = \frac{x}{\sqrt{x^2+y^2}} = 0 = v_y, \quad u_y = \frac{y}{\sqrt{x^2+y^2}} = 0 = -v_x$$

These equations are satisfied nowhere ($u, v$ are not differentiable at $z_0 = 0$), whence $f(z)$ is nowhere differentiable.

3. $f(z) = z^2 = x^2 - y^2 + 2ixy$ has $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We check

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x$$

As expected, $u, v$ satisfy the Cauchy–Riemann equations. Moreover,

$$f'(z) = 2z = 2x + 2iy = u_x + iv_x = v_y - iu_y$$

4. Consider $f(z) = \dfrac{z}{2+|z|^2} = \dfrac{x}{2+x^2+y^2} + \dfrac{iy}{2+x^2+y^2}$. We compute

$$u_x = \frac{2-x^2+y^2}{(2+x^2+y^2)^2} \qquad v_y = \frac{2+x^2-y^2}{(2+x^2+y^2)^2}$$
$$u_y = \frac{-2xy}{(2+x^2+y^2)^2} \qquad -v_x = \frac{2xy}{(2+x^2+y^2)^2}$$

The Cauchy–Riemann equations are satisfied if and only if $xy = 0 = x^2 - y^2$, which is if and only if $x = y = 0$. We conclude that $f$ is not differentiable at any non-zero $z \in \mathbb{C}$.

Since the partial derivatives of $u, v$ are continuous, the converse of the Cauchy–Riemann theorem says that $f$ is differentiable at $z = 0$: indeed

$$f'(0) = u_x(0,0) + iv_x(0,0) = \frac{1}{2}$$

We can alternatively check that this straight from the definition of derivative:

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{1}{2+|z|^2} = \frac{1}{2}$$

The function $f$ is therefore differentiable at precisely one point.

5. The complex exponential function is everywhere differentiable: indeed

$$f(z) = e^z = e^x \cos y + ie^x \sin y$$

satisfies

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin x = -v_x$$

where these are certainly continuous on $\mathbb{C}$. As expected,

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$$

**Exercises 2.3.** 1. Use Theorem 2.24 to find the derivatives of the following functions:

(a) $f(z) = \dfrac{1}{z^2+2z}$ (b) $f(z) = (z^3+2iz+1)^7$ (c) $f(z) = \dfrac{(3z^2-i)^3}{(iz^3+4)^2}$

2. Use the limit definition of the derivative to compute the derivative of the functions:

(a) $f(z) = 3z^3 - iz^2$ (b) $f(z) = \dfrac{1}{z^2}$

3. Give a proof of the quotient rule, directly using the definition of the derivative.

4. Use the quotient rule to prove the power law for negative integer exponents:

$$\forall n \in \mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}z} z^{-n} = -nz^{-n-1}$$

5. (*L'Hôpital's rule*) Suppose $f(z_0) = g(z_0) = 0$, and that $f'(z_0)$ and $g'(z_0) \neq 0$ both exist. Use the definition of the derivative to prove that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

6. Prove that $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not differentiable anywhere.

7. By writing $z = re^{i\theta}$ in polar form, verify directly that $f(z) = |z| = \sqrt{x^2+y^2}$ is not differentiable at $z_0 = 0$.

8. What, if anything, do the Cauchy–Riemann equations allow you to conclude for the following?

(a) $f(z) = (z+i)^2$ (b) $f(z) = \dfrac{1}{\overline{z}-i}$ (c) $f(z) = z^3 - \dfrac{2}{z}$ (d) $f(z) = \left(|z|^2 + z\right)^2$

9. As in Example 2.28.5, prove that $\dfrac{\mathrm{d}}{\mathrm{d}z} e^{kz} = ke^{kz}$ for any complex constant $k = a + ib$.

10. Write a complex function $f(z) = f(z, \overline{z})$ as a function of $z$ and $\overline{z}$. For example,

$$f(z) = |z|^2 = z\overline{z}$$

Noting that $x = \frac{1}{2}(z+\overline{z})$ and $y = \frac{1}{2i}(z-\overline{z})$, use the chain rule

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}}$$

to prove that $f$ satisfies the Cauchy–Riemann equations if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$.

Hence give a quick proof that $f(z) = z\overline{z}^2$ is not differentiable when $z \neq 0$.

11. Suppose $f(z) = u + iv$ satisfies the Cauchy–Riemann equations at $z_0 = x_0 + iy_0$. Use the multivariable linear approximation

$$f(z) \approx f(z_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

to prove that $\frac{f(z)-f(z_0)}{z-z_0} \approx u_x(x_0, y_0) + iv_x(x_0, y_0)$ whenever $z \neq z_0$.

(The existence/continuity assumptions in Theorem 2.27, part 2, guarantee that this approximation becomes equality in the limit, though a rigorous proof requires more work)

## 2.4 Holomorphic and Harmonic Functions

We tend to be most interested in functions which are everywhere differentiable.

**Definition 2.29.** We say that $f : D \to \mathbb{C}$ is:

- *Holomorphic* (or *analytic*[8]) on $D$ if it is differentiable at all points $z_0 \in D$ (necessarily $D$ is open!).
- *Holomorphic at* $z_0 \in D$ if it is differentiable (holomorphic) on some neighborhood of $z_0$.
- *Entire* if it is holomorphic with domain $D = \mathbb{C}$.

**Examples 2.30.** 1. The exponential function $f(z) = e^{4z}$ is entire, as is every polynomial.

2. The rational function $f(z) = \frac{iz}{z^2+4}$ is holomorphic on its implied domain $\mathbb{C} \setminus \{\pm 2i\}$. Indeed, by the quotient rule,

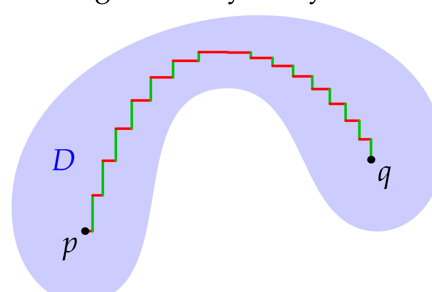
$$f'(z) = \frac{i(z^2+4) - 2iz^2}{(z^2+4)^2} = \frac{i(4-z^2)}{(z^2+4)^2}$$

Our first general result should seem very familiar.

**Theorem 2.31.** *If $f'(z) = 0$ on an open, (path-)connected domain, then $f(z)$ is constant.*

The calculation in the proof should be compared to the corresponding argument from real analysis which also relies on the mean value theorem.

*Proof.* Given $p, q \in D = \operatorname{dom}(f)$, join them with a zig-zag path consisting of finitely many horizontal/vertical segments (Exercise 2.2.10).




On each horizontal segment ($x_1 \le x \le x_2$, $y$ constant), apply the mean value theorem to the single-valued functions

$$x \mapsto u(x, y) \quad \text{and} \quad x \mapsto v(x, y)$$

We therefore obtain values $\widehat{x}, \widetilde{x}$, both in the interval $(x_1, x_2)$, which satisfy

$$\frac{u(x_2, y) - u(x_1, y)}{x_2 - x_1} = u_x(\widehat{x}, y) \quad \text{and} \quad \frac{v(x_2, y) - v(x_1, y)}{x_2 - x_1} = v_x(\widetilde{x}, y)$$

By assumption $f'(z) = u_x + iv_x = 0$, whence both partial derivatives are everywhere zero. It follows that $f(z) = u + iv$ takes the same value at both endpoints of any horizontal segment.

The same holds along at the endpoints of any vertical segment; this time we utilize $u_y = v_y = 0$.

By stepping along the segments in this fashion, we conclude that $f(q) = f(p)$. Since $p, q$ were arbitrary, $f$ is constant on $D$. ∎

[8] We'll use holomorphic/analytic interchangeably though the formal definition of analytic depends on power series. A major part of the course involves showing that these definitions are equivalent.

**Corollary 2.32.** *If $f(z)$ is holomorphic with constant modulus $k = |f(z)|$, then $f(z)$ is constant.*

This is essentially trivial in the real case—think about why! The complex case needs a proof.

*Proof.* Plainly $k^2 = |f(z)|^2 = f(z)\overline{f(z)}$ is constant. If $k = 0$, we are done. Otherwise, $\overline{f(z)} = \frac{k^2}{f(z)}$ is holomorphic (quotient rule!). Write $f(z) = u + iv$, whence $\overline{f(z)} = u - iv$, and consider the Cauchy–Riemann equations for *both*:

$$u_x = v_y, \qquad u_y = -v_x, \qquad u_x = -v_y, \qquad u_y = v_x$$

All partial derivatives are therefore zero: $f'(z) = 0$ and so $f$ is constant. ∎

The next result is worth stating now, since it provides a significant contrast with the real case. We'll prove it later once we've developed contour integration.

**Theorem 2.33.** *If $f(z) = u + iv$ is holomorphic, then $f$ is* **infinitely differentiable.** *Otherwise said:*

- *$f^{(n)}(z)$ exists and is continuous for all $n \in \mathbb{N}$.*
- *$u$ and $v$ have continuous partial derivatives of all orders.*

In real analysis, functions which are everywhere differentiable need not even be *twice* differentiable, let alone infinitely so (Exercise 4).

**Harmonic Functions** The *second* partial derivatives of a holomorphic function satisfy a famous partial differential equation:

$$u_{xx} = \frac{\partial}{\partial x} u_x \overset{CR1}{=} \frac{\partial}{\partial x} v_y = v_{yx} \overset{(*)}{=} v_{xy} = \frac{\partial}{\partial y} v_x \overset{CR2}{=} -\frac{\partial}{\partial y} u_y = -u_{yy}$$

Equality of the mixed partial derivatives $(*)$ follows because all derivatives are continuous (Theorem 2.33 and Clairaut's Theorem). The same equation holds for $v$. We conclude:

**Corollary 2.34.** *If $f = u + iv$ is holomorphic, then $u$ and $v$ are* harmonic functions*; solutions to* Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

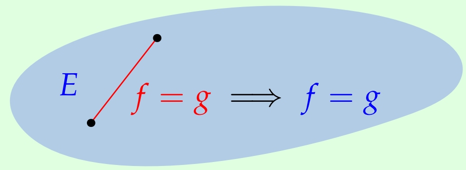
Laplace's equation is one of the most widely applied PDEs in mathematics and physics.

**Example 2.35.** $f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$; its real and imaginary parts are therefore harmonic away from the origin. Indeed,

$$\begin{aligned} u_{xx} + u_{yy} &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{x}{x^2 + y^2} = \frac{\partial}{\partial x} \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{\partial}{\partial y} \frac{2xy}{(x^2 + y^2)^2} \\ &= \frac{-2x(x^2 + y^2) - 4x(y^2 - x^2)}{(x^2 + y^2)^3} - \frac{2x(x^2 + y^2) - 8xy^2}{(x^2 + y^2)^3} = 0 \end{aligned}$$

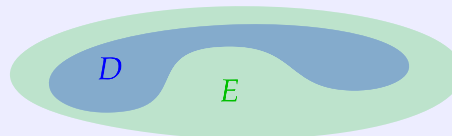
**Analytic Continuations** We finish this introduction to holomorphic functions with another surprising result whose proof will have to wait (this time for the theory of residues).

**Theorem 2.36.** *Suppose $f$ and $g$ are holomorphic functions on an open connected domain $E$ and assume that $f(z) = g(z)$ on some path contained in $E$.*
*Then $f(z) = g(z)$ throughout $E$.*




This is highly counter-intuitive; you need only know the values of a holomorphic function on a tiny path to know the full function on its whole (connected) domain! This leads to a new concept.

**Definition 2.37.** Let $D \subseteq E$ be open connected domains and $g : E \to \mathbb{C}$ holomorphic. Let $f : D \to \mathbb{C}$ be the restriction of $g$ to $D$; that is $f(z) = g(z)$ on $D$.
We call $g$ the *analytic continuation* of $f$ to $E$.




By Theorem 2.36, the analytic continuation of $f$ to $E$ is unique. There are, however, some subtleties, which we explore a little in the next example:

- Given $f$ analytic on $D$, an analytic continuation to $E \supseteq D$ is not guaranteed to exist.
- The choice of extended domain $E$ *really* matters.

**Example 2.38.** Let $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ be the principal square root whose domain $D$ is the first quadrant. Exercise 10 verifies that $f$ is holomorphic on $D$.
The pictures describe *two* analytic continuations of $f$: in both cases the point $w = e^{-\frac{3\pi i}{4}} = e^{\frac{5\pi i}{4}}$ is used for comparison.

1. Let $G$ be the plane omitting the non-positive real axis and define

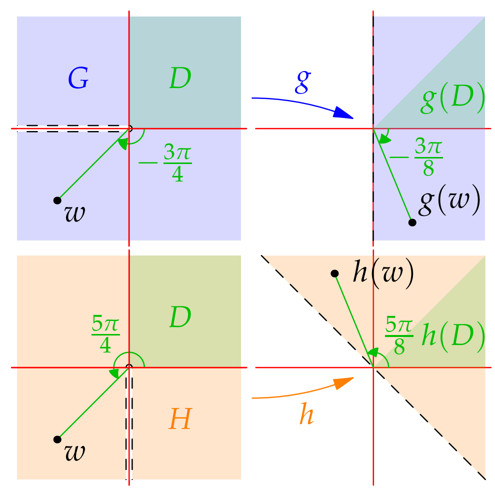
$$g : G \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \operatorname{Arg} z \in (-\pi, \pi)$$

   The codomain of $g$ is the right half-plane. Observe that $g(w) = e^{-\frac{3\pi i}{8}}$ lies in the *fourth quadrant*.

2. Let $H$ be the plane omitting the non-positive imaginary axis and define

$$h : H \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \arg z \in \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$$

   The codomain of $h$ is the upper-right half-plane. Moreover $h(w) = e^{\frac{5\pi i}{8}} = -g(w)$ lies in the *second quadrant*.




The two analytic continuations of $f$ disagree on the intersection of their domains!
In fact the omissions chosen for $G, H$ are necessary: there is no analytic continuation of $f$ to the punctured plane $\mathbb{C} \setminus \{0\}$, or indeed to any domain in which it is possible to loop completely around the origin. We shall return to this topic in Chapter 3...

**Exercises 2.4.** 1. Suppose $g'(z) = h'(z)$ on an open connected domain $D$. Prove that $h(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

*Equivalently: if $g(z)$ and $h(z)$ are anti-derivatives of $f(z)$, then $g(z) - h(z)$ is constant*

2. Verify that $u = e^{nx}\cos ny$ and $v = e^{nx}\sin ny$ are harmonic functions for any $n \in \mathbb{Z}$.

3. Prove or disprove: if $u, v : D \to \mathbb{R}$ are harmonic functions on an open set $D \subseteq \mathbb{R}^2$, then $f(z) := u(x, y) + iv(x, y)$ is holomorphic.

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Show that $f$ is differentiable, but not *twice* so (*an impossible property for complex functions*).

5. Suppose $f(z) = u + iv$ and write $z = x + iy = re^{i\theta}$ in polar form.

   (a) Use the chain rule applied to the polar co-ordinate relations $x = r\cos\theta$, $y = r\sin\theta$ to compute the partial derivatives $u_r, u_\theta, v_r, v_\theta$ in terms of $u_x, u_y, v_x, v_y$.

   (b) Deduce the polar form of the Cauchy–Riemann equations:

$$ru_r = v_\theta \qquad u_\theta = -rv_r, \qquad f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{z}(u_\theta + iv_\theta)$$

6. Prove the polar form of Laplace's equation:

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$$

7. Show that $u = r^n \cos n\theta$ is a harmonic function for any $n \in \mathbb{N}$: find *two* ways to show that this is true!

8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(iy) = -iy^3$ whenever $z = iy$ lies on the imaginary axis. What is the value $f(2)$? Explain your answer.

   (*Hint: Think about analytic continuations*)

9. The function $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Explain why there is no analytic continuation of $f$ that is holomorphic at $z = 0$.

10. (a) Use Exercise 5 to prove that $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ is holomorphic on the first quadrant and find $f'(z)$. Moreover, explain why the functions $g, h$ in Example 2.38 are holomorphic and therefore both analytic continuations of $f$.

    (b) Prove that there exists no analytic continuation of $g$ to any set larger than $G$.

    (*Hint: suppose the extended domain contains $-r \in \mathbb{R}^-$. Now use the fact that an analytic continuation must be continuous at $-r \ldots$*)

# 3 Elementary Functions

We've already considered polynomials, rational functions and, to some extent, $n^{\text{th}}$ roots and the exponential. We now develop the logarithmic and trigonometric functions.

## 3.1 Exponential and Logarithmic Functions

The exponential function $\exp : \mathbb{C} \to \mathbb{C} : z \mapsto e^z$ was defined earlier using Euler's formula

$$\exp(z) = e^z := e^x \cos y + ie^x \sin y \qquad (*)$$

For reference, we collect some basic properties: parts 1–3 are identical to real analysis.

**Lemma 3.1.** *Throughout let $z, w \in \mathbb{C}$.*

1. *The exponential function is entire, with derivative $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$*
2. $e^z \neq 0$
3. *(Exponential laws) $e^{z+w} = e^z e^w$, $e^{z-w} = \frac{e^z}{e^w}$ and $(e^z)^n = e^{nz}$ (whenever $n \in \mathbb{Z}$).*
4. *$e^z$ is* periodic *with period $2\pi i$. Moreover,*

$$e^z = e^w \iff z - w = 2\pi i n \text{ for some } n \in \mathbb{Z}$$

*Sketch Proof.*

1. This is Example 2.28.5: check Cauchy–Riemann and compute $\frac{\mathrm{d}}{\mathrm{d}z}e^z = u_x + iv_x$.
2. This follows trivially from $(*)$: $e^x > 0$ while $\cos y$ and $\sin y$ are never both zero.
3. Apply the multiple-angle formulæ for cosine and sine. For the third law, induct with $z = w$.
4. Certainly $e^{w+2\pi i n} = e^w$ by the periodicity of sine and cosine. Now suppose $e^z = e^w$ where $z = x + iy$ and $w = u + iv$. By considering the modulus and argument, we see that

$$e^x e^{iy} = e^u e^{iv} \implies \begin{cases} e^x = e^u \\ y = v + 2\pi i n \text{ for some } n \in \mathbb{Z} \end{cases}$$

We conclude that $x = u$ and so $z - w = i(y - v) = 2\pi i n$. ∎

**Example 3.2.** Find all $z \in \mathbb{C}$ such that $e^z = 5(-1 + i)$.
Following part 4 of the Lemma, write $z = x + iy$ and take the polar form of $5(-1 + i)$:

$$\begin{aligned} e^z = 5(-1+i) &\iff e^x e^{iy} = 5\sqrt{2}e^{\frac{3\pi i}{4}} = e^{\ln(5\sqrt{2})}e^{\frac{3\pi i}{4}} \\ &\iff \begin{cases} x = \ln(5\sqrt{2}) \\ y = \frac{3\pi}{4} + 2\pi n \quad \text{for some } n \in \mathbb{Z} \end{cases} \\ &\iff z = \ln(5\sqrt{2}) + \left(\frac{3\pi}{4} + 2\pi n\right) i \quad \text{for some} \quad n \in \mathbb{Z} \end{aligned}$$

We see that there are *infinitely many* suitable $z$!

**Duplicate Notation Warning!** When $n \in \mathbb{N}$, the expression $e^{\frac{1}{n}}$ could mean two things. For instance $e^{\frac{1}{3}}$ might mean either:

1. The *set* of cube roots of $e$, namely $\{\sqrt[3]{e}, \sqrt[3]{e}e^{\frac{2\pi i}{3}}, \sqrt[3]{e}e^{-\frac{2\pi i}{3}}\}$;
2. The *real value* $\sqrt[3]{e} \in \mathbb{R}^+$.

Since $e^z$ is such a common function, we default to the second meaning: if you mean the set of $n^{\text{th}}$ roots, say so! Remember you can always write $\exp(z)$ for the function if necessary.

The periodicity of the exponential leads to the more subtle notion of the complex *logarithm*.

**Definition 3.3.** Let $z = re^{i\theta}$ be a non-zero complex number with principal argument $\theta = \operatorname{Arg} z$. The *principal logarithm* of $z$ is the value

$$\operatorname{Log} z := \ln r + i\theta = \ln |z| + i \operatorname{Arg} z$$

where ln is the usual (real!) natural logarithm. The *logarithm* of $z$ is any (and all) of the values[9]

$$\log z = \ln |z| + i \arg z = \ln r + i(\theta + 2\pi n) : n \in \mathbb{Z}$$

**Examples 3.4.** 1. Since $-4 = 4e^{\pi i}$, we see that

$$\operatorname{Log}(-4) = \ln 4 + \pi i \quad \text{and} \quad \log(-4) = \ln 4 + (1 + 2n)\pi i$$

2. Again write in polar form to compute:

$$\operatorname{Log}(\sqrt{3} - i) = \operatorname{Log}(2e^{-\frac{\pi i}{6}}) = \ln 2 - \frac{\pi i}{6} \quad \text{and} \quad \log(\sqrt{3} - i) = \ln 2 - \frac{\pi i}{6} + 2\pi n i$$

These examples involve solving equations of the form $e^w = z$: writing $z = re^{i\theta} = e^{\ln r + i\theta}$ as above, and appealing to part 4 of Lemma 3.1, we instantly see that

$$e^w = z \iff w = \log z$$

Read this carefully, remembering that the logarithm is multi-valued and the exponential periodic:

$$e^{\log z} = z \quad \text{and} \quad \log e^w = w + 2\pi n i \quad \text{where } n \in \mathbb{Z}$$

For reference, we summarize some of the basic properties of the principal logarithm function. All parts should be clear from Definition 3.3.

**Lemma 3.5.** *Throughout, $z$ and $w$ are complex numbers with $z \neq 0$, and $n \in \mathbb{Z}$.*

- $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \{w \in \mathbb{C} : -\pi < \operatorname{Im} w \leq \pi\}$ *is a bijection with inverse* exp.
- $\operatorname{Log} e^w = w + 2\pi n i$ *where $n \in \mathbb{Z}$ is chosen such that* $\operatorname{Im}(\operatorname{Log} e^w) = \operatorname{Im} w + 2\pi n \in (-\pi, \pi]$.
- *If $z \in \mathbb{R}^+$, then* $\operatorname{Log} z = \ln z$ *is the usual* natural *logarithm.*

[9] This is similar to how $\arg z$ means either the *set* $\{\operatorname{Arg} z + 2\pi n i\}$ or some particular value from this set, dependent on context. We'll more formally discuss such *multi-valued* functions in Section 3.2.

**The Logarithm Laws** Just as the standard rules for exponentiation (Lemma 3.1 part 3) apply to the complex exponential, the log laws also translate. However, the multi-valued nature of the logarithm adds some subtlety.

Suppose non-zero $z, w$ are given: since $|zw| = |z|\,|w|$ and $\arg zw = \arg z + \arg w$, we conclude that

$$\begin{aligned}\log zw &= \ln(|z|\,|w|) + i(\arg z + \arg w)\\ &= \ln|z| + i\arg z + \ln|w| + i\arg w\\ &= \log z + \log w\end{aligned}$$

Be very careful, for this expression is *not* an identity of *functions*. What it really means is that the following two *sets* are identical:

$$\begin{aligned}\log z + \log w &= \{\alpha + \beta : \alpha \in \log z, \beta \in \log w\}\\ &= \{|z| + i\operatorname{Arg} z + 2\pi k i + |w| + i\operatorname{Arg} w + 2\pi m i : k, m \in \mathbb{Z}\}\\ \log zw &= \{\ln|zw| + i\operatorname{Arg}(zw) + 2\pi n i : n \in \mathbb{Z}\}\end{aligned}$$

Unless you are sure you won't make a mistake, it is safer to write

$$\boxed{\log zw = \log z + \log w + 2\pi n i \quad \text{for some} \quad n \in \mathbb{Z}}$$

Given its restricted range, we can be more precise for the principal logarithm:

$$\boxed{\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + 2\pi n i \quad \text{for some} \quad n = 0, -1, 1}$$

**Example 3.6.** Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and $w = \sqrt{2}(1+i) = 2e^{\frac{\pi i}{4}}$. Then, for some $k, m, n$,

$$\begin{aligned}&\log z = \ln 2 + \frac{5\pi i}{6} + 2\pi k i, \qquad \log w = \ln 2 + \frac{\pi i}{4} + 2\pi m i\\ &\log zw = \log(4e^{\frac{5\pi i}{6} + \frac{\pi i}{4}}) = \log(4e^{\frac{13\pi i}{12}}) = \ln 4 + \frac{13\pi i}{12} + 2\pi n i\end{aligned}$$

We can choose particular logarithms satisfying $\log zw = \log z + \log w$ provided $n = k + m$. For principal logarithms, we don't get to make a choice:

$$\begin{aligned}&\operatorname{Log} z = \ln 2 + \frac{5\pi i}{6}, \qquad \operatorname{Log} w = \ln 2 + \frac{\pi i}{4}\\ &\operatorname{Log} zw = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \ln 4 - \frac{11\pi i}{12}\\ &\qquad = \ln 4 + \frac{5\pi i}{6} + \frac{\pi i}{4} - 2\pi i = \operatorname{Log} z + \operatorname{Log} w - 2\pi i\end{aligned}$$

We can similarly demonstrate the second log law, with exactly the same caveat:

$$\boxed{\log \frac{z}{w} = \log z - \log w}$$

As before, principal logarithms might require a correction term of $\pm\pi i$.

We can try to translate the final log law ($\log z^n = n \log z$, $n \in \mathbb{N}$), though even more care is needed!

**Example 3.7.** Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and compute what is meant by the *set* $2\log z$:

$$2\log z = \left\{2\left(\ln 2 + \frac{5\pi i}{6} + 2\pi m i\right) : m \in \mathbb{Z}\right\} = \left\{\ln 4 + \frac{5\pi i}{3} + 4\pi m i : m \in \mathbb{Z}\right\}$$

This has only *half* the terms of the set

$$\log z^2 = \left\{\log(4e^{\frac{10\pi i}{6}}) = \ln 4 + \frac{5\pi i}{3} + 2\pi k i : k \in \mathbb{Z}\right\}$$

It is therefore safer to state that $\log z^2 \neq 2\log z$.

Since the principal logarithm is a function rather than a set, we can be more precise: for any $n \in \mathbb{N}$,

$$\operatorname{Log} z^n = n \operatorname{Log} z + 2\pi k i \quad \text{for some integer } k \text{ with } |k| \leq \frac{n}{2}$$

**Example 3.8.** Let $z = e^{-\frac{13\pi i}{16}}$ and consider $z^{16}$. We see that

$$\operatorname{Log} z^{16} = \operatorname{Log} e^{-13\pi i} = \operatorname{Log} e^{\pi i} = i\pi, \quad 16 \operatorname{Log} z = -13\pi i \implies \operatorname{Log} z^{16} = 16 \operatorname{Log} z + 14\pi i$$

In this case $|k| = 7 \leq \frac{16}{2}$.

**Exercises 3.1.** 1. Compute (be careful with (c)!):

(a) $\exp(3 - \frac{\pi}{2} i)$ (b) $\operatorname{Log}(ie)$ (c) $\log(3 - 4i)$ (d) $\operatorname{Log}[(-1 + i)^2]$

2. (a) If $e^z$ is real, show that $\operatorname{Im} z = n\pi$ for some integer $n$.
   (b) If $e^z$ is imaginary, what restriction is placed on $z$?

3. Show in two ways that the function $f(z) = \exp(z^2)$ is entire, and find its derivative.

4. Prove, for any $z \in \mathbb{C}$, that $\left|\exp(z^2)\right| \leq \exp |z|^2$. What must $z$ satisfy if this is to be *equality*?

5. Find $\operatorname{Re} e^{\frac{1}{z}}$ in terms of $x$ and $y$. Why is $\operatorname{Re} e^{\frac{1}{z}}$ harmonic on any domain not containing the origin?

6. Show that $\operatorname{Log} i^3 \neq 3 \operatorname{Log} i$.

7. Show that $\operatorname{Re}(\log(z - 1)) = \frac{1}{2}\ln[(x-1)^2 + y^2]$ whenever $z \neq 1$.

8. Prove the above boxed formula for $\operatorname{Log} z^n$.

9. The square roots of $i$ are $\sqrt{i} = e^{\frac{\pi i}{4}}$ and $-\sqrt{i} = e^{-\frac{3\pi i}{4}}$.
   (a) Compute $\operatorname{Log} \sqrt{i}$ and $\operatorname{Log}(-\sqrt{i})$ and check that $\operatorname{Log} \sqrt{i} = \frac{1}{2} \operatorname{Log} i$.
   (b) Show that the set of all logarithms of all square roots of $i$ is

   $$\log i^{\frac{1}{2}} = \left(n + \frac{1}{4}\right)\pi i \quad \text{where} \quad n \in \mathbb{Z}$$

   Hence deduce that $\log i^{\frac{1}{2}} = \frac{1}{2}\log i$ *as sets*.

### 3.2 Multi-valued Functions, Branch Cuts and the Power Function

Recall that each $\log z$ represents a set of complex numbers; as such, the complex logarithm is termed a *multi-valued function*. We have previously encountered others of this ilk:

- The *argument* of a complex number is any of the values $\arg z = \operatorname{Arg} z + 2\pi n$ where $n \in \mathbb{Z}$. The complex logarithm is merely a modification of this: $\log z = \ln |z| + i \arg z$.
- The $n^{th}$ *root* of $z$ is the set of values $z^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$ where $\omega_n = e^{\frac{2\pi i}{n}}$ is an $n^{\text{th}}$ root of unity and $\sqrt[n]{z}$ the principal $n^{\text{th}}$ root.

It is an abuse of language to refer to a multi-valued *function*, since any function should assign *exactly one* output to each element of its domain. While this problem can be fixed using equivalence classes, another approach is simpler to visualize.

**Definition 3.9.** A *branch* of a multi-valued function $f$ is a single-valued function $F$ on a domain $D$ which is *holomorphic* (differentiable) on $D$ and such that each $F(z)$ is one of the values of $f(z)$.

Let $D = \mathbb{C} \setminus \ell$ where $\ell$ is a line or curve in $\mathbb{C}$. If $F : D \to \mathbb{C}$ is a branch of $f$, we call $\ell$ a *branch cut*. A *branch point* is any point common to all possible branch cuts.

**Branches of the Logarithm** The *principal branch* of the logarithm is a slightly restricted version of the principal logarithm

$$\operatorname{Log} z = \ln r + i\theta \quad \text{where} \quad \theta = \operatorname{Arg} z \in (-\pi, \pi)$$

The branch cut is the non-positive real axis. To check holomorphicity, we verify the Cauchy–Riemann equations:[10]

$$ru_r = r\frac{\partial}{\partial r}\ln r = 1 = \frac{\partial}{\partial \theta}\theta = v_\theta, \qquad u_\theta = 0 = -rv_r$$

The partial derivatives are certainly continuous, whence $\log z$ is holomorphic with derivative

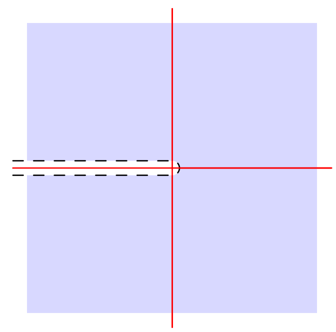
$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

More generally, for any angle $\alpha$ we could take a branch cut $\ell$ to be the line with argument $\alpha$, which defines a new branch of the logarithm:

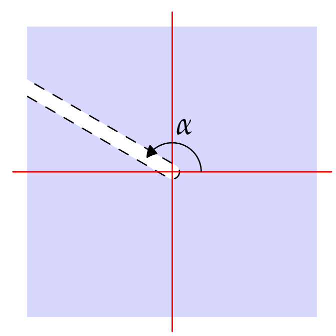
$$\log z = \ln r + i\theta \quad \text{where} \quad \theta \in (\alpha - 2\pi, \alpha)$$

In this description, the principal branch corresponds to $\alpha = \pi$. Note that choosing $\alpha = -\pi$ results in the *same branch cut*, but a *different branch*:

$$\log z = \ln r + i\theta \quad \text{where} \quad \theta \in (-3\pi, -\pi)$$



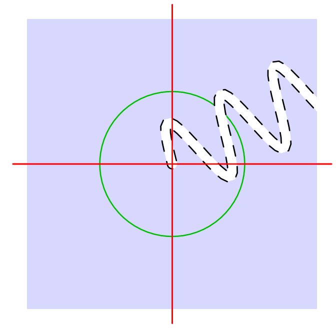
Principal Branch




$\alpha$-Branch

[10]We use the polar form: $ru_r = v_\theta$, $rv_r = -u_\theta$, $f'(z) = e^{-i\theta}(u_r + iv_r)$ (see Exercise 2.4.5).

More esoteric branch cuts are possible, such as the pictured 'squiggle.' At issue is the fact that traversing a counter-clockwise loop around the origin increases the value of a logarithm by $2\pi i$. It is therefore impossible for a branch to be *continuous* (let along *holomorphic*) on any domain containing such a loop; to make $\log z$ single-valued, a branch cut must 'cut' any such path, and thus connect the two *branch points* 0 and $\infty$.



Clarity is crucial here: when you write $\log z$, do you mean a *set*, a particular *element* of that set, or a *branch*? Certain expressions may be true or false depending on the meaning.

**Example 3.10.** Consider $z = \frac{1}{\sqrt{2}}(1+i) = e^{\frac{\pi i}{4}}$. For the principal branch, we have

$$\operatorname{Log} z^2 = \operatorname{Log} e^{\frac{\pi i}{2}} = \frac{\pi i}{2} = 2 \operatorname{Log} z$$

For the branch with $\alpha = \frac{\pi}{3}$ (that is, $\arg z \in (\frac{-5\pi}{3}, \frac{\pi}{3})$), we have

$$z^2 = e^{\frac{\pi i}{2}} = e^{-\frac{3\pi i}{2}} \implies \log z^2 = -\frac{3\pi i}{2} \neq 2 \log z$$

Recall also (Example 3.7), that *as sets*, $\log z^2 \neq 2 \log z$.

For individual branches of the logarithm, specific versions of the logarithm laws (page 30) are available, though they are not worth trying to remember; just carefully think out the possibilities when calculating.

**General Exponential Functions** The logarithm can be used to create exponential functions for any non-zero complex base $c$. Choose a value $\log c$ and define

$$c^z := e^{z \log c}$$

Provided $c$ is not a non-positive real number, the standard is to use the principal logarithm. Regardless of your choice, $\log c$ is constant and the exponential function is holomorphic everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$$

**Example 3.11.** Let $c = i = e^{\frac{\pi i}{2}}$ and use the principal logarithm to define

$$i^z := e^{z \operatorname{Log} i} = \exp\left(\frac{\pi i z}{2}\right) = \exp\left(-\frac{\pi}{2}y + i\frac{\pi}{2}x\right) = e^{-\frac{\pi y}{2}}\left[\cos\frac{\pi}{2}x + i\sin\frac{\pi}{2}x\right]$$

It is simple to check the Cauchy–Riemann equations and see that $i^z$ is entire (holomorphic on $\mathbb{C}$).
If we instead took a different branch of the logarithm with $\arg i = -\frac{3\pi i}{2}$, then

$$i^z = \exp\left(\frac{-3\pi i z}{2}\right) = e^{\frac{3\pi y}{2}}\left[\cos\frac{3\pi}{2}x - i\sin\frac{3\pi}{2}x\right]$$

This is still entire, though it is a completely different function! Note that both choices of $i^z$ agree whenever $z$ is an integer, and for $z = \frac{1}{2}$ they produce the two distinct square roots of $i$!

**Power Functions** Following a similar approach, for any non-zero $z$ and complex number $c$ we may define the (typically multi-valued) function

$$z^c := e^{c \log z}$$

In this case, restricting to the principal branch of the logarithm gives an unambiguous function.

**Definition 3.12.** The *principal value* of $z^c$ is the function

$$\mathrm{P.V.}\, z^c := e^{c \operatorname{Log} z}$$

whose domain is that of the logarithm ($\mathbb{C}$ excluding the non-positive real axis).

**Example 3.13.** Using the principal branch of the logarithm ($\Theta = \operatorname{Arg} z$), we obtain

$$\mathrm{P.V.}\, z^{\frac{1}{3}} = \exp\left(\frac{1}{3}(\ln r + i\Theta)\right) = \exp\left(\ln \sqrt[3]{r} + \frac{i\Theta}{3}\right) = \sqrt[3]{r}e^{\frac{i\Theta}{3}} = \sqrt[3]{z}$$

precisely the principal cube-root of $z$ as defined previously.
If we chose a different branch with $\arg z = \theta$, then

$$z^{\frac{1}{3}} = e^{\frac{1}{3}\log z} = \exp\left(\frac{1}{3}(\ln r + i\theta)\right) = \sqrt[3]{r}\exp\left(\frac{i}{3}(\Theta + (\theta - \Theta))\right) = \sqrt[3]{z}e^{\frac{i(\theta-\Theta)}{3}}$$

Since, for any $z$, the difference in the arguments $\theta - \Theta = 2\pi n$ is a multiple of $2\pi$, this expression really does return one of the cube-roots of $z$.

**Lemma 3.14.** *Choose a branch of the logarithm so that $z^c = e^{c \log z}$ is single-valued. Then $z^c$ is holomorphic on the same domain as the logarithm; moreover $\frac{\mathrm{d}}{\mathrm{d}z} z^c = cz^{c-1}$.*

*Proof.* Since $\log z$ is holomorphic, simply use the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}z}z^c = \frac{\mathrm{d}}{\mathrm{d}z}e^{c\log z} = e^{c\log z}\frac{\mathrm{d}}{\mathrm{d}z}(c\log z) = e^{c\log z}\cdot\frac{c}{z} = ce^{c\log z}e^{-\log z} = ce^{(c-1)\log z} = cz^{c-1}$$

∎

**Example 3.15.** If the principal branch of the logarithm is used, then

$$(zw)^c = \exp(c\operatorname{Log}(zw)) = \exp(c\operatorname{Log} z + c\operatorname{Log} w + 2\pi cni) = z^c w^c e^{2\pi cni}$$

for some $n \in \{0, \pm 1\}$. *We do not expect* simple exponent rules such as $(ab)^c = a^c b^c$ to hold in complex analysis! Note, however, that this does work in the case where $c$ is an integer.
As an example, again using principal values, if $z = w = e^{\frac{3\pi i}{4}}$, then $zw = e^{\frac{3\pi i}{2}} = e^{-\frac{\pi i}{2}}$, whence

$$\mathrm{P.V.}(zw)^{5i} = \exp\left(5i \cdot \frac{-\pi i}{2}\right) = e^{\frac{5\pi}{2}}$$

$$\mathrm{P.V.}\, z^{5i} = \mathrm{P.V.}\, w^{5i} = \exp\left(5i \cdot \frac{3\pi i}{4}\right) = e^{-\frac{15\pi}{4}}$$

$$\implies (zw)^{5i} = e^{\frac{5\pi}{2}} = e^{-\frac{15\pi}{2}}e^{10\pi} = z^{5i}w^{5i}e^{2\pi\cdot 5i\cdot ni} \quad \text{with } n = -1$$

**Exercises 3.2.** 1. Show that the function $f(z) = \operatorname{Log}(z-i)$ is holomorphic everywhere except on the portion $x \le 0$ of the line $y = 1$.

2. Show that the function $f(z) = \frac{1}{z^2+i}\operatorname{Log}(z+4)$ is holomorphic everywhere except at the points $\pm\frac{1}{\sqrt{2}(1-i)}$ and on the portion $x \le -4$ of the real axis.

3. Show that the set $z^{\frac{1}{4}}$ as defined earlier in the course coincides with the set $z^{\frac{1}{4}} := \exp\left(\frac{1}{4}\log z\right)$ as defined in this section.

4. Show that $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$ where $n \in \mathbb{Z}$.

5. Find the principal values of the following:

   (a) $i^{2i}$ (b) $(1-i)^{3i}$ (c) $(-\sqrt{3}+i)^{1+4\pi i}$

6. Suppose $c, c_1, c_2$ and $z$ are complex numbers where $z \neq 0$. If all the powers involved are principal values, show that,

   (a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$ (b) $(z^c)^n = z^{cn}$ for any $n \in \mathbb{N}$.

7. The power function $z^c$ is *usually* multi-valued. However, if $c = m$ is an integer, prove that $z^m$ is single-valued: i.e. it is independent of the branch of logarithm used in its definition.

8. Check the claim at the bottom of Example 3.11: if $m \in \mathbb{Z}$, then $i^m$ is the same value for the two definitions of $i^z$.

9. Continuing the previous question, suppose $c \neq 0$ and define $c^z = e^{z\log c}$ where any choice of the branch of the logarithm is made.

   (a) Let $m \in \mathbb{Z}$. Prove that $c^m$ produces the same value, regardless of the branch of logarithm used to define $\log c$.
   (b) If $z = \frac{1}{m}$, show that $c^z$ really is an $m^{\text{th}}$ root of $c$. If the principal branch of the logarithm is used, show that $c^z$ is the principal $m^{\text{th}}$ root of $c$. For every $m^{\text{th}}$ root of $c$, show that there exists a branch of the logarithm for which $c^z$ equals the given $m^{\text{th}}$ root.

### 3.3 Trigonometric and Inverse Trigonometric Functions

A sensible definition of the basic trigonometric functions comes simply by modifying Euler's formula. For instance, if $y \in \mathbb{R}$, then

$$e^{iy} + e^{-iy} = \cos y + i \sin y + \cos y - i \sin y = 2\cos y$$

This motivates the primary definition.

**Definition 3.16.** For any $z \in \mathbb{C}$ we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

**Example 3.17.** $\cos(\frac{\pi}{4} + i) = \frac{1}{2}\left(e^{\frac{i\pi}{4}-1} + e^{\frac{-i\pi}{4}+1}\right) = \frac{1}{2}\left(\frac{e^{-1}}{\sqrt{2}}(1+i) + \frac{e}{\sqrt{2}}(1-i)\right) = \frac{e+e^{-1}}{2\sqrt{2}} - \frac{e-e^{-1}}{2\sqrt{2}}i$

**Theorem 3.18.** *Sine and cosine are entire with the same derivatives are their real counterparts*

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z$$

*The usual identities, including double and multiple-angle formulæ, are satisfied: for instance*

$$\sin^2 z + \cos^2 z = 1, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w, \quad \cos 2z = 2\cos^2 z - 1, \quad \textit{etc.}$$

*In particular,* $\sin z = \cos(z - \frac{\pi}{2})$ *and* $\cos z = \sin(z + \frac{\pi}{2})$. *Sine and cosine are also* $2\pi$*-periodic and have exactly the same zeros as their real versions:*

$$\sin z = 0 \iff z = n\pi, \quad \cos z = 0 \iff z = \frac{\pi}{2} + n\pi \quad \textit{where} \quad n \in \mathbb{Z}$$

The upshot of the Theorem is that sine and cosine behave exactly as you'd expect.
The proofs are straightforward applications of properties of the exponential function. For instance;

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \frac{\mathrm{d}}{\mathrm{d}z}\frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \cos z$$

and,

$$\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff e^{-2y}(\cos 2x + i \sin 2x) = 1 \iff z = \pi n$$

The remaining trigonometric functions are defined in the expected way: e.g.,

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \quad \text{whenever} \quad z \neq \frac{\pi}{2} + n\pi$$

All trigonometric functions are holomorphic wherever defined and have the usual expressions for their derivatives: e.g., by the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}z}\tan z = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sin z}{\cos z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z$$

**Aside: Hyperbolic Functions** By considering real and imaginary parts,

$$\begin{aligned}\sin z &= \frac{1}{2i}(e^{ix-y}-e^{-ix+y}) = \frac{1}{2i}(e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x)\\ &= \frac{1}{2}(e^{y}+e^{-y})\sin x + \frac{i}{2}(e^{y}-e^{-y})\cos x = \sin x\cosh y + i\cos x\sinh y\\ \cos z &= \cos x\cosh y - i\sin x\sinh y\end{aligned}$$

where $\cosh y = \frac{1}{2}(e^{y}+e^{-y})$ and $\sinh y = \frac{1}{2}(e^{y}-e^{-y})$ are the usual (real) hyperbolic functions. We could have written Example 3.17 this way

$$\cos\left(\frac{\pi}{4}+i\right) = \frac{e+e^{-1}}{2\sqrt{2}} - i\frac{e-e^{-1}}{2\sqrt{2}} = \cos\frac{\pi}{4}\cosh 1 - i\sin\frac{\pi}{4}\sinh 1$$

Hyperbolic functions are a convenient short-cut, but never necessary; use or ignore as you like. All their properties can be derived from their relationship to exponential and trigonometric functions:

$$\cosh z = \frac{e^{z}+e^{-z}}{2} = \cos iz, \qquad \sinh z = \frac{e^{z}-e^{-z}}{2} = -i\sin iz$$

For instance

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\sinh z = -i\frac{\mathrm{d}}{\mathrm{d}z}\sin iz = -i^2\cos iz = \cosh z \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z\\ &\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1\end{aligned}$$

**Inverse Trigonometric Functions** As with logarithms, inverting trigonometric functions results in multi-valued functions.

**Example 3.19.** We find an expression for $\cos^{-1} z$ and compute its derivative.

$$w = \cos^{-1} z \iff z = \cos w \iff 2z = e^{iw} + e^{-iw} \iff (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

which is quadratic in $e^{iw}$. Apply the quadratic formula to see that

$$e^{iw} = \frac{2z + (4z^2-4)^{\frac{1}{2}}}{2} = z + i(1-z^2)^{\frac{1}{2}} \iff \cos^{-1} z = w = -i\log\left[z + i(1-z^2)^{\frac{1}{2}}\right]$$

A branch of $\cos^{-1}$ requires us to choose branches of *both* the square-root *and* the logarithm.
Inverse cosine is holomorphic since it is a composition of holomorphic functions. By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}z}\left[z+i(1-z^2)^{\frac{1}{2}}\right] = \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\left[1 - \frac{iz}{(1-z^2)^{\frac{1}{2}}}\right]\\ &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\cdot\frac{(1-z^2)^{\frac{1}{2}} - iz}{(1-z^2)^{\frac{1}{2}}} = \frac{-1}{(1-z^2)^{\frac{1}{2}}}\end{aligned}$$

If we fix a branch of the square-root (and logarithm) so that $\cos^{-1} z$ is single-valued, this is necessarily the same branch that appears in the expression of the derivative.

Expressions such as these are not worth memorizing; they are not difficult to derive when needed using the method in Example 3.19. Here is the complete list for the three basic functions.

**Theorem 3.20.** *The inverse sine, cosine and tangent functions are given by the expressions*

$$\sin^{-1} z = -i \log\left[iz + (1-z^2)^{\frac{1}{2}}\right] \qquad \cos^{-1} z = -i \log\left[z + i(1-z^2)^{\frac{1}{2}}\right] \qquad \tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$$

*Once branches of the square-root (sine/cosine only) and logarithm are chosen, these are holomorphic on their domains and have familiar derivatives:*

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin^{-1} z = \frac{1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z = \frac{-1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z = \frac{1}{1+z^2}$$

*The branches of the square-root in the derivatives of inverse sine and cosine are identical to those used in the definitions of the original functions.*

**Examples 3.21.** 1. To evaluate $\sin^{-1}\frac{1}{\sqrt{2}}$ as a complex number, we could follow the approach of Example 3.19 (solve $\frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{\sqrt{2}}$ directly), or use the formula in Theorem 3.20:

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log\left[\frac{i}{\sqrt{2}} \pm \sqrt{1-\frac{1}{2}}\right] = -i\log\frac{i\pm 1}{\sqrt{2}}$$

Now evaluate the logarithms separately:

$$-i\log\frac{i+1}{\sqrt{2}} = -i\log e^{\frac{\pi i}{4}} = -i\left[\frac{\pi i}{4} + 2\pi n i\right] = \frac{\pi}{4} + 2\pi n$$
$$-i\log\frac{i-1}{\sqrt{2}} = -i\log e^{\frac{3\pi i}{4}} = -i\left[\frac{3\pi i}{4} + 2\pi n i\right] = \frac{3\pi}{4} + 2\pi n$$

The set of values $\sin^{-1}\frac{1}{\sqrt{2}}$ generated by all branches of the square-root and logarithm is precisely the set we'd have found working entirely within $\mathbb{R}$!

2. We can also evaluate inverse sines that would have no meaning in $\mathbb{R}$. For instance,

$$\sin^{-1} 7 = -i\log[7i \pm \sqrt{-48}] = -i\log(7\pm 4\sqrt{3})i = -i\log(7\pm 4\sqrt{3})e^{\frac{\pi i}{2}}$$
$$= -i\left[\ln(7\pm 4\sqrt{3}) + \frac{\pi i}{2} + 2\pi n i\right] = -i\ln(7\pm 4\sqrt{3}) + \frac{\pi}{2} + 2\pi n$$

Note that $7 > 4\sqrt{3}$, so we are always taking natural log of a positive real number!

3. Compute $\tan^{-1}(i-2\sqrt{3})$. First compute the required fraction in polar form:

$$\frac{i+(i-2\sqrt{3})}{i-(i-2\sqrt{3})} = \frac{2i-2\sqrt{3}}{2\sqrt{3}} = -1 + \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}}e^{\frac{5\pi i}{6}}$$

It follows that

$$\tan^{-1}(i-2\sqrt{3}) = \frac{i}{2}\left(\ln\frac{2}{\sqrt{3}} + \frac{5\pi}{6}i - 2\pi n i\right) = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}} + \pi n : \quad n \in \mathbb{Z}$$

Choosing the principal value of the logarithm ($n = 0$) yields $-\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.

**Exercises 3.3.** 1. Find the real and imaginary parts of $\sin i$, $\cos(1+i)$ and $\tan(2i\ln 5+\frac{\pi}{2})$.

2. As a sanity check, if $w = -i\log\left[z+(z^2-1)^{\frac{1}{2}}\right]$, compute $\cos w = \frac{1}{2}(e^{iw}+e^{-iw})$ directly and verify that you obtain $z$, *irrespective* of which branches are chosen.

3. Using the real and imaginary parts of $\sin z$, directly verify that the Cauchy–Riemann equations are satisfied.

4. Prove the following double/multiple-angle formulæ using the definitions in this section:

   (a) $\cos 2z = 2\cos^2 z - 1$

   (b) $\sin(z-w) = \sin z\cos w - \cos z \sin w$

   (c) $\tan(z+w) = \dfrac{\tan z + \tan z}{1-\tan z\tan w}$

5. Find all the values of $\tan^{-1}(1+i)$.

6. Solve the equation $\cos z = \sqrt{2}$ for $z$.

7. Recall Exercise 3: check explicitly that $\tan w = i - 2\sqrt{3}$ when $w = -\frac{5\pi}{12}+\frac{i}{2}\ln\frac{2}{\sqrt{3}}$.

   (*Hint: use* $\tan w = \frac{e^{2iw}-1}{i(e^{2iw}+1)}$. *Why is this true?*)

8. Suppose $z>1$ is real. Prove that $\operatorname{Re}\sin^{-1} z = \frac{\pi}{2}+2\pi n$ is independent of $z$. What is $\operatorname{Im}\sin^{-1} z$.

9. If the same branch of square-root is chosen in each case, prove that $\sin^{-1} z + \cos^{-1} z$ is constant.

10. Derive the expressions for $\tan^{-1} z$ and its derivative in Theorem 3.20.

11. (a) Given $\cosh z = \cos(-iz)$, find an expression in terms of the complex logarithm for $\cosh^{-1} z$.

    (b) Using your answer to part (a), or otherwise, find all solutions to the equation $\cosh z = \sqrt{3}$.

    (c) Find an expression for the derivative of $\cosh^{-1} z$.

# 4 Integration

At first glance, integration might appear straightforward; surely we can write the following?

$$\int_1^4 z^2\,\mathrm{d}z = \frac{1}{3}z^3\Big|_1^4 = \frac{1}{3}(4^3-1) = 21$$

In *real* analysis such a statement reflects the equivalence of two distinct concepts:

*Anti-derivatives* $\frac{1}{3}z^3$ is an anti-derivative of $z^2$

*Definite Integrals* The 'area' under the curve $y = z^2$ on an interval $[0,z]$ equals $\frac{1}{3}z^3$

This amazing and important equivalence is suitably named the *fundamental* theorem of calculus. But does it hold in *complex* analysis? While anti-derivatives make immediate sense, definite integrals are more delicate. For instance, what should we mean by

$$\int_{3+i}^{4i} z^2\,\mathrm{d}z\ ?$$

A Riemann sum construction requires us to partition some *curve* joining $3+i$ and $4i$; but which curve? Does it matter? These questions lead us to revisit *contour/path integrals* from multi-variable calculus.

## 4.1 Functions of a Real Variable and Contour Integrals

We start by considering complex-valued functions of a *real* variable $w : I \to \mathbb{C}$ where $I \subseteq \mathbb{R}$ is an interval; derivatives and definite integrals are built from those of their real and imaginary parts:

$$w'(t) = u'(t) + iv'(t), \qquad \int_a^b w(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t \qquad (a, b \text{ are } \textit{real} \text{ or } \pm\infty)$$

**Examples 4.1.** 1. If $w(t) = 5t^2 + it$, then

$$w'(t) = 10t + i, \qquad \int_1^2 w(t)\,\mathrm{d}t = \int_1^2 5t^2\,\mathrm{d}t + i\int_1^2 t\,\mathrm{d}t = \frac{35}{3} + \frac{3i}{2}$$

2. If $w(t) = t^2 + e^{it} = t^2 + \cos t + i\sin t$, then

$$w'(t) = 2t - \sin t + i\cos t, \qquad \int_0^{2\pi} w(t)\,\mathrm{d}t = \frac{1}{3}t^3 + \sin t - i\cos t\Big|_0^{2\pi} = \frac{8}{3}\pi^3$$

We see the natural extension of the (real) fundamental theorem here: $\frac{1}{3}t^3 + \sin t - i\cos t$ is plainly an *anti-derivative* of $w(t)$. This requires no proof since it applies separately to the real and imaginary parts of $w(t)$.

Basic rules of integration are easily verified by separately considering real and imaginary parts.

**Lemma 4.2.**
- *Linearity: if $k \in \mathbb{C}$, then $\frac{\mathrm{d}}{\mathrm{d}t}kw(t) = kw'(t)$ and $\int_a^b kw(t)\,\mathrm{d}t = k\int_a^b w(t)\,\mathrm{d}t$*
- *Product rule: $\frac{\mathrm{d}}{\mathrm{d}t}w(t)z(t) = w'(t)z(t) + w(t)z'(t)$*
- *Chain rule: If $s(t)$ is a* real *function then $\frac{\mathrm{d}}{\mathrm{d}t}w\big(s(t)\big) = w'\big(s(t)\big)s'(t)$*

*Complex* substitutions are more subtle, so we provide a proof.

**Lemma 4.3 (Complex Chain Rule).** *Suppose* $w(t) = F\big(z(t)\big)$ *where*

- $z(t) = x(t) + iy(t)$ *is differentiable at* $t$*, and,*
- $F(z) = u(x,y) + iv(x,y)$ *is holomorphic at* $z(t)$

*Then* $w$ *is differentiable at* $t$*, and* $w'(t) = F'\big(z(t)\big)z'(t)$.
*If* $z'(t)$ *is integrable on* $[a,b]$ *and* $F$ *holomorphic on* $z([a,b])$*, we can put this in integral form*

$$\int_a^b F'\big(z(t)\big)z'(t)\,\mathrm{d}t = F\big(z(b)\big) - F\big(z(a)\big)$$

*Proof.* Apply the multi-variable chain rule from real calculus and the Cauchy–Riemann equations:

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\mathrm{d}u}{\mathrm{d}t} + i\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + i\left(\frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right)\\ &= (u_x + iv_x)\frac{\mathrm{d}x}{\mathrm{d}t} + i(v_y - iu_y)\frac{\mathrm{d}y}{\mathrm{d}t} = (u_x + iv_x)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) \qquad \text{(Cauchy–Riemann)}\\ &= F'\big(z(t)\big)z'(t)\end{aligned}$$

■

**Examples 4.4.** 1. Let $w(t) = e^{t-it^2} = F\big(z(t)\big)$ where $F(z) = e^z$ and $z(t) = t - it^2$. Then

$$w'(t) = e^{t-it^2}\frac{\mathrm{d}}{\mathrm{d}t}(t - it^2) = e^{t-it^2}(1 - 2it)$$

Compare with the method of Example 4.1 which gives the same result, if more slowly

$$w'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(e^t\cos t^2 - ie^t\sin t^2) = e^t(\cos t^2 - 2t\sin t^2) - ie^t(\sin t^2 + 2t\cos t^2)$$

2. Since $w(t) = F\big(z(t)\big) = \frac{1}{10}(1 - t^2 + it)^{10}$ has $w'(t) = (i - 2t)(1 - t^2 + it)^9$, we see that

$$\int_0^1 (i - 2t)(1 - t^2 + it)^9\,\mathrm{d}t = \frac{1}{10}(1 - t^2 + it)^{10}\Big|_0^1 = \frac{i^{10} - 1}{10} = -\frac{1}{5}$$

This would be horrific if we instead had to multiply out to find the real and imaginary parts!

3. Sometimes the real and imaginary part approach is simply not tenable,

$$\int_0^1 3it\sqrt{1 + it^2}\,\mathrm{d}t = (1 + it^2)^{3/2}\Big|_0^1 = (1 + i)^{3/2} - 1 = \sqrt[4]{8}e^{\frac{3\pi i}{8}} - 1$$

We evaluate using the principal square root since $1 + it^2$ lies in the first quadrant.

While most of the basic rules of real calculus translate to complex-valued functions of a real variable, not everything goes through. Be particularly careful of existence results such as the mean value theorem which apply perfectly well to real and imaginary parts, but not to the whole...

### Contours and Contour Integrals

We want to integrate complex functions along curves. But what sort of curves?

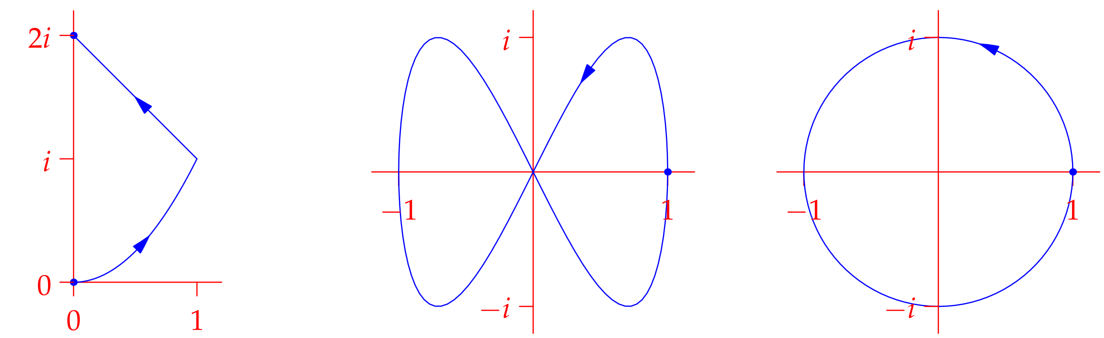
**Definition 4.5.** A *smooth arc* is an oriented curve $C$ in the complex plane for which there exists a *regular parametrization*; a differentiable function $z : [a,b] \to \mathbb{C}$ such that:

1. $z([a,b]) = C$ where $z(a)$ is the *start* of the curve and $z(b)$ is the *end*.
2. $z'(t)$ is *continuous*[11] on $[a,b]$ and *non-zero* on $(a,b)$.

A *contour* is a piecewise smooth arc $C$ consisting of finitely many smooth arcs joined end-to-end. A parametrization $z(t)$ of $C$ is therefore continuous with piecewise continuous derivative.
If we *reverse the orientation* of a contour $C$, the resulting contour is labelled $-C$.
Additionally, we say that a contour is:

- *Closed* if it starts and ends at the same point, $z(a) = z(b)$;
- *Simple* if it does not cross itself (though endpoints can be the same, $z(b) = z(a)$);
- *Positively oriented* if it is simple, closed and traversed counter-clockwise.

**Examples 4.6.** Here are three contours with explicit parametrizations:




Simple, piecewise — $z = \begin{cases} t + it^2 & t \in [0,1] \\ 2 - t + it & t \in [1,2] \end{cases}$

Closed, non-simple — $z = \cos t + i \sin 2t,\ t \in [0, 2\pi]$

Positively oriented — $z = e^{it},\ t \in [0, 2\pi]$

**Definition 4.7.** Let $C$ be a contour parametrized by $z : [a,b] \to \mathbb{C}$ and suppose that $f(z)$ is a complex function defined on the range of $z$. The *contour integral* of $f(z)$ along $C$ is

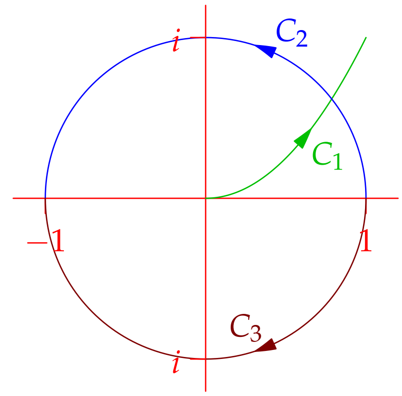
$$\int_C f = \int_C f(z)\,\mathrm{d}z := \int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t$$

This is often written $\oint_C f$ if $C$ is positively oriented (simple and closed).

We plainly require, and assume from now on by default, that the function $f\big(z(t)\big)z'(t)$ is integrable on the *real interval* $[a,b]$; indeed we typically assume that this expression is *piecewise continuous*.

[11]Since $|z'(t)|$ is bounded on $[a,b]$, the arc-length $\int_a^b |z'(t)|\,\mathrm{d}t$ is finite. This extends to any contour.

**Examples 4.8.** We evaluate several contour integrals.




1. For the contour $C_1$ parametrized by $z(t) = t + it^2$, $t \in [0,1]$, we compute

$$\int_{C_1} \overline{z}\,\mathrm{d}z = \int_0^1 \overline{z(t)} z'(t)\,\mathrm{d}t = \int_0^1 (t - it^2)(1 + 2it)\,\mathrm{d}t$$
$$= \int_0^1 t + 2t^3 + it^2\,\mathrm{d}t = \frac{1}{2} + \frac{2}{4} + \frac{i}{3} = 1 + \frac{i}{3}$$

2. For the contour $C_2$ with $z(t) = e^{it}$ with $t \in [0, \pi]$,

$$\int_{C_2} \frac{1}{z}\,\mathrm{d}z = \int_0^\pi \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = \pi i$$

$$\int_{C_2} z^2 + 1\,\mathrm{d}z = \int_0^\pi (e^{2it} + 1) ie^{it}\,\mathrm{d}t = \frac{i}{3i}e^{3it} + \frac{i}{i}e^{it}\Big|_0^\pi = \frac{1}{3}(e^{3\pi i} - 1) + e^{\pi i} - 1 = -\frac{8}{3}$$

3. We compute the same integrals as the previous example, but over the lower semi-circle $C_3$ parametrized by $z(t) = e^{-it}$ with $t \in [0, \pi]$. This time

$$\int_{C_3} \frac{1}{z}\,\mathrm{d}z = \int_0^\pi \frac{-ie^{-it}}{e^{-it}}\,\mathrm{d}t = -\pi i$$
$$\int_{C_3} z^2 + 1\,\mathrm{d}z = \int_0^\pi (e^{-2it} + 1)(-ie^{-it})\,\mathrm{d}t = \frac{-i}{-3i}e^{-3it} + \frac{-i}{-i}e^{-it}\Big|_0^\pi = -\frac{8}{3}$$

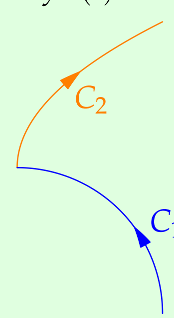
The *sign* of one integral changed but the other did not! We'll return to this problem shortly…

Before considering more examples we develop some of the basic properties of contour integrals. Several are immediate from our earlier discussion, for instance linearity:

$$\int_C \big(af(z) + bg(z)\big)\,\mathrm{d}z = a\int_C f(z)\,\mathrm{d}z + b\int_C g(z)\,\mathrm{d}z$$

Of more importance are the following:

**Theorem 4.9 (Basic rules for contour integrals).** *Suppose $C$ is a contour parametrized by $z(t)$.*




1. *If $C = C_1 \cup C_2$, where $C_1$ and $C_2$ are contours such that the end of the first is the start of the second, then*

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z$$

2. *$\int_C f$ is independent of (orientation-preserving) parametrization.*

3. *Reversing orientation changes the sign of the integral: $\int_{-C} f = -\int_C f$.*

**Example 4.10.** By the previous example and part 3 of the Theorem, if $C$ is the unit circle centered at the origin, then $\oint_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$. This can also be verified directly.

*Proof.* Part 1 follows from the well-known property $\int_a^b = \int_a^c + \int_c^b$ of real integrals. Armed with this, it is enough to check the other parts for a single smooth arc $C$.

2. Suppose $z : [a,b] \to \mathbb{C}$ and $w : [\alpha,\beta] \to \mathbb{C}$ are parametrizations of $C$. Then $w\big(s(t)\big) = z(t)$ for some continuously differentiable $s$ with *positive* derivative. Now compute,

$$\begin{aligned} &z'(t) = w\big(s(t)\big)s'(t), \qquad s(a) = \alpha, \quad s(b) = \beta, \quad \text{from which,} \\ &\int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t = \int_a^b f\big(w(s(t))\big)w'\big(s(t)\big)\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_{s(a)}^{s(b)} f\big(w(s)\big)w'(s)\,\mathrm{d}s \\ &\qquad = \int_\alpha^\beta f\big(w(s)\big)w'(s)\,\mathrm{d}s \end{aligned}$$

3. This is almost identical, except that $-C$ requires a reparametrization $w(s)$ with $s' < 0$, $s(a) = \beta$ and $s(b) = \alpha$. The upshot is that the limits flip on the final integral:

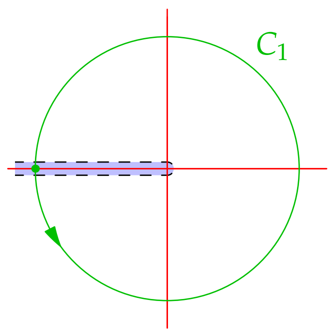
$$\int_C f = \int_a^b f\big(z(t)\big)z'(t)\,\mathrm{d}t = \int_\beta^\alpha f\big(w(s)\big)w'(s)\,\mathrm{d}s = -\int_\alpha^\beta f\big(w(s)\big)w'(s)\,\mathrm{d}s = -\int_{-C} f$$

■

**Contour Integrals of multi-valued functions** If a function is multi-valued, we must specify a *branch* before integrating. It is acceptable to have the contour start and/or finish on the branch cut.[12]

**Examples 4.11.** 1. Compute $\oint_{C_1} z^{1/2}\,\mathrm{d}z$ over the unit circle using the principal square root.

Since the question specifies the principal branch, we must must start and finish the contour at $z = -1$. Parametrize via $z(t) = e^{it}$ where $t \in (-\pi, \pi)$. Since



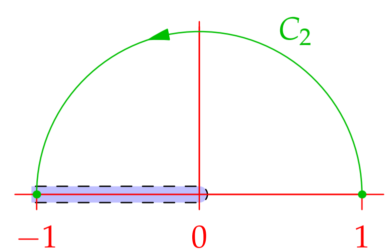

$$\sqrt{z(t)}\,z'(t) = e^{\frac{it}{2}}ie^{it} = ie^{\frac{3it}{2}}$$

is continuous on $[-\pi,\pi]$, we compute

$$\int_{C_1} z^{1/2}\,\mathrm{d}z = \int_{-\pi}^{\pi} ie^{\frac{3it}{2}}\,\mathrm{d}t = \frac{2}{3}e^{\frac{3it}{2}}\Big|_{-\pi}^{\pi} = \frac{2}{3}\big(e^{\frac{3\pi i}{2}} - e^{-\frac{3\pi i}{2}}\big) = -\frac{4i}{3}$$

2. Find $\int_{C_2} z^i\,\mathrm{d}z$ over the unit semi-circle shown using P. V. $z^i$.

Let $z(t) = e^{it}$ where $t \in [0,\pi]$. Since




$$z^i z' = e^{i\,\mathrm{Log}\,z}z' = \exp\big(i(it)\big)\,ie^{it} = ie^{(i-1)t}$$

is continuous when $0 \le t \le \pi$, we have

$$\int_{C_2} z^i\,\mathrm{d}z = \int_0^\pi ie^{(i-1)t}\,\mathrm{d}t = \frac{i}{i-1}\big(e^{(i-1)\pi} - 1\big) = \frac{i-1}{2}\big(1 + e^{-\pi}\big)$$

Warning! Things are more difficult if we allow a contour to *cross* a branch cut, since we'd then have to work with multiple branches simultaneously. We won't do this.

[12] We extend the integrand continuously: recall that if $g(t)$ is *uniformly continuous* on a bounded open interval $(a,b)$, then it has a continuous extension to the closed interval $[a,b]$ and we can therefore define $\int_a^b g(t)\,\mathrm{d}t$.

**Exercises 4.1.** 1. Evaluate the derivatives and integrals:

(a) $\frac{d}{dt}\left[\sin t + i\sqrt{t}\right]$ (b) $\frac{d}{dt}(i+t^3)^2$ (c) $\int_0^1 e^{\pi i t}\,dt$

(d) $\int_0^{\pi/2} e^{it}(1+e^{it})^2\,dt$ (e) $\int_0^1 (i-t)^6\,dt$ (f) $\int_0^{\pi} (i-1)\cos\big((1+i)t\big)\,dt$

2. Show that if $m, n \in \mathbb{Z}$, then

$$\int_0^{2\pi} e^{imt}e^{-int}\,dt = 2\pi\delta_{mn}$$

where $\delta_{mn} = 1$ when $m = n$ and 0 otherwise. Do this two ways:

(a) Using the exponential law and the chain rule/substitution.

(b) By multiplying out and working with real and imaginary parts.

3. Use the chain rule to evaluate the integral $\int_0^x e^{(1+i)t}\,dt$.
Hence find both $\int_0^x e^t \cos t\,dt$ and $\int_0^x e^t \sin t\,dt$ *without* using integration by parts.

4. Prove the product rule for functions of a real variable:

$$\frac{d}{dt}(wz) = w'z + wz'$$

(*Hint: let* $w(t) = u(t) + iv(t)$, $z(t) = x(t) + iy(t)$ *and multiply out...*)

5. Check that the mean value theorem fails for the function $w(t) = \sqrt{t} + it^2$ on the interval $[0, 1]$. That is, there is no $\xi \in (0, 1)$ for which $w'(\xi) = \frac{w(1)-w(0)}{1-0}$.

6. Justify the integral form of the complex chain rule by considering the real and imaginary parts of $F$. What facts from real calculus are you using?

7. Evaluate each contour integral $\int_C f(z)\,dz$ by explicitly parametrizing $C$:

(a) $f(z) = z^2$; $C$ is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = z$; $C$ consists of the straight lines joining $z = 1$ to $1 + i$ to $-1 + i$ to $-1$.

(c) $f(z) = \operatorname{Log} z$; $C$ is the circular arc of radius 3 centered at the origin, oriented counter-clockwise from $-3i$ to $3i$.

(d) $f(z) = \exp \overline{z}$ where $C$ is the straight line from $z = i$ to $z = 3 - i$.

8. Explicitly check that $\int_C z\,dz = \frac{1}{2}(B^2 - A^2)$ along the straight line joining $A$ and $B$.
(*Hint: the line can be parametrized by* $z(t) = (1-t)A + tB$ *where* $t \in [0, 1]$)

9. Let $n \in \mathbb{Z}$ and let $C_0$ be the positively oriented circle centered at $z_0$ with radius $R > 0$. Explicitly parametrize this circle to show that

$$\oint_{C_0} (z - z_0)^{n-1}\,dz = \begin{cases} 2\pi i & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

10. Suppose $z : [a,b] \to \mathbb{C}$ is a regular parametrization of a smooth arc $C$. Then the *arc-length* of the curve is the integral of the *speed* of the parametrization:

$$L = \int_a^b \left| z'(t) \right| \mathrm{d}t$$

   (a) Compute the arc-length of the circle of radius $R$ centered at the origin.
   (b) Compute the arc-length of the simple piecewise curve in Example 4.6.
   (*This requires a tough substitution; perhaps look up a table of integrals...*)
   (c) By commenting on the proof of Theorem 4.9, explain why a reparametrization of $C$ does not change the arc-length.
   (d) Let $s(t) = \int_a^t |z'(\tau)| \, \mathrm{d}\tau$ be the arc-length as a *function* of $t \in [a,b]$. Consider a new parametrization $w(s) = z(t(s))$, where $t(s)$ is the inverse function of $s(t)$.
   Prove that $\left| \frac{\mathrm{d}w}{\mathrm{d}s} \right| = 1$.
   (*This proves that every smooth arc has a unit-speed parametrization*)

The last three questions elaborate a little on the approach in Examples 4.11.

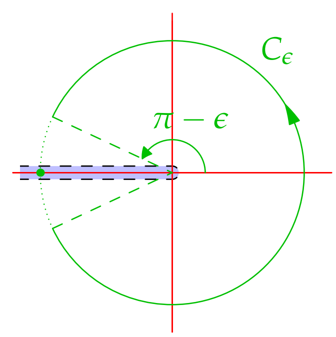
11. (a) Compute the integral of the principal value of $z^{1/3}$ around the positively oriented unit circle starting and finishing at $z = -1$.
   (b) Now consider the branch $z^{1/3} = \exp(\frac{1}{3} \log z)$ where $\arg z \in (0, 2\pi)$. Integrate this around the positively oriented unit circle starting and finishing at $z = 1$. What do you observe?

12. Compute $\oint_C z^{1/2} \, \mathrm{d}z$ where we take the $\alpha$-branch $z^{1/2} = \exp(\frac{1}{2} \log z)$ with $\arg z \in (\alpha, \alpha + 2\pi)$ and the unit circle $C$ traced from angle $\alpha$ to $\alpha + 2\pi$.

13. Let $\epsilon > 0$ be small and suppose that $C_\epsilon$ is the circular arc of radius 1 centered at the origin, traversed counter-clockwise from angle $-\pi + \epsilon$ to $\pi - \epsilon$.
   By parametrizing $C_\epsilon$, explicitly evaluate $\int_{C_\epsilon} \sqrt{z} \, \mathrm{d}z$ and verify that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \sqrt{z} \, \mathrm{d}z = -\frac{4i}{3}$$

   is the value obtained previously for $\oint_C \sqrt{z} \, \mathrm{d}z$.



## 4.2 Path-independence, the Fundamental Theorem & Integral Estimation

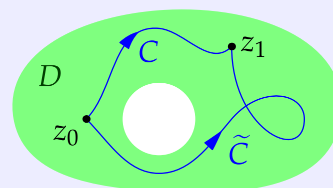
We start by revisiting an earlier example.

**Example (4.8, parts 2 & 3).** Let $F(z) = \frac{1}{3}z^3 + z$ and observe that $F'(z) = z^2 + 1$. If $z : [a, b] \to \mathbb{C}$ parametrizes a smooth arc $C$ such that $z(a) = 1$ and $z(b) = -1$, then Lemma 4.3 shows that

$$\begin{aligned}\int_C z^2 + 1 \, \mathrm{d}z &= \int_a^b \big(z(t)^2 + 1\big) z'(t) \, \mathrm{d}t = \int_a^b F'\big(z(t)\big) z'(t) \, \mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F\big(z(t)\big) \, \mathrm{d}t \\ &= F\big(z(b)\big) - F\big(z(a)\big) = \frac{1}{3}z^3 + z\Big|_1^{-1} \\ &= -\frac{4}{3} - \frac{4}{3} = -\frac{8}{3}\end{aligned}$$

The contour integral is *independent of the choice* of arc $C$, depending only on its *endpoints*.

**Definition 4.12.** Suppose $C$ is a contour in some path-connected $D \subseteq \mathbb{C}$ and that $f : D \to \mathbb{C}$. We say that $\int_C f$ is *path-independent* if its value depends only on the *endpoints* of $C$ and not otherwise on the contour.
Otherwise said, $\int_C f = \int_{\widetilde{C}} f$ for any contour $\widetilde{C}$ with the same endpoints.

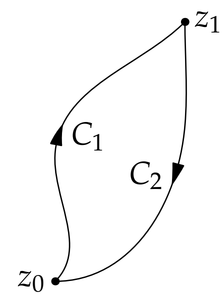



When $\int_C f$ is path-independent (and *only* in such cases), it is permissible to write a contour integral with explicit limits: to write $\int_{z_0}^{z_1} f$ is to assert that the integral evaluates identically along any contour with endpoints $z_0, z_1$.

Before tackling the main result, we tidy up a connection between path independence and closed curves. You should have seen exactly this result in multi-variable calculus.

**Lemma 4.13.** *Let $f$ defined on a domain $D \subseteq \mathbb{C}$ be given.* ***Every*** *contour integral $\int_C f(z) \, \mathrm{d}z$ over a contour in $D$ is path-independent if and only if $\int_C f(z) \, \mathrm{d}z = 0$ round* ***every*** *closed contour.*

*Proof.* ($\Rightarrow$) Assume path-independence for all integrals $\int_C f$ and let $C$ be a given *closed* contour in $D$. Choose any points $z_0, z_1 \in C$ and decompose $C = C_1 \cup C_2$ into two contours: from $z_0$ to $z_1$ and back again. Then,




$$\begin{aligned}\int_C f(z) \, \mathrm{d}z &= \int_{C_1} f(z) \, \mathrm{d}z + \int_{C_2} f(z) \, \mathrm{d}z && \text{(Theorem 4.9, part 1)} \\ &= \int_{C_1} f(z) \, \mathrm{d}z - \int_{-C_2} f(z) \, \mathrm{d}z && \text{(Theorem 4.9, part 3)}\end{aligned}$$

Since $C_1$ and $-C_2$ share the same endpoints, path-independence tells us that the integrals are equal, whence $\int_C f = 0$.

($\Leftarrow$) Conversely, suppose $\int_C f(z) \, \mathrm{d}z = 0$ round any closed contour in $D$, and let $C_1, -C_2$ be given contours in $D$ with the same endpoints $z_0, z_1$.
Plainly $C = C_1 \cup C_2$ is a closed contour, and the previous calculation demonstrates path-independence: $\int_{C_1} f(z) \, \mathrm{d}z = \int_{-C_2} f(z) \, \mathrm{d}z$. ∎

In the above example, the existence of an *anti-derivative* $F(z) = \frac{1}{3}z^2 + z$ of $f(z) = z^2 + 1$ demonstrated path-independence, thus facilitating easy computation of the integral. This should seem familiar...

**Theorem 4.14 (Fundamental Theorem).** *Let $f$ be continuous on an open domain. Then,*

$$f \text{ has an anti-derivative} \iff \text{all contour integrals } \int_C f \text{ are path-independent}$$

*In such situations, $\int_C f(z)\,dz = F(z_1) - F(z_0)$ where $F(z)$ is any anti-derivative of $f(z)$.*

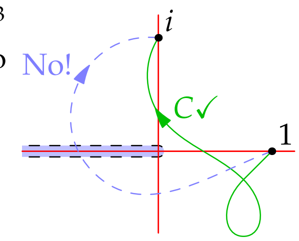
The openness and continuity assumptions are necessary and will be used in the proof. In multi-variable calculus, the ($\Rightarrow$) direction is often known as the Fundamental Theorem of Line Integrals.

**Examples 4.15.** 1. The function $f(z) = (z+i)^3$ has anti-derivative $F(z) = \frac{1}{4}(z+i)^4$. It follows that

$$\int_0^{1-2i} (z+i)^4 = \frac{1}{4}(z+i)^4\Big|_0^{1-2i} = \frac{1}{4}\left[(1-i)^4 - i^4\right] = \frac{1}{4}(-4-1) = -\frac{5}{4}$$

*regardless* of the contour used to travel from $z = 0$ to $1 - 2i$.

2. The principal root $f(z) = \sqrt{z}$ has anti-derivative $F(z) = \frac{2}{3}z^{3/2} = \frac{2}{3}(\sqrt{z})^3$ (also using the principal branch!). If $C$ is any contour from $z_0 = 1$ to $z_1 = i$ which *doesn't cross the branch cut*, then




$$\int_C \sqrt{z}\,dz = \frac{2}{3}(\sqrt{z})^3\Big|_1^i = \frac{2}{3}((\sqrt{i})^3 - (\sqrt{1})^3) = \frac{2}{3}(e^{\frac{3\pi i}{4}} - 1)$$

3. Since $f(z) = \sin z$ has an anti-derivative $F(z) = -\cos z$, valid everywhere, we see that $\int_C \sin z\,dz = 0$ round any closed curve $C$.

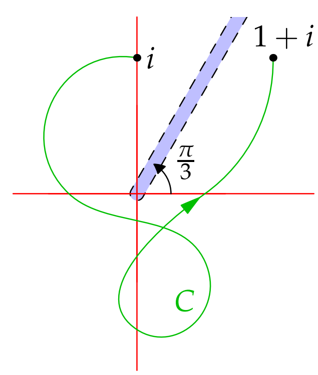
4. Around the unit circle $\oint_C \bar{z}\,dz = \int_{-\pi}^{\pi} e^{-it}ie^{it}\,dt = 2\pi i \neq 0$. Thus $f(z) = \bar{z}$ has no anti-derivative on any domain including the unit circle!

5. (Example 4.10, cont) Around the unit circle, $\oint_C \frac{1}{z}\,dz = 2\pi i$ is non-zero. By the fundamental theorem, $f(z) = \frac{1}{z}$ cannot have an anti-derivative on any domain containing said circle. But this is obvious from our discussion in Section 3.2: If an anti-derivative $F(z)$ existed, then

$$\frac{d}{dz}F(z) = \frac{1}{z} = \frac{d}{dz}\log z \implies F(z) = \log z + c \qquad \text{(Theorem 2.31)}$$

for some branch of the logarithm, which contradicts the fact that no branch can be made single-valued on a path encircling the origin.

We can, however, use the fundamental theorem to evaluate $\int_C \frac{1}{z}\,dz$ over a contour within the domain of a single branch. In the picture, given the contour $C$, we choose the branch cut shown and the logarithm with $-\frac{5\pi}{3} < \arg z < \frac{\pi}{3}$:




$$\int_C \frac{1}{z}\,dz = \log(1+i) - \log i = \log\sqrt{2}e^{\frac{\pi i}{4}} - \log e^{-\frac{3\pi i}{2}}$$
$$= \ln\sqrt{2} + \frac{\pi i}{4} - \frac{3\pi i}{2} = \frac{1}{4}(\ln 4 + 7\pi i)$$

**Proving the Fundamental Theorem** The forward direction is straightforward, though note the importance of a contour consisting of only *finitely* many smooth arcs.

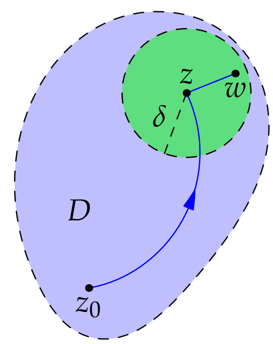
*Proof.* ($\Rightarrow$) Suppose $f$ has an anti-derivative $F$. Given a contour $C$ parametrized by a piecewise smooth function $z(t)$, write $C = C_1 \cup \cdots \cup C_n$ where each $C_k$ is a smooth arc between endpoints $z_{k-1} = z(t_{k-1})$ and $z_k = z(t_k)$. By Theorem 4.9 and Lemma 4.3, we see that

$$\int_C f = \sum_{k=1}^n \int_{C_k} f = \int_{t_{k-1}}^{t_k} F'(z(t))z'(t)\,dt = \sum_{k=1}^n \left[F(z_k) - F(z_{k-1})\right] = F(z_n) - F(z_0)$$

The converse is similar to a standard proof of part I of the fundamental theorem from real analysis:

$$f \text{ continuous} \implies \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

There are, however, a couple of extra subtleties: we must choose how to travel from one endpoint of an integral to the other (path-independence says it doesn't matter how); we also need to bound the modulus of a contour integral. This last isn't as trivial as it may seem...




($\Leftarrow$) We prove when the domain $D$ is connected.[13] Fix $z_0 \in D$ and *define* $F(z) := \int_{z_0}^z f(\zeta)\,d\zeta$ where the integral is taken along *any* curve (path-independence). We need to show that $\lim_{w\to z} \frac{F(w)-F(z)}{w-z} = f(z)$ on $D$.

Fix $z \in D$ and let $\epsilon > 0$ be given. Since $f$ is continuous and $D$ open,

$$\exists \delta > 0 \text{ such that } |\zeta - z| < \delta \implies \zeta \in D \text{ and } |f(\zeta) - f(z)| < \frac{\epsilon}{2}$$

Let $w \in D$ be such that $0 < |w - z| < \delta$. Evaluating along any curve joining $z, w$ (path-independence again), we obtain

$$F(w) - F(z) = \int_z^w f(\zeta)\,d\zeta$$

For simplicity, we *choose* the straight line segment from $z$ to $w$. The proof is almost complete:

$$\begin{aligned}\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| &= \left|\frac{1}{w-z}\int_z^w f(\zeta) - f(z)\,d\zeta\right| = \frac{1}{|w-z|}\left|\int_z^w f(\zeta) - f(z)\,d\zeta\right| \\ &\le \frac{1}{|w-z|}\,|w-z|\,\frac{\epsilon}{2} < \epsilon\end{aligned}$$

■

The second last inequality should give you pause. At first glance it appears as if we argued

$$\left|\int_z^w f(\zeta) - f(z)\,d\zeta\right| \le \int_z^w |f(\zeta) - f(z)|\,d\zeta \le \int_z^w \frac{\epsilon}{2}\,d\zeta = |w-z|\,\frac{\epsilon}{2}$$

This is fine in *real* analysis (if $z \le w$), but is utter nonsense in *complex*-land: the middle term

$$\int_z^w |f(\zeta) - f(z)|\,d\zeta = \int_a^b \left|f(\zeta(t)) - f(z)\right| \zeta'(t)\,dt$$

*need not be a real number*! We had better tidy this up...

[13] If not, repeat the argument choosing a new $z_0$ for each connected component ('lump') of $D$.

**Theorem 4.16 (Integral Estimation).** 1. *Suppose $w : [a, b] \to \mathbb{C}$ is piecewise continuous. Then*

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| \le \int_a^b |w(t)|\,\mathrm{d}t$$

2. *Suppose $C$ is a contour with length $L$, and let $f$ be piecewise continuous on $C$. Then $|f(z)|$ is bounded by some $M \ge 0$ on $C$, and*

$$\left| \int_C f(z)\,\mathrm{d}z \right| \le ML$$

Part 2 justifies the suspect inequality in the proof of the Fundamental Theorem: $f(\zeta) - f(z)$ is bounded by $M = \frac{\epsilon}{2}$ and the path of integration is the straight line of length $L = |w - z|$.

*Proof.* 1. Let $\int_a^b w(t)\,\mathrm{d}t = re^{i\theta}$. Since $\theta$ is constant, $r = \int_a^b e^{-i\theta} w(t)\,\mathrm{d}t$ is *real*. In particular,

$$r = \int_a^b \operatorname{Re}\big(e^{-i\theta} w(t)\big) + i \operatorname{Im}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t = \int_a^b \operatorname{Re}\big(e^{-i\theta} w(t)\big)\,\mathrm{d}t$$

Appealing to $\operatorname{Re} z \le |z|$, we see that

$$\left| \int_a^b w(t)\,\mathrm{d}t \right| = r = \int_a^b \operatorname{Re}(e^{-i\theta} w(t))\,\mathrm{d}t \le \int_a^b \left| e^{-i\theta} w(t) \right|\,\mathrm{d}t = \int_a^b |w(t)|\,\mathrm{d}t$$

2. Parametrize the contour integral by $z : [a, b] \to \mathbb{C}$. Since $f(z(t))$ is piecewise continuous on a closed bounded interval, it is bounded and thus $M$ exists. But now,

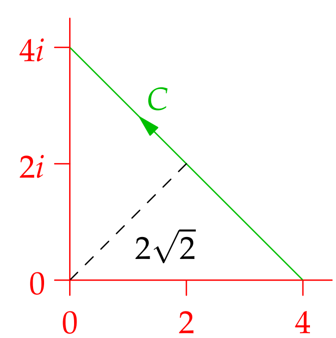
$$\left| \int_C f(z)\,\mathrm{d}z \right| = \left| \int_a^b f\big(z(t)\big) z'(t)\,\mathrm{d}t \right| \le \int_a^b \left| f\big(z(t)\big) z'(t) \right|\,\mathrm{d}t \qquad \text{(part 1)}$$
$$\le \int_a^b M \left| z'(t) \right|\,\mathrm{d}t = ML$$

■

While an explicit computation of arc-length is usually impractical, it is straightforward for circular or straight-line arcs. Indeed the estimation of integrals around circles is crucial for later sections.

**Examples 4.17.** 1. On the straight line $C$ joining $z = 4$ to $4i$, we see that

$$2\sqrt{2} = |2(1+i)| \le |z| \le 4 \implies |z+1| \le |z| + 1 \le 5$$




By the extended triangle inequality,

$$\left| z^4 + 4 \right| \ge \left| |z|^4 - 4 \right| \ge \left| (2\sqrt{2})^4 - 4 \right| = 60$$

Since $C$ has length $4\sqrt{2}$, it follows that

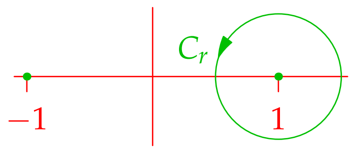
$$\left| \int_C \frac{z+1}{z^4+4}\,\mathrm{d}z \right| \le \frac{20\sqrt{2}}{60} = \frac{\sqrt{2}}{3}$$

2. For the same function $f(z) = \frac{z+1}{z^4+4}$, consider the circle $C_R$ with radius $R > \sqrt[4]{4}$ centered at the origin. On $C_R$, we have $|z|^4 > 4$; the triangle inequality tells us that

$$\left|\frac{z+1}{z^4+4}\right| \le \frac{|z|+1}{|z^4|-4} = \frac{R+1}{R^4-4} \implies \left|\oint_{C_R} \frac{z+1}{z^4+4}\,dz\right| \le \frac{2\pi R(R+1)}{R^4-4}$$

In particular, this approaches zero as $R \to \infty$.

3. Let $C_r$ be the circle with radius $r < 2$ centered at $z = 1$. Then




$$|z-1| = r \text{ and } |z+1| = |2-(1-z)| \ge 2-r$$

from which

$$\left|\oint_{C_r} \frac{1}{z^2-1}\,dz\right| \le \frac{2\pi r}{(2-r)r} = \frac{2\pi}{2-r} \xrightarrow[r\to 0]{} \pi$$

In Exercise 11, it will be verified that $\oint_{C_r} \frac{1}{z^2-1}\,dz = i\pi$ for *any* $r < 2$.

Examples 2 & 3 are typical of the calculations dominating later sections.

**Exercises 4.2.** 1. Evaluate each contour integral $\int_C f(z)\,dz$ using the fundamental theorem:

(a) $f(z) = z^5$ where $C$ is the straight line from $z = 1$ to $z = i$.

(b) $f(z) = \dfrac{1}{z}$ where $C$ is the pair of straight lines from $z = 1$ to $-1-i$ to $-i$.

(c) $f(z) = iz\sin z^2$, where $C$ is the straight line from the origin to $z = i\sqrt{\pi}$.

(d) $f(z) = \dfrac{1}{1+z^2}$ where $C$ is the straight line from $z = 1$ to $2+i$.

(e) $f(z) = \dfrac{1}{\sqrt{z}}$ where $C$ is any path $z(t)$ with $\operatorname{Re} z > 0$ joining $z = 1+i$ and $z = 4$.

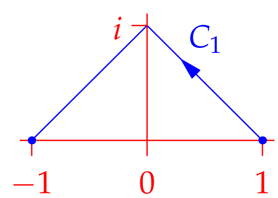
(f) $f(z) = \text{P.V.}\, z^{-1-2i}$ along the quarter circle $z(t) = e^{it}$ where $0 \le t \le \frac{\pi}{2}$.

2. Let $n \in \mathbb{N}_0$. Prove that for every contour $C$ from $z_0$ to $z_1$

$$\int_C z^n\,dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

3. If $C$ is a closed curve not containing $z_0$, and $n \in \mathbb{Z}\setminus\{0\}$, prove that $\int_C (z-z_0)^{n-1}\,dz = 0$.

4. Let $f(z) = z^{1/3}$ be the branch where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Evaluate the integral $\int_{C_1} f(z)\,dz$ where $C_1$ is the curve shown in the picture.




5. Evaluate $\int_{2i}^{1+i} \operatorname{Log} z\,dz$ where the curve $C$ lies in the upper half-plane. (*Hint: use integration by parts*)

6. Evaluate $\int_{-1}^{1} z^{2-i}\,dz$ where $z^{2-i}$ is the principal branch, and the integral is over any contour which, apart from its endpoints, lies above the real axis.

7. Let $C$ be the arc of the circle $|z| = 2$ from $z = 2$ to $2i$. Without evaluating the integral, show that

(a) $\left| \int_C \frac{z+4}{z^3-1} \, dz \right| \leq \frac{6\pi}{7}$ (b) $\left| \int_C \frac{dz}{z^2-1} \right| \leq \frac{\pi}{3}$

8. If $C$ is the straight line joining the origin to $1+i$, show that $\left| \int_C z^3 e^{2iz} \, dz \right| \leq 4$

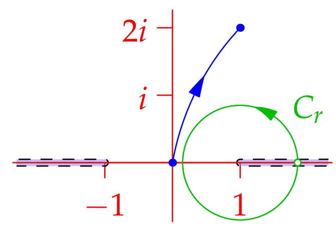
9. If $C$ is the boundary of the triangle with vertices $0, 3i$ and $-4$, prove that $\left| \oint_C (e^z - \overline{z}) \, dz \right| \leq 60$
(*Hint: show that* $|e^z - \overline{z}| \leq e^x + \sqrt{x^2+y^2} \ldots$)

10. Let $C_R$ be the circle of radius $R > 1$ centered at the origin. Prove that

$$\left| \oint_{C_R} \frac{\operatorname{Log} z}{z^2} \, dz \right| < 2\pi \left( \frac{\pi + \ln R}{R} \right)$$

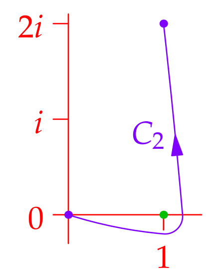
and thus conclude that $\lim_{R\to\infty} \oint_{C_R} \frac{\operatorname{Log} z}{z^2} \, dz = 0$.

11. (Hard) We continue and expand Example 4.17.3. Let $f(z) = \frac{1}{z^2-1}$.




(a) Explain why $F(z) = \frac{1}{2}\big(\operatorname{Log}(1-z) - \operatorname{Log}(1+z)\big)$ is an anti-derivative of $f(z)$ on the domain $D = \mathbb{C} \setminus \ell$ where $\ell$ consists of the two segments of the real axis where $|z| \geq 1$.
Use $F(z)$ to evaluate $\int_0^{1+2i} f(z) \, dz$ along any curve in $D$.

(b) Show that $\oint_{C_r} f(z) \, dz = i\pi$ where $C_r$ is a circle of radius $r < 2$ centered at $z = 1$.
(*Hint: parametrize* $z(t) = 1 + re^{it}$ *where* $-\pi < t < \pi$ *and use* $F(z)$)



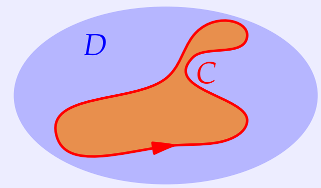

(c) Evaluate $\int_{C_2} f(z) \, dz$ along a curve $C_2$ passing to the *right* of $z = 1$ as in the picture. Compare your answer with parts (a) and (b).
(*Hint: you need a* ***different*** *branch of the log for one of the terms in* $F(z)$)

### 4.3 The Cauchy–Goursat Theorem and Cauchy's Integral Formula

In this section we start to extend the fundamental theorem with the goal of more fully understanding and evaluating holomorphic functions. We first require another piece of topology.

**Definition 4.18.** A *simply-connected* region $D$ is a path-connected set such that any simple closed curve $C \subseteq D$ may be shrunk to a point *without leaving* $D$. Otherwise said, the inside of any such $C$ also lies in $D$.

A non-simply connected region has *holes*: e.g., the set $D$ pictured at the bottom of the page.




**Theorem 4.19 (Cauchy–Goursat, version 1).** *Suppose $C$ is a closed contour in a simply-connected region $D$. If $f$ is holomorphic on $D$, then $\int_C f(z)\,dz = 0$.*

A basic proof invokes *Green's Theorem* (from multi-variable calculus) after assuming that the real and imaginary parts of $f$ have continuous partial derivatives and that $C$ is simple. A proof without these assumption is significantly longer and more challenging.

**Lemma 4.20 (Green's Theorem).** *Suppose $D$ is a closed bounded simply-connected domain with boundary $C$ and that $P, Q : D \to \mathbb{R}$ have continuous partial derivatives. Then*

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

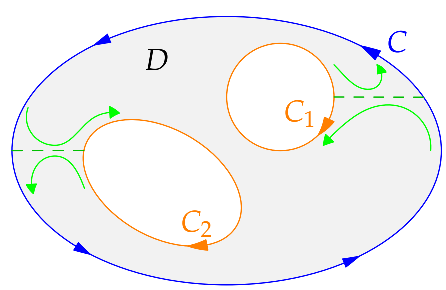
*Sketch Proof of Cauchy–Goursat.* Suppose $f(z) = u + iv$ has continuous partial derivatives and that $C$ is simple (assume WLOG that $C$ is positively oriented). Parametrize $C$ by $z(t)$, then

$$\begin{aligned}
\oint_C f(z)\,dz &= \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \big(u(z(t)) + iv(z(t))\big)\big(x'(t) + iy'(t)\big)\,dt \\
&= \int_a^b ux' - vy'\,dt + i\int_a^b vx' + uy'\,dt \\
&= \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy && \text{(definition of line integral in } \mathbb{R}^2\text{)} \\
&= \iint_D -v_x - u_y\,dA + i\iint_D u_y - v_x\,dA && \text{(Green's Theorem)}
\end{aligned}$$

Both double integrals are zero by the Cauchy–Riemann equations. ∎

To extend Cauchy–Goursat, we perform a sneaky trick. From the interior of a simple closed contour $C$, remove the regions inside simple closed non-intersecting contours $C_1, \ldots, C_k$. Orient these clockwise so that the interior region $D$ lies to all contours' *left*.

By cutting $D$ as indicated and traversing each cut twice in opposite directions, $C, C_1, \ldots, C_k$ and the cuts may be joined to form a new simple closed contour. We may now apply Cauchy–Goursat…



**Corollary 4.21 (Cauchy–Goursat, version 2).** *Suppose $C$ is a simple closed contour, oriented counter-clockwise. Let $C_1, \ldots, C_k$ be non-intersecting simple closed contours in the interior of $C$, oriented clockwise. If $f(z)$ is holomorphic on the region between and including $C$ and the interior boundaries $C_1, \ldots, C_k$, then*

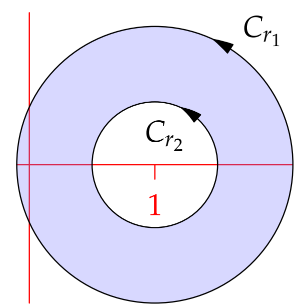
$$\int_C f(z)\,dz + \sum_{j=1}^{k} \int_{C_j} f(z)\,dz = 0$$

The power of this result lies in how it allows us to *compare* integrals around different contours.

**Corollary 4.22.** *Suppose $C_1, C_2$ are non-intersecting positively oriented simple closed contours. If $f$ is holomorphic on the region between and including the curves, then*

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

**Examples 4.23.** 1. (Example 4.17.3, cont.) The function



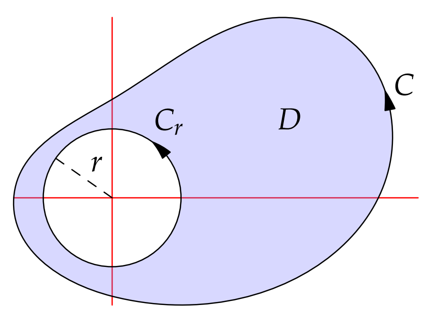

$$f(z) = \frac{1}{z^2 - 1}$$

is holomorphic on and between any two circles $C_r$ centered at $z = 1$ with radius $r < 2$. Cauchy–Goursat thus confirms part of the conclusion of Exercise 4.2.11, that $\oint_{C_r} \frac{1}{z^2-1}\,dz$ is *independent of the radius $r$*. We will shortly develop a better method to evaluate this integral.

2. We compute the integral of $f(z) = \frac{1}{z}$ around *any* simple closed contour $C$ staying away from the origin.

   - If the origin is *outside* $C$, then $f$ is holomorphic on and inside $C$. By Cauchy–Goursat, we conclude that $\oint_C \frac{1}{z}\,dz = 0$.

   - If the origin is *inside* $C$, then there is some minimum distance $d$ of $C$ to the origin. Choose any circle $C_r$ with radius $r < d$ centered at the origin. Since $f$ is holomorphic on the region $D$ between and including $C$ and $C_r$, we conclude that




$$\oint_C \frac{1}{z}\,dz = \oint_{C_r} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it}\,d\theta = 2\pi i$$

More generally, if $C$ is a positively-oriented simple closed contour not containing $z_0$, then

$$\oint_C \frac{1}{z - z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ lies inside } C \\ 0 & \text{if } z_0 \text{ lies outside } C \end{cases}$$

This example generalizes to perhaps the most powerful result in complex analysis…

**Theorem 4.24 (Cauchy's Integral Formula).** *Suppose $f$ is holomorphic everywhere on and inside a simple closed contour $C$. If $z_0$ is any point inside $C$, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

*More generally, $f$ is infinitely differentiable at $z_0$ with $n^{th}$ derivative*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

**Example 4.25.** If $f(z)$ is a polynomial, we can check the integral formula explicitly by appealing to the Cauchy–Goursat Theorem and Exercise 4.1.9: if $C$ is a simple closed contour encircling $z_0$, then

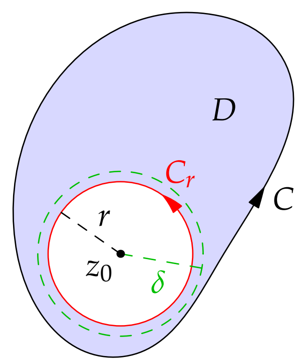
$$\oint_C (z-z_0)^{n-1}\,dz = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

For instance, if $f(z) = 3z^2+2$ and $z_0 = 0$, then

$$\frac{2!}{2\pi i}\oint_C \frac{3z^2+2}{z^3}\,dz = \frac{2}{2\pi i}\oint_C \frac{3}{z}\,dz + \frac{2}{2\pi i}\oint_C \frac{2}{z^3}\,dz = 6+0 = f''(0)$$

*Proof of the basic Integral formula*[14]. Let $\epsilon > 0$ be given. Denote by $D$ the open region interior to $C$. Since $f$ is holomorphic it is also continuous. Combining these:

$$\exists\delta > 0 \text{ such that } |z-z_0| < \delta \implies z \in D \text{ and } |f(z) - f(z_0)| < \frac{\epsilon}{2}$$




Draw a circle $C_r$ of radius $r < \delta$ centered at $z_0$. This lies entirely in $D$. Since $\frac{f(z)}{z-z_0}$ is holomorphic between and on $C_r$ and $C$, Corollary 4.22 tells us that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_r} \frac{f(z)}{z-z_0}\,dz$$

We need only bound an integral to complete the proof:

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz - f(z_0)\right| &= \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(z)}{z-z_0}\,dz - \frac{f(z_0)}{2\pi i}\oint_{C_r}\frac{1}{z-z_0}\,dz\right| \\ &= \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \overset{\text{Thm}}{\underset{4.16}{\leq}} \frac{1}{2\pi}\cdot\frac{\epsilon}{2r}\cdot 2\pi r < \epsilon\end{aligned}$$

since $|z-z_0| = r$ and $|f(z)-f(z_0)| < \frac{\epsilon}{2}$ on $C_r$, which has circumference $2\pi r$. ∎

[14]Informally, it appears as if the general formula follows by repeated differentiation (induction):

$$f^{(n+1)}(z_0) = \frac{d}{dz_0} f^{(n)}(z_0) \overset{???}{=} \frac{n!}{2\pi i}\oint_C \frac{\partial}{\partial z_0}\frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{(n+1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+2}}\,dz$$

One cannot blindly bring a derivative inside an integral like this, so a formal proof is required; Exercise 13 offers a sketch.

**Examples 4.26.** We can use the integral formula to evaluate certain integrals that would be difficult if not impossible to evaluate by parametrization.

1. The integral formula makes Example 4.17.3 and its extensions very easy: apply the formula to $f(z) = \frac{2\pi i}{z+1}$ around (say) the circle with radius 1 centered at $z = 1$ to see that

$$\oint_C \frac{1}{z^2-1}\,dz = \frac{1}{2\pi i}\oint_C \frac{2\pi i}{(z+1)(z-1)}\,dz = f(1) = i\pi$$

   Indeed $\oint_C \frac{1}{z^2-1}\,dz = i\pi$ around any simple closed curve encircling $z = 1$ but not $z = -1$.

2. If $C$ is the circle centered at $z = i$ with radius 1, then $f(z) = \frac{3z\sin z}{z+i}$ is holomorphic on and inside $C$, whence

$$\begin{aligned}\oint_C \frac{3z\sin z}{z^2+1}\,dz &= \oint_C \frac{3z\sin z}{(z+i)(z-i)}\,dz = 2\pi i f(i) = 2\pi i\frac{3i\sin i}{2i} = 3\pi i\sin i\\ &= \frac{3}{2}(e^{-1}-e^{1})\end{aligned}$$

   Contrast this with parametrizing $z(t) = i + e^{it}$ and attempting to evaluate directly!

$$\int_0^{2\pi} \frac{3(i+e^{it})\sin(i+e^{it})ie^{it}}{(i+e^{it})^2+1}\,dt\ldots$$

3. Let $C$ be a simple closed contour staying away from $z_0 = 4$. Since $f(z) = \frac{3z^2+7}{e^z}$ is entire,

$$\oint_C \frac{3z^2+7}{e^z(z-4)}\,dz = \begin{cases} 0 & \text{if } z_0 = 4 \text{ is outside } C\\ 2\pi i f(4) = 110\pi i e^{-4} & \text{if } z_0 = 4 \text{ is inside } C\end{cases}$$

4. Let $C$ be the circle of radius 2 centered at $z = 1+i$. Then $g(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ is holomorphic on and inside $C$, except at $z = i$. We conclude that

$$\oint_C g(z)\,dz = \oint_C \frac{1}{(z+i)^3(z-i)^3}\,dz = \frac{2\pi i}{2!}\left.\frac{d^2}{dz^2}\right|_i (z+i)^{-3} = 12\pi i(2i)^{-5} = \frac{3\pi}{8}$$

We finish with a previously mentioned corollary of the integral formula.

**Corollary 4.27 (Theorem 2.33).** *Holomorphic functions are infinitely differentiable with all derivatives holomorphic. In particular, their real and imaginary parts have continuous partial derivatives of all orders.*

*Proof.* If $f(z)$ is holomorphic at $w$, then it is holomorphic on an open set $D$ containing $w$. We may therefore choose a circle $C_r$ centered at $w$ lying inside $D$.
The integral formula with $n = 2$ says that the second derivative $f''(z_0)$ exists at every point $z_0$ inside $C_r$. Otherwise said, the derivative $f'(z)$ is holomorphic at $w$.
Now induct to see the $f''(z)$ is holomorphic at $w$, etc. ∎

Finally, note that if $f$ has an anti-derivative $F$, then $F$ is necessarily holomorphic and so, by the Corollary, is $f$ itself. Combining our results yields the following summary.

**Theorem (Summary).** *Suppose $f$ is continuous on an open domain $D$ and that curves $C$ lie in $D$.*

$$\begin{array}{ccc} \text{all } \oint_C f(z)\,\mathrm{d}z = 0 & \xLeftarrow[\text{(if } D \text{ simply-connected)}]{\text{Cauchy–Goursat}} & f \text{ holomorphic on } D \\ \text{Lemma 4.13} \Updownarrow & & \Uparrow \text{Cauchy Integral Formula} \\ \text{all } \int_C f(z)\,\mathrm{d}z \text{ path-independent} & \xLeftrightarrow[\text{Fundamental Thm}]{} & f \text{ has an anti-derivative on } D \end{array}$$

**Exercises 4.3.** 1. Apply the Cauchy–Goursat Theorem to show that $\oint_C f(z)\,\mathrm{d}z = 0$ when the contour $C$ is the unit circle $|z| = 1$.

(a) $f(z) = \dfrac{z^2}{z+3}$ (b) $f(z) = ze^{-z}$ (c) $f(z) = \mathrm{Log}(z+2)$

2. Let $C_1$ be the square with sides the lines $x = \pm 1, y = \pm 1$, and $C_2$ the circle $|z| = 4$. Explain why

$$\oint_{C_1} \frac{1}{3z^2+1}\,\mathrm{d}z = \oint_{C_2} \frac{1}{3z^2+1}\,\mathrm{d}z$$

3. Let $C$ be the square with sides $x = 0, 1$ and $y = 0, 1$. Evaluate the integral $\oint_C \frac{1}{z-a}\,\mathrm{d}z$ when:

(a) $a$ is *exterior* to the square;

(b) $a$ is *interior* to the square.

4. Let $C$ be the positively-oriented boundary of the half-disk $0 \le r \le 1$, $0 \le \theta \le \pi$ and define $f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}$ and $f(0) = 0$ using the branch of $z^{1/2}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Prove that

$$\oint_C f(z)\,\mathrm{d}z = 0$$

by evaluating three contour integrals: over the semicircle, and over two segments of the real axis joining 0 to $\pm 1$. Why does Cauchy–Goursat not apply here?

5. If $C$ is a positively-oriented simple closed contour, prove that the area enclosed by $C$ is

$$\frac{1}{2i}\oint_C \overline{z}\,\mathrm{d}z$$

(*Hint: Mirror the sketch proof of Cauchy–Goursat, even though $\overline{z}$ isn't holomorphic…*)

6. Let $C$ denote the boundary of the square with sides $x = \pm 2, y = \pm 2$. Evaluate the following:

(a) $\displaystyle\oint_C \frac{e^{-\frac{\pi z}{2}}\,\mathrm{d}z}{z-i}$ (b) $\displaystyle\oint_C \frac{e^z + e^{-z}}{z(z^2+10)}\,\mathrm{d}z$ (c) $\displaystyle\oint_C \frac{z\,\mathrm{d}z}{3z+i}$ (d) $\displaystyle\oint_C \frac{\sec(z/2)}{(z-1-i)^2}\,\mathrm{d}z$

7. Evaluate the integral $\oint_C g(z)\,\mathrm{d}z$ around the circle of radius 3 centered at $z = i$ when:

(a) $g(z) = \dfrac{1}{z^2+9}$ (b) $g(z) = \dfrac{1}{(z^2+9)^2}$

8. Prove that if $f$ is holomorphic on and inside a simple closed contour $C$ and $z_0$ is not on $C$, then

$$\oint_C \frac{f'(z)}{z-z_0}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

9. Properly justify the claim at the end of Example 4.26.1, that

$$\oint_C \frac{1}{z^2-1}\mathrm{d}z = i\pi$$

when $C$ is *any* simple closed curve encircling $z=1$ but not $z=-1$. What happens if instead $C$ encircles $z=-1$ but not $z=1$?

10. Suppose we have a polynomial $p(z) = \sum\limits_{k=0}^{n} a_k(z-z_0)^k$ centered at $z_0$. Use Cauchy's integral formula to prove that $a_k = \frac{p^{(k)}(z_0)}{k!}$ is the usual Taylor coefficient.

11. Let $C$ be the unit circle $z = e^{i\theta}$ where $-\pi < \theta \le \pi$ and suppose $a \in \mathbb{R}$ is constant. By first evaluating $\oint_C z^{-1}e^{az}\,\mathrm{d}z$, prove that

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,\mathrm{d}\theta = \pi$$

12. (a) Suppose that $f(z)$ is *continuous* on and inside a simple closed contour $C$. Prove that the function $g(z)$ defined by

$$g(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z$$

is holomorphic at every point $z_0$ inside $C$ and that

$$g'(z_0) := \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

(*Hint: consider the formula in part (b) of the next question, after replacing the first two $f$'s by $g$*)

(b) If $f(z) = x(1-x)(1-y)$ and $C$ is the square with vertices $0, 1, 1+i, i$ compute $g(z_0)$.

13. We prove the 1st derivative version of Cauchy's integral formula. As previously, let $\delta > 0$ be such that $|w - z_0| < \delta \implies w \in D$.

(a) If $|\Delta z| < \delta$ and $z \in C$ explain why

i. $|z - z_0| \ge \delta$

ii. $|z - z_0 - \Delta z| > 0$

(b) Use the basic integral formula on $C$ to evaluate $f(z_0+\Delta z) - f(z_0)$ and thus prove that

$$\left|\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\right| \le \frac{ML}{2\pi(\delta-\Delta z)\delta^2}\,|\Delta z|$$

where $M$ is an upper bound for $|f(z)|$ on $C$, and $L$ is the length of $C$. Hence conclude that $f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$.

If you want a challenge, try to do the same thing for higher derivatives!

## 4.4 Liouville's Theorem and The Maximum Modulus Principle

In this section we deduce several powerful corollaries of Cauchy's integral formula. The first is a straightforward combination of the integral formula with integral estimation (Theorem 4.16).

**Lemma 4.28 (Cauchy's Inequality).** *If $f$ is holomorphic on and inside the circle $C$ of radius $R$ centered at $z_0$, and $|f(z)| \le M$ on $C$, then*

$$\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z\right| \le \frac{n!M}{R^n}$$

This seems innocuous, but it has surprising applications. If $f$ is entire and bounded ($|f(z)| \le M$ for all $z$), then Cauchy's inequality applies for *any* circle centered at *any* $z_0$:

$$\forall z_0 \in \mathbb{C}, R > 0,\ \left|f'(z_0)\right| \le \frac{M}{R} \implies \forall z_0,\ f'(z_0) = 0 \overset{\text{Thm}}{\underset{2.31}{\implies}} f(z) \text{ constant}$$

**Theorem 4.29 (Liouville).** *Every bounded, entire function is constant.*

Liouville's Theorem is the key ingredient in perhaps the simplest proof of a very famous result.

**Corollary 4.30 (Fund. Thm. of Algebra).** *Every non-constant polynomial has a root in $\mathbb{C}$.*

*Proof.* WLOG suppose a non-constant polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ is monic. We assume, for contradiction, that $p(z) = 0$ has no solutions. It follows that is $\frac{1}{p(z)}$ is *entire*. Our strategy is to establish that $\frac{1}{p(z)}$ is also *bounded*; since $p(z)$ is non-constant, this contradicts Liouville's Theorem.
In fact it is enough to bound $\frac{1}{p(z)}$ for all *large* $z$: that is, we want to find some $R > 0$ such that

$$|z| > R \implies \frac{1}{p(z)} \text{ bounded}$$

This is since $\frac{1}{p(z)}$, being continuous, is automatically bounded on any disk $|z| \le R$ (compactness, Theorem 2.21). So how do we find a suitable $R$? Everything follows from the triangle inequality,

$$|z| > R \implies |p(z)| = |z|^n \left|1 + \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \ge |z|^n \left|1 - \left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right|\right|$$

To complete the proof, we need only choose $R$ such that $\left|\sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}}\right| \le \sum_{k=0}^{n-1} \frac{|a_k|}{|z|^{n-k}} \le \frac{1}{2}$, for then

$$|z| > R \implies |p(z)| \ge \frac{1}{2}R^n \implies \frac{1}{|p(z)|} \le \frac{2}{R^n}$$

is bounded. It is sufficient to force each term in the above sum to be $\le \frac{1}{2n}$, as may be accomplished (see Exercise 8) by defining

$$R := \max\left\{(2n\,|a_k|)^{\frac{1}{n-k}} : 0 \le k < n\right\}$$

■

The proof used the fact that a continuous function $f$ on a compact (closed, bounded) domain $K$ is bounded (Theorem 2.21). As you should recall from real analysis, the least upper bound is achieved:

$$\exists z_0 \in K \text{ such that } |f(z_0)| = \sup\{|f(z)| : z \in K\}$$

For *holomorphic* functions, a more restrictive and surprising result holds: the maximum modulus of a non-constant such function on a compact domain is always and only achieved at an *edge point*.

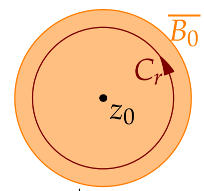
**Example 4.31.** Let $f(z) = e^z$ on the unit disk $|z| \le 1$. Then $|f(z)| = e^x$, whence the maximum modulus $|f(1)| = e$ occurs at $z = 1$, on the edge of the disk.

The theorem establishing this important observation is usually stated a little differently.

**Theorem 4.32 (Maximum Modulus Principle).** *Suppose $f$ is holomorphic and non-constant on a* connected, open *domain $D$. Then $|f(z)|$ has no maximum value on $D$.*

Holomorphicity is critical: see, for instance, Exercise 5.

*Proof.* We start by demonstrating a special case.




Fix $\delta > 0$, assume $f$ is holomorphic on and inside $\overline{B_0} := \{z : |z - z_0| \le \delta\}$ and that $|f(z)|$ attains its maximum at $z_0$. We prove that $f(z)$ is *constant* on $\overline{B_0}$.
Let $r \le \delta$ and apply the integral formula to the circle $C_r$ $(z(t) = z_0 + re^{it})$:

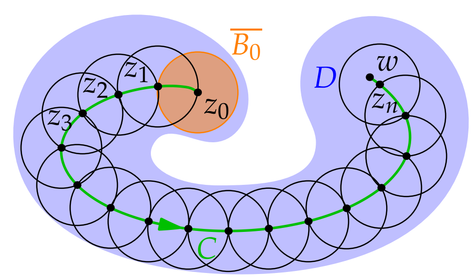
$$\begin{aligned}
|f(z_0)| &= \frac{1}{2\pi}\left|\oint_{C_r} \frac{f(z)}{z - z_0}\,\mathrm{d}z\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\,\mathrm{d}t\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right| \mathrm{d}t && \text{(Theorem 4.16)} \\
&\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,\mathrm{d}t = |f(z_0)| && (|f(z_0)| \text{ is the maximum of } |f(z)|)
\end{aligned}$$

Both inequalities are therefore *equalities*, the second of which now becomes

$$\int_0^{2\pi} |f(z_0)| - \left|f(z_0 + re^{it})\right| \mathrm{d}t = 0$$

Since the integrand is non-negative and continuous, it must be zero. But then $|f(z)| = |f(z_0)|$ on *all such circles* $C_r$, whence $|f(z)|$ is *constant* on $\overline{B_0}$. By Corollary 2.32, $f(z)$ is also constant on $B_0$.

Now suppose $f$ is holomorphic on some connected open set $D$ and that $|f(z)|$ attains its maximum at $z_0$. Let $w \in D$ be any other point and join $z_0$ to $w$ by a smooth arc $C$.

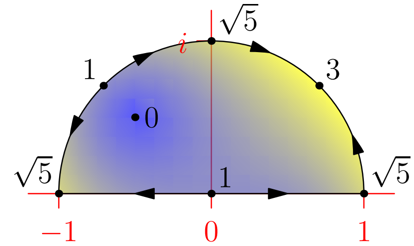



Let $\delta$ be the tube-radius from Exercise 2.2.10. We may therefore cover $C$ with *finitely many* closed disks $\overline{B_0}, \overline{B_1}, \ldots, \overline{B_n} \subset D$ of radius $\delta$, indexed so that each center $z_k$ lies in one of the previous disks (at most $\frac{\text{length}(C)}{\delta}$ disks are needed).

By the special case, $f(z)$ is constant on $\overline{B_0}$. Since $z_1 \in C \cap \overline{B_0}$, we see that $|f(z_1)| = |f(z_0)|$ is also the maximum modulus on $D$. The special case now says that $f(z)$ is constant on $\overline{B_0} \cup \overline{B_1}$.

Continuing in this fashion (essentially induction), we conclude that $f(z) = f(z_0)$ is constant on $\overline{B_0} \cup \cdots \cup \overline{B_n}$. In particular $f(w) = f(z_0)$. Since $w$ was arbitrary, $f(z)$ is constant on $D$. ∎

**Example 4.33.** Let $f(z) = 2z^2 + i$ on the upper semi-disk with radius 1. There are two boundaries:




- $y = 0$: plainly $|f(z)| = \sqrt{4x^4 + 1}$ is maximal at $z = \pm 1$.
- $r = 1$: write $f(z) = 2e^{2i\theta} + i$ from which

$$|f(z)| = \sqrt{(2\cos 2\theta)^2 + (2\sin 2\theta + 1)^2} = \sqrt{5 + 4\sin 2\theta}$$

which is maximal when $\theta = \frac{\pi}{4}$ with $\left|f(e^{\frac{i\pi}{4}})\right| = 3$.

The color indicates the value of $|f(z)|$, and the arrows its direction of increase on the boundary.

**Exercises 4.4.** 1. (a) Suppose $f$ is entire and that $|f(z)| \le c\,|z|$ for some constant $c \in \mathbb{R}^+$. Prove that $f(z) = kz$ where $k \in \mathbb{C}$ satisfies $|k| \le c$.

(b) What can you say about $f$ if it is entire and there exists some linear polynomial $cz + d$ with $c \ne 0$ such that $|f(z)| \le |cz + d|$ for all $z \in \mathbb{C}$?

2. If $f(z) = u + iv$ is entire and $u(x, y)$ is bounded above, apply Liouville's Theorem to $\exp(f(z))$ to prove that $u$ is constant.

3. Let $f(z)$ be a *non-zero* holomorphic function on a closed bounded domain. By considering $g(z) = \frac{1}{f(z)}$, show that the minimum value of $|f(z)|$ also occurs on the boundary.

4. Find the maximum and minimum values of $\left|z^2 + 4i\right|$ on the unit disk $|z| \le 1$.

5. Consider $f(z) = \exp(-|z|^2)$ defined on the unit disk $|z| \le 1$. What is its maximum modulus, and where is it found? Why doesn't this contradict the maximum modulus principle?

6. (a) On the rectangle $0 \le x \le \pi, 0 \le y \le 1$, show that $|\sin z|$ is maximal at the point $z = \frac{\pi}{2} + i$. (*Hint: first show that* $|\sin z|^2 = \sin^2 x + \sinh^2 y$)

(b) Prove that there exists a sequence $(z_n) \subset \mathbb{C}$ such that $\lim \sin z_n = \infty$.

7. Revisit the standard method from multivariable calculus (compute $\nabla\,|f(z)|^2$) to check that the maximum value of $\left|2z^2 + i\right|$ in Example 4.33 really does occur on the boundary.

8. Complete the proof of the fundamental theorem of algebra:

(a) Verify that $R := \max\left\{(2n\,|a_k|)^{\frac{1}{n-k}} : 0 \le k < n\right\}$ is *positive.*

(b) Prove that $|z| > R \implies \forall k,\ \frac{|a_k|}{|z|^{n-k}} < \frac{1}{2n}$.

9. (a) Prove the factor theorem: if $p(z_1) = 0$, then $p(z) = (z - z_1)q(z)$ for some polynomial $q(z)$. (*Hint: This follows easily from the division algorithm: if* $\deg p \ge \deg g$*, then there exist unique polynomials* $q(z), r(z)$ *for which*

$$p(z) = g(z)q(z) + r(z) \quad \textit{and} \quad \deg r < \deg g$$

*For a challenge, prove some version of this sufficient for our needs...*)

(b) Prove that a degree $n \ge 1$ polynomial $p(z)$ factors uniquely over $\mathbb{C}$: up to order, there exist unique $a, z_1, \ldots, z_n \in \mathbb{C}$ such that

$$p(z) = a(z - z_1)\cdots(z - z_n)$$

# 5 Power Series, Taylor's Theorem & Analytic Functions

The theory of series in complex analysis differs significantly from the real situation, particularly with regard to two concepts.

- Taylor's Theorem: Holomorphic functions *equal* their Taylor series. This is false in real analysis where differentiable functions need not have, nor equal, a Taylor series.
- Laurent Series (chapter 6): series with negative powers $(z^{-1} + 3z^{-2} + \cdots)$ are a powerful tool.

We start be reviewing the basics of both infinite and power series: as we saw for sequences in Section 2.2, this is essentially identical to the real situation.

## 5.1 A Brief Review of Power Series

**Definition 5.1 (Infinite Series).** The $n^{th}$ *partial sum* of a sequence $(z_n)_{n=0}^{\infty}$ is the complex number

$$s_n = \sum_{k=0}^{n} z_k = z_0 + \cdots + z_n$$

The *(infinite) series* $\sum z_n := \lim s_n$ is said to converge/diverge if the sequence $(s_n)$ converges/diverges.
The series *converges absolutely* if $\sum |z_n|$ converges.
The series *converges conditionally* if it converges but not absolutely.

By convention, the initial term of the sequence/series is $z_0$. This isn't required: it could be $z_1$, etc.

**Theorem 5.2 (Basic Series Facts).** *Let $\sum z_n$ and $\sum w_n$ be series of complex numbers.*

1. *If $z_n = x_n + iy_n$, then $\sum z_n$ converges if and only if $\sum x_n$ and $\sum y_n$ both converge, in which case*
$$\sum z_n = \sum x_n + i \sum y_n$$
2. *If $a, b \in \mathbb{C}$, and $\sum z_n$ and $\sum w_n$ converge, then $\sum az_n + bw_n$ converges, in which case*
$$\sum az_n + bw_n = a \sum z_n + b \sum w_n$$
3. *($n^{th}$ term/divergence test) If $\sum z_n$ converges, then $\lim z_n = 0$.*
4. *The (real!) comparison, ratio and root tests apply to the series $\sum |z_n|$.*
5. *Absolute convergence implies convergence; moreover $|\sum z_n| \leq \sum |z_n|$.*

*Proof.* 1. This is Theorem 2.8, part 1.

2, 3. These follow from 1 and the corresponding results for the real series $\sum x_n, \sum y_n$.

4. This requires no proof: $\sum |z_n|$ is a series of non-negative real numbers, so the real version apply!

5. Since $|x_n|, |y_n| \leq |z_n|$, the (real) comparison test says that $\sum x_n$ and $\sum y_n$ are absolutely convergent and thus convergent. By part 1, $\sum z_n$ converges. Finally, apply the triangle inequality $\left|\sum_{k=0}^{m} z_k\right| \leq \sum_{k=0}^{m} |z_k| \leq \sum_{n=0}^{\infty} |z_n|$ and take limits as $m \to \infty$. ■

**Definition 5.3 (Power Series & Analyticity).** A *power series centered at* $z_0$ is a function of the form

$$p(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad (z_0 \text{ and the } \textit{coefficients } a_n \text{ are constants})$$

A function $f : D \to \mathbb{C}$ is *analytic* if every $z_0 \in D$ has a neighborhood on which $f(z)$ equals a power series centered at $z_0$. That is,

$$\forall z_0 \in D,\ \exists \delta > 0,\ (a_n) \text{ such that } |z-z_0| < \delta \implies f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

To be analytic at a point $z_0$ is to be analytic on some neighborhood of $z_0$.

**Example 5.4 (Geometric series).** By the $n^{\text{th}}$ term test, the power series $\sum z^n$ diverges when $|z| \geq 1$. Otherwise, inside the unit circle $|z| < 1$ we have $z^{n+1} \to 0$, from which

$$s_n - zs_n = 1 - z^{n+1} \implies s_n = \frac{1-z^{n+1}}{1-z} \implies \sum_{n=0}^{\infty} z^n = \lim s_n = \frac{1}{1-z}$$

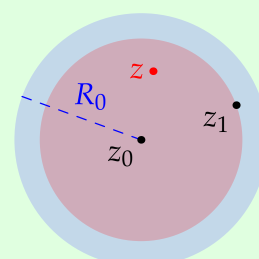
In fact $f(z) = \frac{1}{1-z}$ is analytic on its whole domain $\mathbb{C} \setminus \{1\}$: the substitution $z \mapsto \frac{z-z_0}{1-z_0}$ ($z_0 \neq 1$) to see that $\frac{1}{1-z}$ equals a power series centered at $z_0$:

$$\frac{1}{1-z} = \frac{1}{1-z_0} \cdot \frac{1}{1-\frac{z-z_0}{1-z_0}} = \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0}\right)^n \quad \text{whenever} \quad |z-z_0| < |1-z_0|$$

Regardless of the center $z_0$, observe how the geometric series converges on a *disk* (radius $|1-z_0|$). This behavior in fact happens for *every* power series, analogous to the interval/radius of convergence discussion from real analysis.

**Theorem 5.5 (Radius of Convergence).** *Suppose* $p(z) = \sum a_n(z-z_0)^n$.

1. *If* $p(z)$ *converges at* $z_1 \neq z_0$*, then it is absolutely convergent at every every* $z$ *satisfying* $|z-z_0| < |z_1-z_0|$.
2. *Define the* ***radius of convergence*** $R_0 := \sup\{|z-z_0| : p(z) \text{ converges}\}$. *Then* $p(z)$ *converges absolutely whenever* $|z-z_0| < R_0$*, and diverges whenever* $|z-z_0| > R_0$.




*Proof.* 1. By the $n^{\text{th}}$ term test, the sequence $\left(a_n(z_1-z_0)^n\right)$ converges (to 0); it is therefore bounded by some $M \in \mathbb{R}^+$. But then

$$|a_n|\,|z-z_0|^n = |a_n|\,|z_1-z_0|^n \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n \leq Mr^n \quad \text{where} \quad r = \frac{|z-z_0|}{|z_1-z_0|} < 1$$

Since $\sum Mr^n$ converges, the comparison test says that $\sum |a_n|\,|z-z_0|^n$ converges.

2. If $|z-z_0| < R_0$, then $\exists z_1$ such that $|z-z_0| < |z_1-z_0| \leq R_0$ and $p(z_1)$ converges; now apply part (a). The divergence condition is an exercise.

■

A power series therefore has a *disk of convergence.* As in real analysis, we have to test convergence separately on the boundary circle $|z - z_0| = R_0$; a key technique for this is *Abel's Test* (Exercise 4). Note particularly the two extreme cases:

- If $R_0 = \infty$, the series is absolutely convergent on $\mathbb{C}$.
- If $R_0 = 0$, the series converges only when $z = z_0$.

As in real analysis, we could also compute $R_0 = \liminf |a_n|^{-1/n}$, though for us this will mostly be redundant since the condition observed in Exercise 1 (proved generally in Section 5.3) is often obvious.

**Exercises 5.1.** 1. Sketch the disks of convergence with centers $z_0 = -1, 1+i, 3-2i$ for the function $f(z) = \frac{1}{1-z}$ (Example 5.4). Complete the following observation:

$R_0 = |1 - z_0|$ is the distance from $z_0$ to the nearest point at which $f(z)$ is ____________

2. By mimicking Example 5.4, find a power series centered at $z_0 \neq i$ which equals the function $g(z) = \frac{2}{1+iz}$. What is its radius of convergence?

3. Revisit the proof of Theorem 5.5.

   (a) Complete part 2: If $|z - z_0| > R_0$, prove that $p(z) = \sum a_n (z - z_0)^n$ diverges.

   (b) In part 1, explain why we couldn't simply use the comparison test to say

   $$|a_n| \, |z - z_0|^n < |a_n| \, |z_1 - z_0|^n \implies \sum |a_n| \, |z - z_0|^n \text{ converges}$$

   (*Hint: Think carefully about the hypothesis!*)

4. (*Abel's Test*) In real analysis, the alternating series test was often used to decide convergence at the endpoints of an interval of convergence. Here is a generalization to the complex situation.

   Consider the power series $\sum a_n z^n$ where $(a_n)$ is a *real* sequence such that

   $$a_n \geq 0, \quad a_{n+1} \leq a_n, \quad \lim_{n \to \infty} a_n = 0$$

   (a) Write $s_n(z) = \sum_{k=0}^{n} a_k z^k$ for the partial sum and prove that

   $$(1 - z) s_n(z) = a_0 - a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k$$

   (b) Prove *Abel's Test*: $\sum a_n z^n$ converges on the *closed* disk $|z| \leq 1$, *except perhaps* when $z = 1$.
   (*Hint: Show that* $\sum (a_k - a_{k-1}) z^k$ *converges absolutely by comparison with a telescoping series*)

   (c) i. Find the disk of convergence of $\sum \frac{z^n}{n}$ (all $z \in \mathbb{C}$ at which the series converges).

   ii. Prove that the real series $\sum \frac{\cos n\theta}{n}$ converges except when $\theta$ is divisible by $2\pi$. For what values of $\theta$ does the series $\sum \frac{\sin n\theta}{n}$ converge?
   (*Hint: Use part (i)...*)

   (d) Find all values of $z$ for which the series $\sum \frac{1+i}{(n+i)(4+3i)^n} (z - 1 + 2i)^n$ converges and sketch the disk of convergence.
   (*Hint: Let* $w = \frac{z-1+2i}{4+3i}$ *and think about real and imaginary parts*)

## 5.2 Taylor Series and Taylor's Theorem

The overarching goal of the next two sections is the establishment of a key result:

A function is holomorphic if and only if it is analytic

Otherwise said, $f(z)$ is differentiable on an open domain $D$ if and only if for each $z_0 \in D$ there is some neighborhood of $z_0$ on which it equals a power series $f(z) = \sum a_n(z-z_0)^n$.

If a holomorphic function is to equal a power series, it is natural ask *which one*? The answer revisits a familiar definition and leads to a startling difference between the real and complex cases.

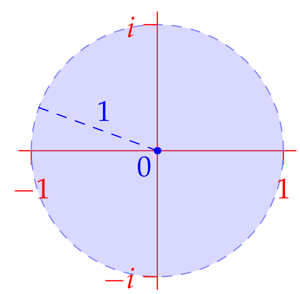
**Definition 5.6.** If $f(z)$ is infinitely differentiable at $z_0$, then its *Taylor series* is the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

The *Taylor coefficients* are $a_n = \frac{f^{(n)}(z_0)}{n!}$. A *Maclaurin series* is a Taylor series centered at $z_0 = 0$.

**Example (5.4, cont.).** On the disk $|z| < 1$, the function $f(z) = \frac{1}{1-z}$ has

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n = f(z)$$




The Maclaurin series is precisely the geometric power series representation of $f(z)$ on the open disk $|z| < 1$!

In complex analysis, this situation is completely general...

**Theorem 5.7 (Taylor's Theorem).** *Suppose $f(z)$ is holomorphic on a disk $|z-z_0| < R$. Then,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \textit{whenever} \quad |z-z_0| < R$$

In comparison to real analysis, this is a *very* strong statement: in the real case, Taylor's Theorem is usually stated with one of several awkward remainder terms, and there exist infinitely differentiable functions which do not equal their Taylor series (see Exercise 4).

Plainly $R$ cannot be larger than the radius of convergence $R_0$ of the Taylor series. If $f$ is entire, then the result holds for all positive $R$ and the series has infinite radius of convergence.

**Examples 5.8.** Familiar Maclaurin series are identical to real analysis:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Since $e^z$, $\sin z$ and $\cos z$ are entire, we may take $R = \infty$ in Taylor's Theorem: each function equals its Maclaurin series everywhere on $\mathbb{C}$.

Before seeing the proof of Taylor's Theorem, we apply it to quickly deduce half of our key result.

If $f : D \to \mathbb{C}$ is holomorphic, then, for every $z_0 \in D$, there exists some disk $|z - z_0| < R$ on which $f$ is holomorphic. By Taylor's Theorem, $f(z)$ equals its Taylor series on this disk. Otherwise said:

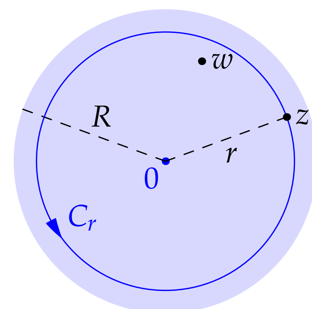
**Corollary 5.9.** *Every holomorphic function is analytic.*

We'll obtain the converse in the next section (Corollary 5.15).

Why is Taylor's Theorem so much more specific in complex analysis? The reason is that we have can apply a powerful tool unavailable in real analysis: Cauchy's integral formula.

*Proof of Taylor's Theorem.* By relabelling $\tilde{f}(z) = f(z - z_0)$, it is enough to prove when $z_0 = 0$, that is for Maclaurin series. We therefore suppose $f(z)$ is holomorphic when $|z| < R$.

Let $w$ be given where $|w| < R$. Apply the geometric series formula (Example 5.4), to see that if $z \neq 0$, then




$$\frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k = \frac{1-\left(\frac{w}{z}\right)^n}{z-w} \implies \frac{1}{z-w} = \frac{1}{z}\sum_{k=0}^{n-1}\frac{w^k}{z^{k+1}} + \frac{1}{z-w}\left(\frac{w}{z}\right)^n$$

Let $C_r$ be a circle centered at the origin with radius satisfying $|w| < r < R$. Since 0 and $w$ both lie inside $C_r$, we may apply Cauchy's integral formula *twice*, using the previous expression to substitute.

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } w) \\ &= \sum_{k=0}^{n-1}\frac{w^k}{2\pi i}\oint_{C_r}\frac{f(z)}{z^{k+1}}\,\mathrm{d}z + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(subsititute for } \tfrac{1}{z-w}) \\ &= \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } 0) \end{aligned}$$

To finish the proof, we need to control the final integral. Since $f$ is holomorphic on $C_r$, it is bounded by some $M > 0$. Moreover, for $z \in C_r$ we have $|z - w| \geq ||z| - |w|| = r - |w|$. It follows that

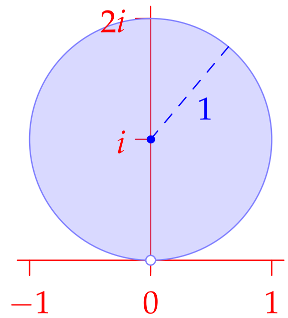
$$\begin{aligned} \left|f(w) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k\right| &= \frac{|w|^n}{2\pi}\left|\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z\right| \\ &\leq \frac{|w|^n M\cdot 2\pi r}{2\pi r^n(r-|w|)} = \frac{Mr}{r-|w|}\left(\frac{|w|}{r}\right)^n \end{aligned}$$

Since $|w| < r$, this last converges to zero. Otherwise said

$$f(w) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n$$

so that $f(w)$ equals its Maclaurin series whenever $|w| < R$. ∎

**Example 5.10.** The principal logarithm $f(z) = \operatorname{Log} z$ is holomorphic on the open disk $|z-i| < 1$.




Whenever $n \geq 1$, we have

$$f^{(n)}(i) = \left.\frac{(-1)^{n-1}(n-1)!}{z^n}\right|_{z=i} = -i^n(n-1)!$$

By Taylor's Theorem, $f(z)$ equals its Taylor series on the disk:

$$\operatorname{Log} z = \operatorname{Log} i - \sum_{n=1}^{\infty} \frac{i^n}{n}(z-i)^n = \frac{\pi i}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n}$$

In case you're feeling skeptical, convergence *inside* the disk ($|z-i| < 1$) can be verified directly using the comparison test:

$$|z-i| = r < 1 \implies \frac{|z-i|^n}{n} \leq r^n \implies \sum \frac{i^n(z-i)^n}{n} \text{ converges absolutely}$$

At $z = 0$, we recognize the divergent harmonic series $\sum \frac{1}{n}$, so the radius of convergence of the Taylor series is in fact $R_0 = 1$. Exercise 3 shows that the series converges everywhere else on the boundary circle $|z-i| = 1$.

**Exercises 5.2.** 1. (a) Compute the Maclaurin series of $\cos z$ directly from the definition.

(b) Evaluate the Taylor series of $\sin z$ about $z_0 = \frac{\pi}{2}$ and confirm that it equals your answer to part (a) when $z$ is replaced with $z - \frac{\pi}{2}$.

(c) Find the Taylor series of $\cos z$ centered at $z_0 = i$.

2. Consider $f(z) = \frac{1}{z}$. For any $z_0 \neq 0$, find the Taylor series of $f(z)$ about $z_0$. What is its disk of convergence?

3. Use Abel's test (Exercise 5.1.4) to verify that the Taylor series for $\operatorname{Log} z$ centered at $z_0 = i$ (Example 5.10) converges everywhere on the boundary circle $|z-i| = 1$, except when $z = 0$.

4. Consider the function

$$f(z) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

When $z \in \mathbb{R}$ this provides the classic example of an infinitely differentiable function whose Maclaurin series (being identically zero) does not equal the original function except at the origin. When $z \in \mathbb{C}$, explain why $f(z)$ does not contradict Taylor's Theorem.

### 5.3 Uniform Convergence: Continuity, Integrability and Differentiability

As in real analysis, we'd like to establish some useful facts:

**Uniqueness of representation** Equal power series have identical coefficients. Otherwise said, on a given disk $|z - z_0| < R$, a function can only equal at most one power series $\sum a_n(z - z_0)^n$ (its Taylor series).

**Computability** Power series are continuous, differentiable and integrable inside their disk of convergence. They can moreover be differentiated and integrated term-by-term like polynomials.

The arguments are intertwined. Since the first results are identical to the real case, we will be brief, postponing all examples until the end. The critical ingredient is uniform convergence.

**Definition 5.11.** Suppose $f(z) = \sum a_n(z - z_0)^n$ is a power series with $n^{\text{th}}$ partial sum $s_n(z)$ and remainder $\rho_n(z) = f(z) - s_n(z)$. We say that the series *converges uniformly* on a domain $D$ if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N,\ z \in D \implies |\rho_n(z)| < \epsilon$$

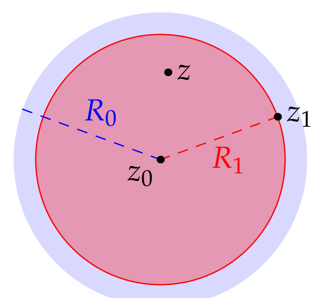
*Uniformity* means that $N = N(\epsilon)$ is independent of the location $z \in D$. If $N = N(\epsilon, z)$ were permitted to depend on $z$, we'd refer to the convergence as *pointwise.*

**Theorem 5.12.** *Let $R_0$ be the radius of convergence of a power series centered at $z_0$. If $R_1 < R_0$, then the series converges uniformly on the closed disk $|z - z_0| \le R_1$.*

This first argument also applies in real analysis.

*Proof.* As preparation, suppose $z_1$ satisfies $|z_1 - z_0| = R_1$. Since $R_1 < R_0$, the series converges absolutely at $z_1$ (Theorem 5.5). Denote by $\sigma_n$ the $n^{\text{th}}$ remainder of this absolutely convergent series:

$$\sigma_n = \sum_{k=n+1}^{\infty} |a_k|\,|z_1 - z_0|^k = \sum_{k=n+1}^{\infty} |a_k|\,R_1^k$$




Now suppose $\epsilon > 0$ is given. Since the above series converges, the $n^{\text{th}}$-term test says that $\lim_{n\to\infty} \sigma_n = 0$: that is,

$$\exists N \text{ such that } n > N \implies \sigma_n < \epsilon$$

By the comparison test, if $z$ satisfies $|z - z_0| \le R_1$, then

$$|\rho_n(z)| = \left|\sum_{k=n+1}^{\infty} a_k(z - z_0)^k\right| \le \sum_{k=n+1}^{\infty} |a_k|\,|z - z_0|^k \le \sum_{k=n+1}^{\infty} |a_k|\,R_1^k = \sigma_n < \epsilon$$

Since $N$ depends only on $\epsilon$ (not on $z$), we conclude that the convergence is uniform.[15] ∎

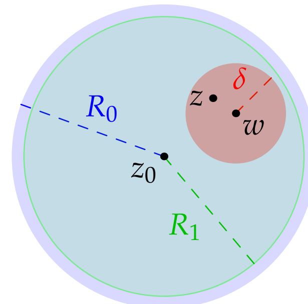
That $R_1$ is *strictly less* than the radius of convergence $R_0$ is important. In Exercise 9, we'll see that a power series need not converge uniformly on the full open disk of convergence $|z - z_0| < R_0$.

[15] It looks as if $N$ might also depend on the choice of $z_1$ in the first line. However, any suitable $z_1$ produces the same value $R_1 = |z_1 - z_0|$ and thus the same sequence $(\sigma_n)$. It is from the convergence of this sequence that we get $N$.

**Theorem 5.13 (Continuity).** *Suppose $f(z) = \sum a_n (z-z_0)^n$ has radius of convergence $R_0$. Then $f(z)$ is continuous on the open disk of convergence $|z-z_0| < R_0$.*

This is identical to the famous $\frac{\epsilon}{3}$-proof encountered in real analysis.

*Proof.* Fix $w$ and $R_1$ such that $|w-z_0| < R_1 < R_0$. Let $\epsilon > 0$ be given. Observe:




- Uniform convergence whenever $|z-z_0| \le R_1$ (Theorem 5.12):

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{3} \text{ and } |\rho_n(w)| < \frac{\epsilon}{3}$$

- Openness and continuity ($s_n$ is a polynomial!): for any $n > N$,

$$\exists \delta > 0 \text{ such that } |z-w| < \delta \implies \begin{cases} |z-z_0| < R_1 \\ |s_n(z) - s_n(w)| < \frac{\epsilon}{3} \end{cases}$$

Put everything together to see that $f(z)$ is continuous at $w$: for any $n > N$,

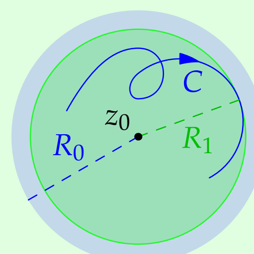
$$\begin{aligned} |z-w| < \delta \implies |f(z) - f(w)| &= |f(z) - s_n(z) + s_n(z) - s_n(w) + s_n(w) - f(w)| \\ &\le |\rho_n(z)| + |s_n(z) - s_n(w)| + |\rho_n(w)| < \epsilon \end{aligned}$$

∎

The discussion now differs from the real approach via our employ of *contour integrals*. The remaining results follow from a general version of term-by-term integration.

**Theorem 5.14.** *Suppose $f(z) = \sum a_n (z-z_0)^n$ has radius of convergence $R_0$. Let $g(z)$ be continuous on some contour $C$ in the open disk of convergence $|z-z_0| < R_0$. We may then integrate term-by-term:*

$$\int_C g(z) f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,\mathrm{d}z$$




*Proof.* The integral $\int_C g(z) f(z)\,\mathrm{d}z$ exists since $f, g$ are continuous on $C$. Since $C$ is a compact set:

- $C$ lies in some closed disk $|z-z_0| \le R_1 < R_0$ on which the series $f(z)$ converges uniformly.
- $g(z)$ is bounded on $C$ by some $M > 0$.

Let $C$ have length $L$ and let $\epsilon > 0$ be given. Since $f(z)$ converges uniformly when $|z-z_0| \le R_1$,

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{ML}$$

Now take integrals and moduli; if $n > N$, then

$$\begin{aligned} \left| \int_C g(z) f(z)\,\mathrm{d}z - \sum_{k=0}^{n} a_k \int_C g(z)(z-z_0)^k\,\mathrm{d}z \right| &= \left| \int_C g(z) \left( f(z) - \sum_{k=0}^{n} a_k (z-z_0)^k \right) \mathrm{d}z \right| \\ &= \left| \int_C g(z) \rho_n(z)\,\mathrm{d}z \right| < M \cdot \frac{\epsilon}{ML} \cdot L = \epsilon \end{aligned}$$

∎

Everything we want now follows by choosing specific functions $g(z)$ in Theorem 5.14!

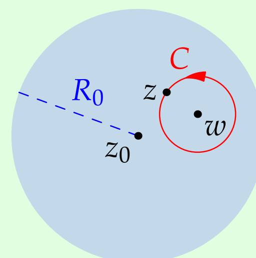
**Corollary 5.15.** *Suppose $f(z) = \sum a_n(z-z_0)^n$ has positive radius of convergence $R_0$.*

1. *(Term-by-term integration) Let $g(z) = 1$ to see that*

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}\Big|_{C(start)}^{C(end)}$$

2. *(Holomorphicity) By part 1, $\int_C f$ is path-independent for any contour $C$ in the open disk of convergence. We conclude (Summary, page 57) that $f$ is holomorphic on said disk. In particular,* ***every analytic function is holomorphic****.*

3. *(Term-by-term differentiation) Given $|w-z_0| < R_0$, let $g(z) = \frac{1}{2\pi i(z-w)^2}$ and apply Cauchy's integral formula on a small circle around $w$:*




$$f'(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-w)^2}\,\mathrm{d}z = \sum \frac{a_n}{2\pi i}\oint_C \frac{(z-z_0)^n}{(z-w)^2}\,\mathrm{d}z$$
$$= \sum a_n \frac{\mathrm{d}}{\mathrm{d}z}\Big|_{z=w}(z-z_0)^n = \sum a_n n(w-z_0)^{n-1}$$

4. *(Unique representation) The power series is the Taylor series of $f(z)$: that is, $a_n = \frac{f^{(n)}(z_0)}{n!}$.*

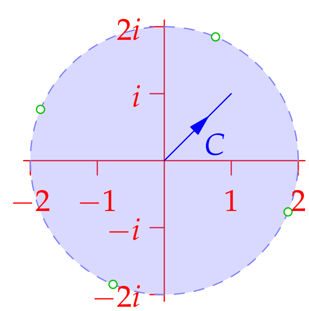
In Exercise 7 proves unique representation. Since analytic and holomorphic are now equivalent, we'll retire the latter term for the rest of these notes.

**Examples 5.16.** We may now compute Taylor & Maclaurin series using algebra, integration and differentiation: if a function equals a series, that's the one we want regardless of how we found it!

1. $f(z) = z^3 e^{z^2} = z^3 \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n+3}}{n!}$ is the Maclaurin series of $f(z)$. Since the radius of convergence is infinite, the function equals its Maclaurin series everywhere on $\mathbb{C}$.

2. The function $f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$ is entire since it equals the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$.

3. We find the Maclaurin series of $f(z) = \frac{1}{z^4+16i}$ algebraically:




$$f(z) = \frac{1}{16i\left(1 - \frac{z^4}{-16i}\right)} = \frac{1}{16i}\sum_{n=0}^{\infty}\left(\frac{z^4}{-16i}\right)^n = \sum_{n=0}^{\infty} \frac{i^{n-1}}{16^{n+1}} z^{4n}$$

This converges whenever $\left|\frac{z^4}{-16i}\right| < 1 \iff |z| < 2$, equalling the distance from the center to the nearest point(s) where $f(z)$ is undefined. If $C$ is the straight line from $z = 0$ to $z = 1+i$, then

$$\int_C f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{i^{n-1}}{16^{n+1}} \int_C z^{4n}\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{i^{n-1}(1+i)^{4n+1}}{16^{n+1}(4n+1)} = \sum_{n=0}^{\infty} \frac{1-i}{16(4n+1)}\left(\frac{-i}{4}\right)^n$$

**Zeros of Analytic Functions & the Analytic Continuation**

The theory we've developed has huge implications. As an example, we apply our discussion to the zeros of an analytic function. If a function is analytic at $z_0$, then (Definition 5.3, Theorem 5.7) there is some disk on which it equals its Taylor series:

$$|z-z_0|<R \implies f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad (*)$$

Suppose $f(z_0)=0$. We ask if/when the Taylor series starts to get interesting...

**Definition 5.17.** Suppose $z_0$ is a zero of an analytic function $f(z)$.

1. We say that $z_0$ is a *zero of order* $m\geq 1$ if $f^{(m)}(z_0)$ is the first non-zero derivative. A zero of order 1 is also called a *simple zero.*
2. If all derivatives are zero, $z_0$ is a *non-isolated zero*: plainly $f(z)\equiv 0$ on some disk $|z-z_0|<R$.

We are used to the idea of a polynomial having a zero $z_0$ if and only if we can factorize out $z-z_0$. The tight link-up with Taylor series makes essentially this observation hold for *any* analytic function!

**Theorem 5.18.** *An analytic function $f(z)$ has a zero $z_0$ of order $m$ if and only if $f(z)=(z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic at $z_0$ and $\psi(z_0)\neq 0$. Indeed, on some disk $|z-z_0|<R$,*

$$f(z)=\sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n=(z-z_0)^m\psi(z)$$

*Proof.* ($\Rightarrow$) Suppose $z_0$ is a zero of order $m$. Define

$$\psi(z)=\begin{cases}\dfrac{f(z)}{(z-z_0)^m} & \text{whenever } z\neq z_0\\[2ex] \dfrac{f^{(m)}(z_0)}{m!} & \text{when } z=z_0\end{cases}$$

On the disk $|z-z_0|<R$, $(*)$ says that this function equals the power series

$$\psi(z)=\sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

and is therefore analytic. Moreover, $f(z)=(z-z_0)^m\psi(z)$ everywhere.

($\Leftarrow$) If $\psi(z)$ is analytic at $z_0$, then it equals its Taylor series on some disk $|z-z_0|<R$. But then

$$f(z):=(z-z_0)^m\psi(z)=\sum_{n=0m}^{\infty}\frac{\psi^{(n)}(z_0)}{n!}(z-z_0)^{n+m}$$

is a power series on this disk. By uniqueness of representation (Corollary 5.15), this is the Taylor series of $f(z)$. If $\psi(z_0)\neq 0$, then $f^{(m)}(z_0)=m!\psi(z_0)$ is plainly the first non-zero derivative. ∎

**Examples 5.19.** 1. $f(z) = z^4(z-2i)^{10} = z^4\psi_1(z) = (z-2i)^{10}\psi_2(z)$ has two zeros:

*Order four at* $z_1 = 0$: $\psi_1(z) = (z-2i)^{10}$ is analytic with $\psi_1(0) = -1024 \neq 0$.

*Order ten at* $z_2 = 2i$: $\psi_2(z) = z^4$ is analytic with $\psi_2(2i) = 16 \neq 0$.

2. $g(z) = 17(z-4i)^3 \sin z$ has a zero of order three at $4i$, and simple zeros at each integer multiple of $\pi$. For instance, the Taylor series for sine centered at $z_0 = \pi$ yields

$$\begin{aligned} g(z) &= 17(z-4i)^3 \sin(z-\pi+\pi) = -17(z-4i)^3 \sin(z-\pi) \\ &= -17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-\pi)^{2n+1} = (z-\pi)\psi(z) \end{aligned}$$

where $\psi(z) = -17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-\pi)^{2n}$ is analytic and $\psi(\pi) = -17(\pi-4i)^3 \neq 0$.

The examples are typical in that the zeros of an analytic function tend to be *isolated*. Indeed, analytic functions with non-isolated zeros are very boring...

**Theorem 5.20.** *Suppose $z_0$ is a zero of an analytic function $f(z)$.*

1. *If $z_0$ has order $m$, then there exists a punctured disk $0 < |z-z_0| < R$ on which $f(z) \neq 0$. Otherwise said: $z_0$ is* ***isolated.***
2. *If $z_0$ is non-isolated and the domain $D$ of $f(z)$ is open and connected, then $f(z) \equiv 0$ on $D$.*

**Example 5.21.** Analyticity is crucial! For instance, the non-analytic function $f(z) = z + \bar{z} = 2x$ is zero precisely along the real axis: the zero is not isolated but neither is the function identically zero on any open domain. There really is something to prove here!

*Proof.* 1. Write $f(z) = (z-z_0)^m\psi(z)$, where $\psi$ is analytic and non-zero at $z_0$. Since $\psi$ is continuous, it is non-zero on some disk $|z-z_0| < R$.

2. As observed in Definition 5.17, $f(z) \equiv 0$ on some disk $|z-z_0| < R$.
To finish, we extend by an argument similar to that used to prove the Maximum Modulus Principle (Theorem 4.32). Given any $w \neq z_0$, join $z_0$ to $w$ by a path in $D$ which we cover by finitely many disks; since $f(z) \equiv 0$ on all these disks, we conclude that $f(w) = 0$. ∎

By applying part 2 to a function $f - g$, we obtain the proof of a powerful and previously heralded result.

**Corollary 5.22 (Theorem 2.36—Analytic Continuation).** *Suppose $f(z), g(z)$ are analytic on an open connected domain $D$, and that $f(z) = g(z)$ on some contour $C$ in $D$. Then $f(z) = g(z)$ on $D$.*

As the proof of Theorem 5.20 shows, it is enough that the set of points where $f(z) = g(z)$ have a limit point $z_0 \in D$ (i.e., that $f - g$ has a non-isolated zero).

**Example 5.23.** The function $g(z) = \frac{1}{1-z}$ is analytic on the punctured plane $D = \mathbb{C} \setminus \{1\}$ and equals $f(z) = \sum z^n$ on the disk $|z| < 1$. Even though the power series $f(z)$ does not converge outside the disk, we conclude that $g(z)$ is the unique *analytic continuation* of $f(z)$ to the larger domain $D$.

**Exercises 5.3.** 1. Find a power series representation and the radius of convergence:

(a) $f(z) = \frac{z}{4-z}$ about $z_0 = 0$ (b) $f(z) = z \sin z^2$ about $z_0 = 0$

(c) $f(z) = \cosh 3z$ about $z_0 = \frac{i\pi}{9}$

2. (a) *Without computing derivatives*, find the Taylor series for $f(z) = \frac{1}{z}$ about $z_0 \neq 0$.

   (b) Differentiate your answer term-by-term to find the Taylor series of $\frac{1}{z^2}$ about $z_0$.

3. By expressing $f(z)$ as a Maclaurin series, show that it is entire:

$$f(z) = \begin{cases} \frac{1}{z^2}(1 - \cos z) & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{cases}$$

4. (a) By integrating the Taylor series for $z^{-1}$ about $z_0 = 1$, prove that

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \quad \text{whenever } |z-1| < 1$$

   (b) Prove that the following function is analytic on the domain $0 < |z|$, $\operatorname{Arg} z \in (-\pi, \pi)$:

$$f(z) = \begin{cases} \frac{\operatorname{Log} z}{z-1} & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

5. Suppose $f(z)$ is analytic and non-constant at $z_0$ ($f'(z_0) \neq 0$). Prove that

$$\exists R > 0 \text{ such that } 0 < |z - z_0| < \epsilon \implies f(z) \neq f(z_0)$$

   To what extent can you weaken the hypothesis $f'(z_0) \neq 0$?

6. Consider the Maclaurin series $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ on the disk $|z| < 1$. Show that $h(z) = \frac{1}{z^2+1}$ is the analytic continuation of $f(z)$ to $\mathbb{C} \setminus \{i, -i\}$.

7. (a) Prove part 4 of Corollary 5.15: if $f(z) = \sum a_n(z - z_0)^n$, prove that $f^{(m)}(z_0) = m! a_m$ so that the series really is the Taylor series of $f(z)$.
   (*Hint: let* $g(z) = \frac{m!}{2\pi i (z-z_0)^{m+1}}$ *in Theorem 5.14*)

   (b) Explain carefully why every power series defines an analytic function.
   (*Think carefully about the definitions and what we've proved in the last two sections!*)

8. Suppose that the series $\sum a_n(z - z_0)^n$ has radius of convergence $R_0$ and that $f(z) = \sum a_n(z - z_0)^n$ whenever $|z - z_0| < R_0$. Prove that

$$R_0 = \inf\{|\hat{z} - z_0| : f(z) \text{ non-analytic or undefined at } \hat{z}\}$$

   (*$R_0$ is essentially the distance from $z_0$ to the nearest point at which $f(z)$ is non-analytic*)

9. (Hard) Consider $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ on $|z| < 1$.

   (a) Verify uniform convergence when $|z| \leq R_1 < 1$: given $\epsilon > 0$, find an explicit $N$ such that

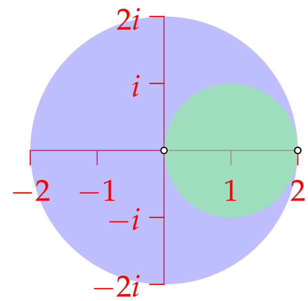
$$n > N \implies |\rho_n(z)| = \left| f(z) - \sum_{k=0}^{n} z^k \right| < \epsilon \text{ whenever } |z| \leq R_1$$

   (b) Prove that $f(z)$ is *not* uniformly convergent on $|z| < 1$ (*Let $\epsilon = 1$, then...*).

# 6 Laurent Series, Residues and Poles

While Taylor series are undeniably useful, they also have weaknesses, not least because their domains are *disks.* We motivate a more general construction with an example.

**Example 6.1.** $f(z) = \frac{1}{z(2-z)}$ can be written as a Taylor series centered at $z = 1$:




$$f(z) = \frac{1}{1-(z-1)^2} = \sum_{n=0}^{\infty}(z-1)^{2n} \quad \text{whenever } |z-1| < 1$$

However, we are also interested in the behavior of $f(z)$ near the points $z = 0, 2$. Due to their disk-domains, we can't use Taylor series to loop around these points.

As an alternative, start with the Maclaurin series for $\frac{1}{2-z}$ (valid when $|z| < 2$) and divide through by $z$ to obtain a new expression:

$$f(z) = \frac{1}{2z(1-\frac{z}{2})} = \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=-1}^{\infty}\frac{z^n}{2^{n+2}} = \frac{1}{2z} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \frac{z^3}{32} + \cdots$$

By construction, this 'series with negative terms' is valid on the punctured disk $0 < |z| < 2$.

## 6.1 Laurent Series

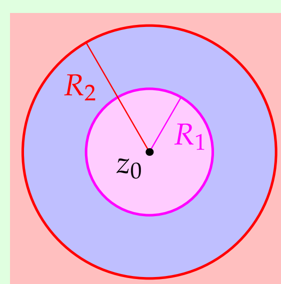
Series with negative terms are very useful in complex analysis. In Example 6.1, the larger domain encircling the origin provides an obvious advantage over the Taylor series. That such a series converges on an annulus rather than a disk is typical. We omit the full details, but by splitting a general series into positive and negative powers, substituting $w = (z-z_0)^{-1}$, and applying Theorems 5.5, 5.12 and 5.13 to the resulting power series in $w$

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=-\infty}^{-1} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}w^n$$

one obtains the natural extensions of these results.

**Theorem 6.2 (Annulus of convergence).** *Given a series* $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, *define*

$$R_1 = \inf\{|z-z_0| : f(z) \text{ converges}\}$$
$$R_2 = \sup\{|z-z_0| : f(z) \text{ converges}\}$$




1. *The series converges absolutely to a continuous function on the (open) annulus of convergence* $R_1 < |z-z_0| < R_2$. *The convergence is moreover uniform on any closed sub-annulus.*
2. *If* $|z-z_0| < R_1$ *or* $|z-z_0| > R_2$*, the series diverges.*

*As with power series, convergence on the boundary circles must be checked separately.*

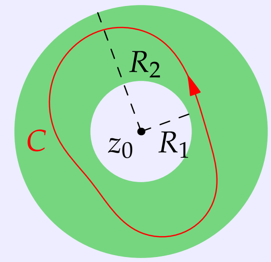
As in Example 6.1, the inner radius can be $R_1 = 0$ and the domain a punctured disk. Moreover, the outer radius can be $R_2 = \infty$.

As with Taylor series, our goal is often to find a series representation of a given function.

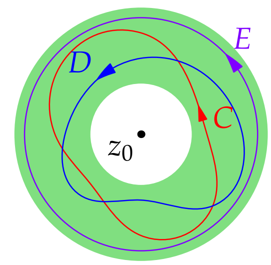
**Definition 6.3.** Suppose $f(z)$ is analytic on an *annulus* $R_1 < |z - z_0| < R_2$. Its *Laurent series* centered at $z_0$ is the expression

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and $C$ is any simple closed contour encircling $z_0$ within the annulus.




- If you prefer, write $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $b_n = \frac{1}{2\pi i} \oint_C (z - z_0)^{n-1} f(z) \, \mathrm{d}z$.
- The coefficients $a_n$ are independent of the choice of contour $C$. To see this, suppose $D$ is another simple closed curve encircling $z_0$, and choose a circle $E$ outside both $C$ and $D$. Since $\frac{f(z)}{(z-z_0)^{n+1}}$ is analytic on the annulus, two applications of Cauchy–Goursat yield




$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \oint_E \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \oint_D \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

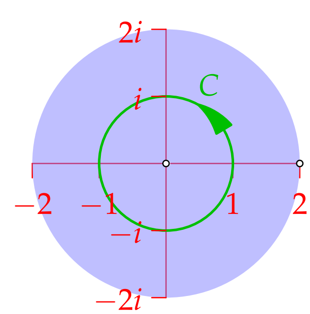
- The Laurent coefficients are inspired by our discussion of Taylor series. If $f(z)$ happens to be analytic on the *disk* $|z - z_0| < R_2$, then its Laurent series equals the Taylor series:

$$\begin{cases} n \geq 0 \implies a_n = \frac{f^{(n)}(z_0)}{n!} & \text{(Cauchy's integral formula)} \\ n < 0 \implies a_n = 0 & \text{(Cauchy–Goursat)} \end{cases}$$

Typically, however, $f(z)$ is not defined (or analytic) at $z_0$. Laurent series therefore generalize and replace Taylor series in such situations.

**Example (6.1, cont.).** $f(z) = \frac{1}{z(2-z)}$ is certainly analytic on the annulus $0 < |z| < 2$.

By first writing $f(z) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{2-z}\right)$ using partial fractions, we compute the Laurent series for $f$ on the annulus using the unit circle centered at the origin:




$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, \mathrm{d}z = \frac{1}{4\pi i} \oint_C \frac{1}{z^{n+2}} + \frac{1}{(2-z)z^{n+1}} \, \mathrm{d}z$$

The first integral evaluates to $\frac{1}{2}$ when $n = -1$ and to zero otherwise. The second integral may be found using Cauchy's integral formula:

$$\frac{1}{4\pi i} \oint_C \frac{1}{(2-z)z^{n+1}} \, \mathrm{d}z = \begin{cases} 0 & \text{if } n \leq -1 \\ \frac{1}{2(n!)} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\Big|_{z=0} (2-z)^{-1} = \frac{1}{2^{n+2}} & \text{if } n \geq 0 \end{cases}$$

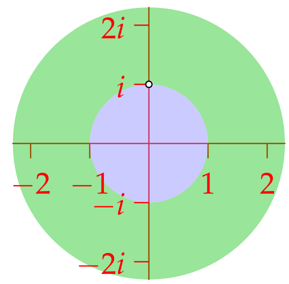
We conclude that $a_n = \frac{1}{2^{n+2}}$: the Laurent series of $f(z)$ is precisely the series computed previously!

The Laurent series centered at $z_0 = 2$ can be computed similarly, both using the integral method and our original approach.

The example is typical. Computing a Laurent series directly from the definition is usually ugly given how many contour integrals must be evaluated! Thankfully, as we'll see shortly, all the standard facts regarding Taylor series translate to this new situation. In particular, if $f(z) = \sum a_n(z - z_0)^n$ equals a series with negative terms, then the series will turn out to be the Laurent series of $f(z)$. We'll deal with the theory shortly, but first a few more examples following from this observation to get used to the idea.

**Examples 6.4.** 1. On the disk $|z| < 1$, we have the Maclaurin series

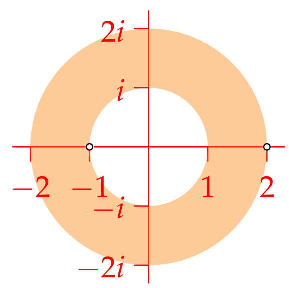
$$\frac{1}{z-i} = \frac{1}{-i(1-\frac{z}{i})} = i\sum_{n=0}^{\infty}(-iz)^n = i + z - iz^2 - z^3 + iz^4 + \cdots$$




On the annulus $|z| > 1$, we have the Laurent series

$$\frac{1}{z-i} = \frac{z}{(1-\frac{i}{z})} = \sum_{n=0}^{\infty} i^n z^{-n-1} = \frac{i}{z} - \frac{1}{z^2} - \frac{i}{z^3} + \frac{1}{z^4} + \cdots$$

2. On the $1 < |z| < 2$, we have the Laurent series




$$\begin{aligned}\frac{3}{(2-z)(1+z)} &= \frac{1}{2-z} + \frac{1}{1+z} = \frac{1}{2(1-\frac{z}{2})} + \frac{1}{z(1+\frac{1}{z})} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{z}\sum_{m=0}^{\infty}(-z)^{-m} \\ &= \cdots + z^{-3} - z^{-2} + z^{-1} + \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots\end{aligned}$$

3. Since $e^z = \sum \frac{z^n}{n!}$ is valid on the entire complex plane, we obtain the Laurent series expansion

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

valid on the punctured plane $z \neq 0$.

4. Again by substituting in a known Maclaurin series, we obtain another Laurent series valid on the punctured plane $z \neq 0$:

$$\frac{1}{z^7}\sin z^2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{4n-5} = z^{-5} - \frac{1}{6}z^{-1} + \frac{1}{120}z^3 - \frac{1}{5040}z^7 + \cdots$$

5. Multiplying term-by-term, and since we need *both* Maclaurin series to be valid, we obtain (the first few terms of) a Laurent series valid on the punctured disk $0 < |z| < 1$:

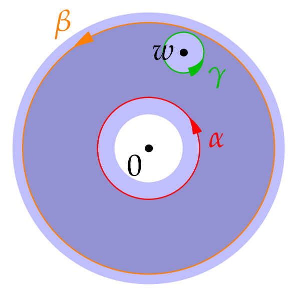
$$\begin{aligned}\frac{1}{z(z-1)(z-2i)} &= \frac{1}{z}\left(\sum_{n=0}^{\infty}(-1)^n z^n\right)\left(\sum_{m=0}^{\infty}\left(\frac{i}{2}\right)^m z^m\right) \\ &= \frac{1}{z}(1 - z + z^2 - z^3 + \cdots)\left(1 + \frac{i}{2}z - \frac{1}{4}z^2 - \frac{i}{8}z^3 + \cdots\right) \\ &= \frac{1}{z} + \left(-1 + \frac{i}{2}\right) + \left(\frac{3}{4} - \frac{i}{2}\right)z + \left(-\frac{3}{4} + \frac{3i}{8}\right)z^2 + \cdots\end{aligned}$$

We now state and prove the main properties of Laurent series. These are very similar to the corresponding statements & arguments for Taylor series; the additional challenge mostly comes from keeping track of two series at once.

**Theorem 6.5 (Laurent's Theorem).** *An analytic function on an open annulus equals its Laurent series.*

*Proof.* It is enough to prove when the annulus is centered at $z_0 = 0$. Let $w$ in the annulus be given.

Since the annulus is open, we may choose three non-overlapping circles $\alpha, \beta, \gamma$ with radii $R_\alpha, R_\beta, R_\gamma$ as in the picture:




- $\gamma$ a small circle centered at $w$ inside the annulus;
- $\alpha, \beta$ centered at 0, $\alpha$ inside and $\beta$ outside $w$ (thus $R_\alpha < |w| < R_\beta$).

Since $\frac{f(z)}{z-w}$ is analytic on/inside the closed region with boundaries $\alpha, \beta, \gamma$, Cauchy–Goursat and the integral formula tell us that

$$\left(\oint_\beta - \oint_\alpha - \oint_\gamma\right) \frac{f(z)}{z-w}\,\mathrm{d}z = 0 \implies f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = \frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,\mathrm{d}z \qquad (*)$$

As in the proof of Taylor's theorem, we expand $\frac{1}{z-w}$, this time in two ways:

$$\frac{1}{z-w} = \frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k + \frac{1}{z-w}\left(\frac{w}{z}\right)^n = -\frac{1}{w}\sum_{k=1}^{n}\left(\frac{z}{w}\right)^{k-1} + \frac{1}{z-w}\left(\frac{z}{w}\right)^n$$

These expansions allow us to attack the two integrals in $(*)$:

$$\frac{1}{2\pi i}\oint_\beta \frac{f(z)}{z-w}\,\mathrm{d}z = \sum_{k=0}^{n-1}\underbrace{\frac{w^k}{2\pi i}\oint_\beta \frac{f(z)}{z^{k+1}}\,\mathrm{d}z}_{a_k w^k} + \frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z$$

$$\frac{-1}{2\pi i}\oint_\alpha \frac{f(z)}{z-w}\,\mathrm{d}z = \sum_{k=1}^{n}\underbrace{\frac{1}{2\pi i w^k}\oint_\alpha z^{k-1}f(z)\,\mathrm{d}z}_{a_{-k}w^{-k}} - \frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,\mathrm{d}z$$

We finish by summing and estimating, using the facts that $z \in \alpha \cup \beta \implies |z-w| > R_\gamma$, and that $f(z)$ is bounded (by some $M > 0$) on the closed bounded annulus between $\alpha, \beta$:

$$\begin{aligned}
\left|f(w) - \sum_{k=-n}^{n-1} a_k w^k\right| &= \left|\frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z - \frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,\mathrm{d}z\right| \\
&\overset{\triangle}{\le} \frac{|w|^n}{2\pi}\left|\oint_\beta \frac{f(z)}{z^n(z-w)}\,\mathrm{d}z\right| + \frac{1}{2\pi |w|^n}\left|\oint_\alpha \frac{z^n f(z)}{z-w}\,\mathrm{d}z\right| \\
&\le \frac{|w|^n}{2\pi}\cdot\frac{M}{R_\beta^n R_\gamma}\cdot 2\pi R_\beta + \frac{1}{2\pi |w|^n}\cdot\frac{R_\alpha^n M}{R_\gamma}\cdot 2\pi R_\alpha \qquad \text{(Theorem 4.16)} \\
&= \frac{M}{R_\gamma}\left[R_\beta\left(\frac{|w|}{R_\beta}\right)^n + R_\alpha\left(\frac{R_\alpha}{|w|}\right)^n\right] \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

■

Our final corollary summarizes the remaining core properties of Laurent series.

**Corollary 6.6.** *Let $f(z) = \sum a_n(z-z_0)^n$ be a series. Then:*

1. *(Term-by-term Integration) If $g(z)$ is continuous on a contour $C$ lying inside the annulus, then*

$$\int_C g(z)f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \int_C g(z)(z-z_0)^n\,\mathrm{d}z$$

   *In particular, $f(z)$ may be integrated term-by-term along $C$.*

2. *(Analyticity/Derivatives) $f(z)$ is analytic on the annulus and $f'(z) = \sum_{n=-\infty}^{\infty} a_n n(z-z_0)^{n-1}$*

3. *(Uniqueness) $\sum a_n(z-z_0)^n$ is the Laurent series of $f(z)$ (as in Definition 6.3).*

The remainder are the analogues of Theorems 5.14 and Corollary 5.15: some details are in Exercise 7.

**Examples 6.7.** The corollary formally justifies Examples 6.4. Here are two extensions.

1. In accordance with part 2 of the corollary,

$$\begin{aligned}\frac{1}{z^2}e^{1/z} &= -\frac{\mathrm{d}}{\mathrm{d}z}e^{1/z} = \frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \sum_{n=1}^{\infty}\frac{z^{-1-n}}{(n-1)!} = \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{z^{-(n-1)}}{(n-1)!} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}\\ &= \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{2z^4}+\frac{1}{6z^5}+\frac{1}{24z^6}+\cdots\end{aligned}$$

   We need not have differentiated: by part 3 we could instead simply multiply the known series for $e^{1/z}$ term-by-term to obtain the result!

2. We use part 1 to compute the integral around a contour $C$ encircling the origin:

$$\oint_C \frac{1}{z^7}\sin z^2\,\mathrm{d}z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\oint_C z^{4n-5}\,\mathrm{d}z = \oint_C \frac{(-1)^1}{(2+1)!}z^{4-5} = -\frac{1}{3}\pi i$$

   since all but one of the integrals evaluates to zero.

**Exercises 6.1.** 1. (Example 6.1, cont.) The function $f(z) = \frac{1}{z(2-z)}$ is analytic everywhere except at $z = 0$ and 2.

   (a) Use Definition 6.3 to directly compute the Laurent series of $f(z)$ on the punctured disk $0 < |z-2| < 2$: use the circle radius 1 centered at $z_0 = 2$.

   (b) Use the Taylor series of $\frac{1}{z}$ centered at $z_0 = 2$ to more rapidly compute the Laurent series in part (a).

2. Find a Laurent series representation for each function. Also find $\oint_C f(z)\,\mathrm{d}z$ where $C$ is a simple closed curve in the given domain encircling the origin.

   (a) $f(z) = \frac{3}{z^2}e^{2z}$ when $|z| > 0$

   (b) $f(z) = \cos\frac{i}{z}$ when $|z| > 0$

   (c) $f(z) = \frac{1}{1+z^3}$ when $1 < |z|$ (*Hint: let $w = z^{-1}$*)

3. On each domain, find a Laurent series centered at $z_0 = 0$ for the function

$$f(z) = \frac{1}{z(z-2i)} = \frac{i}{2}\left(\frac{1}{z} - \frac{1}{z-2i}\right)$$

(a) $D_1 = \{z : 0 < |z| < 2\}$ (b) $D_2 = \{z : |z| > 2\}$ (*again let* $w = z^{-1}$)

4. Repeat the previous question for

$$f(z) = \frac{1-2i}{(z-1)(z-2i)} = \frac{1}{z-1} - \frac{1}{z-2i}$$

Also find $\oint_C f(z)\,dz$ where $C$ is a simple closed curve in the given domain encircling the origin.

(a) $D_1 = \{z : 0 < |z| < 1\}$ (b) $D_2 = \{z : 1 < |z| < 2\}$ (c) $D_3 = \{z : |z| > 2\}$

5. Show that when $0 < |z-1| < 2$, we have

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$$

6. Let $a$ be complex number. Show that

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad \text{whenever } |a| < |z|$$

7. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a series as in Corollary 6.6.

   (a) Suppose part 1 has been proved. Explain why the function $f(z) - a_{-1}(z-z_0)^{-1}$ is analytic on the annulus. Hence conclude that $f(z)$ is analytic on the annulus.
   (*This is different to Corollary 5.15 since $a_{-1}(z-z_0)^{-1}$ has no anti-derivative on the annulus!*)

   (b) In order to mimic the proof of Corollary 5.15 to show that $f(z)$ is differentiable term-by-term, what properties must the curve $C$ have?

   (c) Prove part 3.
   (*Hint: Recall Exercise 5.3.7 - the same hint works!*).

### 6.2 Singularities and Cauchy's Residue Theorem

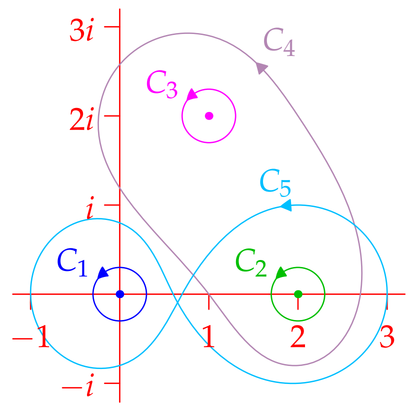
One practical purpose of this chapter is the development of an efficient method for computing contour integrals of analytic functions. Essentially everything depends on a few crucial facts:

1. If $C$ encircles $z_0$, then $\oint_C (z-z_0)^n \,dz = 0$ when $n \neq -1$, and $\oint_C (z-z_0)^{-1}\,dz = 2\pi i$.
2. Cauchy–Goursat (Theorem 4.19) ($f$ analytic on/inside $C \implies \oint_C f = 0$) and its extension (Corollary 4.21) to regions with finitely many interior boundary curves.
3. The existence of Taylor/Laurent series expansions.

Make sure you are familiar with these ideas before proceeding!

**Example 6.8.** The rational function

$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$$




is analytic except at the points $z_1 = 0$, $z_2 = 2$, $z_3 = 1+2i$.
Several curves are drawn. Integrating round the small circle $C_1$ is easy using the first fact (above):

$$\oint_{C_1} f(z)\,dz = 3\oint_{C_1}\frac{dz}{z} + \oint_{C_1}\frac{dz}{z^2} + \oint_{C_1}\frac{5i\,dz}{z-2} + \oint_{C_1}\frac{dz}{z-1-2i}$$
$$= 3\cdot 2\pi i + 0 + 0 + 0 = 6\pi i$$

since the latter three integrands are analytic on/inside $C_1$. Similarly,

$$\oint_{C_2} f(z)\,dz = 5i\oint_{C_2}\frac{dz}{z-2} = -10\pi, \qquad \oint_{C_3} f(z)\,dz = \oint_{C_3}\frac{dz}{z-1-2i} = 2\pi i$$

The curves $C_4$ and $C_5$ are more interesting. Since $f(z)$ is analytic on and between $C_4$ and $C_2/C_3$, Cauchy–Goursat tells us that

$$\oint_{C_4} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz = -10\pi + 2\pi i = 2\pi(i-5)$$

$C_5$ appears trickier, though it becomes easy once you visualize it as *two* contours: the first encircles $z_2 = 2$ *counter-clockwise* while the second passes *clockwise* around $z_1 = 0$. We conclude that

$$\int_{C_5} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = -10\pi - 6\pi i = -2\pi(5+3i)$$

The example suggests that the value of any integral round a simple closed contour can be evaluated as a linear combination

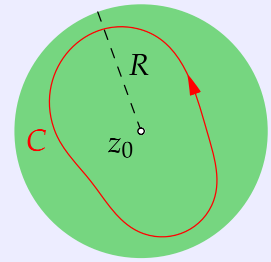
$$\int_C f(z)\,dz = \lambda_1\oint_{C_1} f + \lambda_2\oint_{C_2} f + \lambda_3\oint_{C_3} f = 6\pi i\lambda_1 - 10\pi\lambda_2 + 2\pi i\lambda_3$$

where $\lambda_k$ denotes the number of times $C$ orbits $z_k$ in a counter-clockwise direction.

To properly develop the idea in the example, some new language is helpful.

**Definition 6.9 (Isolated Singularities).** We say that $z_0$ is an *isolated singularity* of $f(z)$ if the function is analytic on a punctured disk $0 < |z - z_0| < R$, but not at $z_0$ itself.

By Laurent's Theorem (6.5), $f(z)$ equals its Laurent series on this domain:




$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and $C$ encircles $z_0$. The *residue* of $f(z)$ at $z_0$ is the $-1$-coefficient

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \, \mathrm{d}z$$

The type of isolated singularity is determined by the rest of the Laurent series:

*Removable Singularity* The Laurent series is a Taylor series (the residue is necessarily zero). There are no negative powers; the series $f(z)$ extends analytically to $z_0$.

*Pole of order m* The highest negative power in the Laurent series is $(z - z_0)^{-m}$. A pole of order 1 is typically called a *simple pole*, order 2 a *double pole*, etc.

*Essential Singularity* The Laurent series has infinitely many negative terms.

**Examples 6.10.** 1. The series $f(z) = \sum_{n=0}^{\infty} 3^{-n}(z - 2i)^n$ defined on the punctured disk $0 < |z - 2i| < 3$ has a removable singularity at $z_0 = 2i$ (residue $\operatorname*{Res}_{z=2i} f(z) = 0$). Indeed the function is a geometric series and thus equals

$$f(z) = \frac{1}{1 - \frac{z-2i}{3}} = \frac{3}{3 + 2i - z}$$

on the punctured disk. Certainly this extends analytically to $f(2i) = 1$.

2. $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$ has an essential singularity at zero with $\operatorname*{Res}_{z=0} e^{1/z} = 1$.

3. (Example 6.8) The function $f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$ is analytic on the punctured disk $0 < |z| < 0.3$ (inside the circle $C_1$). Since $\frac{5i}{z-2} + \frac{1}{z-1-2i}$ is also analytic at zero, the Laurent series of $f(z)$ about $z_1 = 0$ has the form

$$f(z) = \frac{1}{z^2} + \frac{3}{z} + \sum_{n=0}^{\infty} a_n z^n$$

We conclude that $f(z)$ has a pole of order 2 at $z_1 = 0$ and residue $\operatorname*{Res}_{z=0} f(z) = 3$. Similarly, $f(z)$ has simple poles (order 1) at $z_2 = 2$ and $z_2 = 1 + 2i$ with

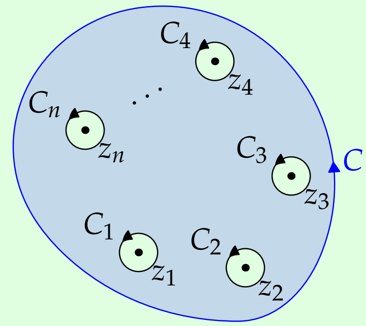
$$\operatorname*{Res}_{z=2} f(z) = 5i, \qquad \operatorname*{Res}_{z=1+2i} f(z) = 1$$

**Theorem 6.11 (Cauchy's Residue Theorem).** *Let $f(z)$ be analytic on and inside a simple closed contour $C$, except at finitely many singularities $z_1, \ldots, z_n$. Then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

*More generally, if $C$ is closed and orbits the point $z_k$ counter-clockwise $\lambda_k$ times (the* winding number *of $C$ about $z_k$), then*

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \lambda_k \operatorname*{Res}_{z=z_k} f(z)$$



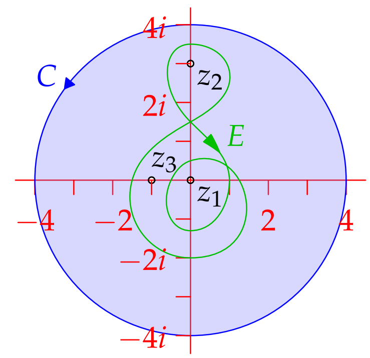

*Proof.* First, center a circle $C_k$ at each $z_k$ such that no other singularities lie on/inside $C_k$ and apply Cauchy–Goursat. The general argument follows by considering paths like $C_5$ in Example 6.8. ∎

**Examples 6.12.** Let $C$ be the circle with radius 4 centered at the origin and $E$ the green curve drawn.

1. The rational function $f(z) = \frac{3(1+iz)}{z(z-3i)}$ has simple poles at $z_1 = 0$ and $z_2 = 3i$. There are several ways to compute the residues and thus the integrals $\oint_C f(z)\mathrm{d}z$ and $\oint_E f(z)\mathrm{d}z$.

   *Partial Fractions* It's just high-school algebra!

$$f(z) = \frac{i}{z} + \frac{2i}{z-3i} \implies \operatorname*{Res}_{z=0} f(z) = i, \quad \operatorname*{Res}_{z=3i} f(z) = 2i$$
$$\implies \oint_C f(z)\,\mathrm{d}z = 2\pi i(i+2i) = -6\pi$$




   The curve $E$ orbits *twice clockwise* around $z_1$ and *once counter-clockwise* around $z_2$. Thus

$$\int_E f(z)\,\mathrm{d}z = 2\pi i \left[-2\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=3i} f(z)\right] = 0$$

   *Laurent series* Remember that we only need to find the $z^{-1}$ terms for the residues!

$$\frac{3(1+iz)}{z(z-3i)} = \frac{i-z}{z(1-\frac{iz}{3})} = \left(\frac{i}{z}-1\right)\sum_{n=0}^{\infty}\left(\frac{iz}{3}\right)^n = \frac{i}{z} + \text{power series}$$
$$\frac{3(1+iz)}{z(z-3i)} = \frac{z-3i+2i}{(1+\frac{z-3i}{3i})(z-3i)} = \left(\frac{2i}{z-3i}+1\right)\sum_{n=0}^{\infty}\left(\frac{3i-z}{3i}\right)^n = \frac{2i}{z-3i} + \text{power series}$$

   *Cauchy's integral formula* Let $C_k$ be a small circle around $z_k$, then

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_1} \frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z-3i}\right|_{z=0} = i$$
$$\operatorname*{Res}_{z=3i} f(z) = \frac{1}{2\pi i}\oint_{C_2} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_2} \frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z}\right|_{z=3i} = 2i$$

2. $f(z) = z^2 \sin\frac{1}{z}$ has one isolated singularity at the origin. Using the Maclaurin series for $\sin z$, we see that this is an essential singularity. Moreover,

$$z^2 \sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n} \implies \oint_C z^2 \sin\frac{1}{z}\, dz = 2\pi i \operatorname*{Res}_{z=0}\left(z^2 \sin\frac{1}{z}\right) = -\frac{\pi i}{3}$$

Since $E$ loops twice clockwise around the origin, we obtain

$$\int_E z^2 \sin\frac{1}{z}\, dz = 2\pi i \cdot (-2) \operatorname*{Res}_{z=0}\left(z^2 \sin\frac{1}{z}\right) = \frac{2\pi i}{3}$$

3. $f(z) = 3e^{1/z} + \frac{4}{z-7i} + \frac{2i}{z+1}$ has an essential singularity at the origin and two simple poles at $-1$ and $7i$. Since the last of these lies *outside* the curves $C, E$, it does not contribute to either integral. Moreover, since $E$ loops twice clockwise around the origin and once clockwise around $z_3 = -1$. We therefore have

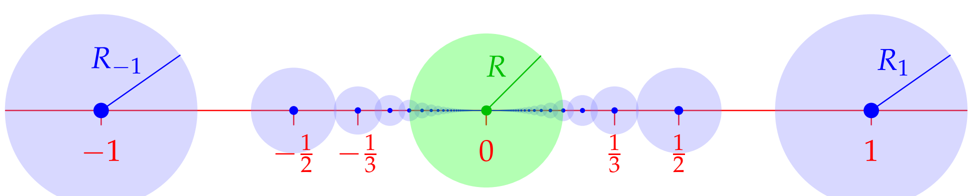
$$\oint_C f(z)\, dz = 2\pi i \left(\operatorname*{Res}_{z=0} 3e^{1/z} + \operatorname*{Res}_{z=-1} \frac{2i}{z+1}\right) = 2\pi i(3+2i)$$

$$\oint_E f(z)\, dz = 2\pi i \left(-2 \operatorname*{Res}_{z=0} 3e^{1/z} + (-1) \operatorname*{Res}_{z=-1} \frac{2i}{z+1}\right) = 2\pi i(-6-2i) = 4\pi(1-3i)$$

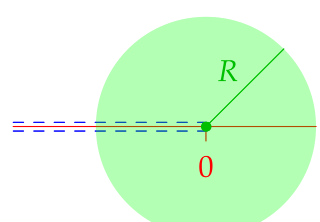
### Non-isolated Singularities

Revisit Definition 6.9. A point $z_0$ is said to be a *non-isolated singularity* of $f(z)$ if the function is non-analytic (including undefined) at $z_0$ and at some point on *every* punctured disk $0 < |z - z_0| < R$. Such a function is necessarily non-analytic at infinitely many points and does not have a Laurent series centered at $z_0$. Non-isolated singularities typically appear in two flavors.

**Examples 6.13.** 1. $f(z) = \left(e^{\frac{2\pi i}{z}} - 1\right)^{-1}$ has singularities at $z_0 = 0$ and at $z_n = \frac{1}{n}$ for each non-zero integer $n$. Each non-zero singularity is isolated (choose e.g., $R_n = \frac{1}{(|n|+1)^2}$ in Definition 6.9). The *limit point* $z_0 = 0$ is non-isolated: for any $R > 0$, the punctured disk $0 < |z| < R$ contains other singularities.




2. The square-root function $f(z) = z^{1/2}$ has a *branch point* $z_0 = 0$. For $f$ to be analytic, we need to make a branch cut: for instance the non-positive real axis for the principal branch.
Since any punctured disk $0 < |z| < R$ contains other points of the branch cut (where $f$ is non-analytic), the branch point is a non-isolated singularity *for any branch* of $f$.



**Exercises 6.2.** 1. For each type of singularity, what, if anything, can be said about the value of the residue $\operatorname*{Res}_{z=z_0} f(z)$? Choose from 'Equals zero,' 'Non-zero,' or 'No restriction'.

(a) Removable singularity.
(b) Simple pole.
(c) Pole of order $m \geq 2$.
(d) Essential singularity.

2. Find the residue at $z = 0$ of each function:

(a) $f(z) = \dfrac{1}{z+3z^2}$ (b) $g(z) = z\cos\dfrac{1}{z}$ (c) $h(z) = \dfrac{z - \sin z}{z}$

3. Let $C$ be the circle $|z| = 3$. Evaluate the integrals using Cauchy's residue theorem:

(a) $\displaystyle\oint_C \frac{e^{-z}}{z^2}\,\mathrm{d}z$ (b) $\displaystyle\oint_C \frac{e^{-z}}{(z-1)^2}\,\mathrm{d}z$ (c) $\displaystyle\oint_C z^2 e^{1/z}\,\mathrm{d}z$ (d) $\displaystyle\oint_C \frac{z+1}{z^2-2z}\,\mathrm{d}z$

4. Suppose a closed contour $C$ loops twice counter-clockwise around $z = i$ and three times clockwise around $z = 2$. Use residues to compute the integral

$$\int_C \frac{z+3}{(z-2)^2(z-i)}\,\mathrm{d}z$$

5. Identify the type of singular point of each function and determine the residue:

(a) $\dfrac{1-\cosh z}{z^3}$ (b) $\dfrac{1-e^{2z}}{z^4}$ (c) $\dfrac{e^{2z}}{(z-1)^2}$

6. Suppose $f(z)$ is analytic at $z_0$ and define $g(z) = (z - z_0)^{-1} f(z)$. Prove:

(a) If $f(z_0) \neq 0$, then $z_0$ is a simple pole of $g(z)$ with $\operatorname*{Res}_{z=z_0} g(z) = f(z_0)$;

(b) If $f(z_0) = 0$, then $z_0$ is a removable singularity of $g(z)$.

7. A function $f(z)$ is said to have an *isolated singularity at* $\infty$ if it is analytic on some infinite annulus $|z| > R$. On this domain, we have a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

(a) What properties do you think this series should have if we were to describe the singularity at $\infty$ to be:

i. Removable? ii. A pole of order $m$? iii. Essential?

(b) The *residue at* $\infty$ of the above function is defined to be

$$\operatorname*{Res}_{z=\infty} f(z) := -\operatorname*{Res}_{z=0}\left(z^{-2} f(z^{-1})\right) \tag{$*$}$$

i. Prove that $\operatorname*{Res}_{z=\infty} f(z) = -a_{-1}$, where $a_{-1}$ is the coefficient in the above Laurent series.

ii. Suppose $f(z)$ is analytic on $\mathbb{C}$ except at finitely many singularities. By substituting $w = \frac{1}{z}$ in $(*)$, prove that the *sum* of all residues, *including at infinity*, is zero.

8. Let $P(z)$ and $Q(z)$ be polynomials and assume $C$ is a simple closed contour such that all zeros of $Q(z)$ lie interior to $C$.

(a) If $\deg Q \geq 2 + \deg P$, prove that $\oint_C \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0$

(b) What can you conclude if $\deg Q = 1 + \deg P$?

(*Hint: Combine Exercises 6 & 7, or try the substitution* $w = \frac{1}{z}$ *directly*)

## 6.3 Poles & Zeros

If the order of a pole is known, residues may often be computed quite efficiently. For instance, if $f(z)$ has a pole of order $m$, Laurent's Theorem tells us that

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n = (z-z_0)^{-m}\phi(z), \quad \text{where} \quad \phi(z) = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \tag{$*$}$$

Plainly $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) = a_{-m} \neq 0$. In fact this property categorizes poles of order $m$ (analogous to Theorem 5.18 for zeros), providing a simple formula for the computation of residues.

**Theorem 6.14.** *A function $f(z)$ has a pole of order $m$ at $z_0$ if and only if $f(z) = (z-z_0)^{-m}\phi(z)$ where $\phi(z)$ is analytic at $z_0$ and $\phi(z_0) \neq 0$. In such situations, the residue is a Taylor coefficient of $\phi(z)$:*[16]

$$f(z) = (z-z_0)^{-m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \implies \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\phi^{(m-1)}(z_0)$$

*This specializes to $\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$ for a simple pole.*

*Proof.* The ($\Rightarrow$) direction is above. For the converse: $\phi(z)$ equals its Taylor series and so ($*$) describes the Laurent series of $f(z) = (z-z_0)^{-m}\phi(z)$ (Corollary 6.6, uniqueness), which moreover has a pole of order $m$ since $a_{-m} = \phi(z_0) \neq 0$. ∎

**Examples 6.15.** 1. (Example 6.12 revisited) $f(z) = \frac{3(1+iz)}{z(z-3i)}$ has two poles:

*Simple pole at $z_1 = 0$:* Write $f(z) = z^{-1}\phi_1(z)$, where $\phi_1(z) = \frac{3(1+iz)}{z-3i}$. This is analytic and non-zero at $z_1 = 0$, whence

$$\operatorname*{Res}_{z=0} f(z) = \phi_1(0) = \frac{3}{-3i} = i$$

*Simple pole at $z_2 = 3i$:* This time, $f(z) = (z-3i)^{-1}\phi_2(z)$, where $\phi_2(z) = \frac{3(1+iz)}{z}$. Thus

$$\operatorname*{Res}_{z=3i} f(z) = \phi_2(3i) = \frac{3(1-3)}{3i} = 2i$$

2. $f(z) = \frac{1-2iz}{(z-1)(z-2i)^3}$ also has two poles.

*Simple pole at $z_1 = 1$:* Write $f(z) = (z-1)^{-1}\phi_1(z)$, where $\phi_1(z) = \frac{1-2iz}{(z-2i)^3}$. Thus

$$\operatorname*{Res}_{z=1} f(z) = \phi_1(1) = \frac{1-2i}{(1-2i)^3} = \frac{1}{(1-2i)^2} = \frac{4i-3}{25}$$

*Pole of order three at $z_2 = 2i$:* $f(z) = (z-2i)^{-1}\phi_2(z)$, where $\phi_2(z) = \frac{1-2iz}{z-1} = -2i + \frac{1-2i}{z-1}$. Thus

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(3-1)!}\phi_2''(2i) = \frac{1-2i}{(z-1)^3}\bigg|_{z=2i} = \frac{-1}{(2i-1)^2} = \frac{3-4i}{25}$$

[16] The formula works whenever $f(z) = (z-z_0)^{-m}\phi(z)$ with $\phi$ analytic, **even if $\phi(z_0) = 0$**. However, a naïve application means you won't know the order of the pole and you'll have to differentiate more times than necessary!

As the previous examples show, the method is very effective for rational with low-order poles; as a bonus, it saves us from partial fractions! Its utility is more variable for other functions...

**Examples 6.16.** 1. The function $f(z) = \frac{e^z}{(z-1)^2(z+1)} = (z-1)^{-2}\phi(z)$ has a double pole at $z_0 = 1$, with

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\phi'(1) = \left.\frac{ze^z}{(z+1)^2}\right|_{z=1} = \frac{1}{4}e$$

2. Don't let the denominator fool you! At first glance we appear to have a pole of order *six*:

$$f(z) = \frac{6\sin z - 6z + z^3}{z^6} = \frac{\widetilde{\phi}(z)}{z^6} \overset{??}{\Longrightarrow} \operatorname*{Res}_{z=0} f(z) = \frac{1}{5!}\widetilde{\phi}^{(5)}(0) = \frac{6}{120} = \frac{1}{20}$$

However, if we apply the Maclaurin series for sine, we instead find a *simple pole*:

$$f(z) = \frac{1}{z^6}\left(6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} - 6z + z^3\right) = z^{-6}\sum_{n=2}^{\infty}\frac{6(-1)^n}{(2n+1)!}z^{2n+1}$$
$$= z^{-1}\sum_{m=0}^{\infty}\frac{6(-1)^m}{(2m+5)!}z^{2m} = \frac{1}{20z} + \frac{1}{840} + \cdots \Longrightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{20}$$

Even though the first calculation produced the correct residue (see footnote 16), the function $\widetilde{\phi}$ was incorrect ($\widetilde{\phi}(0) = 0$). The correct function is the series $\phi(z) = 6\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+5)!}z^{2m}$.

For certain functions with *simple poles*, the method is even easier.

**Corollary 6.17.** *Suppose $p(z), q(z)$ are analytic at $z_0$, that $p(z_0) \neq 0$ and that $q(z)$ has a simple zero. Then $f(z) = \frac{p(z)}{q(z)}$ has a simple pole at $z_0$, and*

$$\operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

*Proof.* By Theorem 5.18, $q(z) = (z-z_0)\psi(z)$ where $\psi$ is analytic and $\psi(z_0) \neq 0$. Take $\phi(z) = \frac{p(z)}{\psi(z)}$ in Theorem 6.14 ($\phi$ is analytic and non-zero at $z_0$), to see that

$$f(z) = \frac{p(z)}{q(z)} = (z-z_0)^{-1}\frac{p(z)}{\psi(z)} \quad \text{and} \quad \operatorname*{Res}_{z=z_0}\frac{p(z)}{q(z)} = \phi(z_0) = \frac{p(z_0)}{\psi(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

■

**Examples 6.18.** 1. The function $f(z) = \frac{p(z)}{q(z)} = \frac{\sin z}{z^2+4} = \frac{\sin z}{(z-2i)(z+2i)}$ has simple poles at $\pm 2i$ with

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=-2i} f(z) = \frac{p(\pm 2i)}{q'(\pm 2i)} = \frac{\sin 2i}{4i} = \frac{1}{8}(e^2 - e^{-2}) = \frac{1}{4}\sinh 2$$

The reciprocal $g(z) = \frac{q(z)}{p(z)} = \frac{z^2+4}{\sin z}$ has simple poles at $z = n\pi$ for every $n \in \mathbb{Z}$, with

$$\operatorname*{Res}_{z=n\pi}\frac{z^2+4}{\sin z} = \frac{q(n\pi)}{p'(n\pi)} = \frac{n^2\pi^2+4}{\cos n\pi} = (-1)^n(n^2\pi^2+4)$$

2. Since $q(z)=e^{2z}-1=\sum_{n=1}^{\infty}\frac{2^nz^n}{n!}=z\sum_{n=0}^{\infty}\frac{2^nz^n}{(n+1)!}$ has a simple zero at $z=0$, we see that

$$f(z)=\frac{\sqrt{z+4i}}{(z+i)^2\operatorname{Log}(z+2)(e^{2z}-1)}=\frac{p(z)}{q(z)} \qquad \left(p(z)=\tfrac{\sqrt{z+4i}}{(z+i)^2\operatorname{Log}(z+2)}\right)$$

has a simple pole at $z=0$ and may easily compute

$$\operatorname*{Res}_{z=0} f(z)=\frac{p(0)}{q'(0)}=\frac{e^{\frac{i\pi}{4}}}{\ln 2}=\frac{1+i}{\sqrt{2}\ln 2}$$

We could instead have chosen $q(z)=(z+i)^2\operatorname{Log}(z+2)(e^{2z}-1)$, but computing $q'(z)$ would have been disgusting!

**Counting the Number of Poles and Zeros**

A sneaky application of our pole/zero categorization permits us to *count* them.

Suppose $f$ has a zero of order $m$ at $z_0$. Write $f(z)=(z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic and *non-zero* on some closed disk $|z-z_0|\le R$. Let $C_0$ be the boundary of the disk and compute:

$$\begin{aligned}\oint_{C_0}\frac{f'(z)}{f(z)}\,\mathrm{d}z&=\oint_{C_0}\frac{m(z-z_0)^{m-1}\psi(z)+(z-z_0)^m\psi'(z)}{(z-z_0)^m\psi(z)}\,\mathrm{d}z\\&=\oint_{C_0}\frac{m}{z-z_0}+\frac{\psi'(z)}{\psi(z)}\,\mathrm{d}z=2\pi i m\end{aligned}$$

where we used the fact that $\frac{\psi'(z)}{\psi(z)}$ is analytic on and inside $C_0$. The integral $\frac{1}{2\pi i}\oint_{C_0}\frac{f'(z)}{f(z)}\,\mathrm{d}z$ therefore counts the *multiplicity* of the zero. By performing a similar calculation for poles and combining the results for several poles and zeros, we recover a famous result.

**Theorem 6.19 (Cauchy's Argument Principle).** *Suppose $f$ is analytic except at poles,*[17] *on and inside a simple closed curve $C$, and that $f(z)$ has no poles or zeros on $C$. Then*

$$\frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}\,\mathrm{d}z=Z-P$$

*where $Z$ and $P$ are the number of zeros and poles of $f$ inside $C$, counted up to multiplicity.*

**Example 6.20.** Consider $f(z)=\frac{(z-i)^2\sin z}{(z-5)^4}$ where $C$ is a large circle surrounding the points $0$, $i$ and $5$. Plainly $Z=2+1=3$ and $P=4$. We may compute the integral explicitly,

$$\frac{1}{2\pi i}\oint_C\frac{f'(z)}{f(z)}\,\mathrm{d}z=\frac{1}{2\pi i}\oint_C\frac{2}{z-i}+\frac{\cos z}{\sin z}-\frac{4}{z-5}\,\mathrm{d}z=2+\cos 0-4=-1=Z-P$$

(Use logarithmic differentiation unless you're feeling masochistic!)

[17] Such a function is termed *meromorphic*. The result is called the argument principle because $\frac{f'(z)}{f(z)}=\frac{\mathrm{d}}{\mathrm{d}z}\log f(z)$ means that the integral is calculating the net change in the logarithm, and thus *argument*, of $f(z)$ round the curve $C$.

**Properties of Isolated Singularities**

Recall (Definition 6.9) that a function $f(z)$ analytic on a punctured disk $0 < |z - z_0| < R$ can have one of three types of isolated singularity at $z_0$: removable, pole, and essential. We finish by considering some conditions/properties for each type.

**Theorem 6.21 (Removable Singularities).** *Suppose $f(z)$ has an isolated singularity at $z_0$. The following are equivalent:*

1. *The singularity is removable.*
2. *$\lim_{z \to z_0} f(z)$ exists and is finite.*
3. *There exists a punctured disk $0 < |z - z_0| < \delta$ on which $f(z)$ is bounded.*

*Proof.* For simplicity, suppose that $z_0 = 0$ and that $f(z)$ is analytic on the punctured disk $0 < |z| < R$.

($1 \Rightarrow 2$) Since $z_0$ is removable, the Laurent series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $0 < |z| < R$ has no negative terms. It follows that $\lim_{z \to 0} f(z) = a_0$.

($2 \Rightarrow 3$) This is almost tautological but bears repeating: If $\lim_{z \to 0} f(z) = a_0$ is finite then choose $\epsilon = |a_0|$ in the definition;

$$\begin{aligned} \exists \delta > 0 \text{ such that } 0 < |z| < \delta \implies & |f(z) - a_0| < |a_0| \\ \implies & |f(z)| = |f(z) - a_0 + a_0| \overset{\triangle}{<} 2\,|a_0| \end{aligned}$$

($3 \Rightarrow 1$) Suppose $f(z)$ is bounded on $0 < |z| < \delta$, and consider

$$g(z) = \begin{cases} z^2 f(z) & \text{if } 0 < z < \delta \\ 0 & \text{if } z = 0 \end{cases}$$

Since $f$ is bounded, we may compute the limit

$$\lim_{z \to 0} \frac{g(z) - g(0)}{z} = \lim_{z \to 0} z f(z) = 0$$

whence $g(z)$ is differentiable at zero! Since $g(z)$ is already differentiable on the punctured disk $0 < |z| < \min(\delta, R)$, it is therefore analytic on the *disk* $|z| < \min\{\delta, R\}$ and equals its Maclaurin series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=2}^{\infty} b_n z^n = z^2 \sum_{m=0}^{\infty} b_{m-2} z^m \qquad (b_0 = g(0) = 0 \text{ and } b_1 = g'(0) = 0)$$

We conclude that

$$f(z) = \sum_{m=0}^{\infty} b_{m-2} z^m \quad \text{whenever} \quad 0 < |z| < \min(\delta, R)$$

whence $f$ has a removable singularity at zero. ∎

We leave the next result as an exercise.

**Theorem 6.22.** *Suppose $f$ has an isolated singularity at $z = z_0$.*

1. *$z_0$ is a pole if and only if $\lim_{z \to z_0} f(z) = \infty$.*
2. *Suppose $z_0$ is essential and that $w \in \mathbb{C} \cup \{\infty\}$ is given. Then there exists a sequence $(z_n)$ converging to $z_0$ for which $\lim_{n \to \infty} f(z_n) = w$.*

Combined with the previous result, we see that $z_0$ is an essential singularity if and only if $\lim_{z \to z_0} f(z)$ does not exist.

The second result is the *Casorati–Weierstrass Theorem*; the range of $f(z)$ is *dense* in any neighborhood of an essential singularity. A stronger result is available, through its proof is beyond us.

**Theorem 6.23 (Picard).** *If $z_0$ is an essential singularity of $f(z)$, then, on any neighborhood of $z_0$, $f(z)$ takes every complex value* except at most one.

**Example 6.24.** We verify Picard's Theorem for the essential singularity of $f(z) = e^{1/z}$ at $z_0 = 0$. Let $w \in \mathbb{C} \setminus \{0\}$ be given, and write $w = re^{i\theta}$ where $0 \le \theta < 2\pi$. Then

$$f(z) = w \iff e^{1/z} = e^{\ln r + i\theta} \iff \frac{1}{z} = \ln r + i\theta + 2\pi i n$$

for some integer $n$. If $n > 0$, observe that

$$\left|\frac{1}{z}\right| = \sqrt{(\ln r)^2 + (\theta + 2\pi n)^2} \ge 2\pi n \implies |z| < \frac{1}{2\pi n}$$

whence $|z|$ can be chosen arbitrarily small. A suitable value $z$ therefore exists in any punctured disk $0 < |z| < \delta$. The function $f(z) = e^{1/z}$ thus takes on every complex value except one (namely $w = 0$).

**Exercises 6.3.** 1. Determine the order of each pole and its residue.

(a) $f(z) = \dfrac{z+1}{z^2+9}$ (b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

2. Verify the value of each residue:

(a) $\operatorname*{Res}_{z=-1} \dfrac{z^{1/4}}{z+1} = \dfrac{1+i}{\sqrt{2}}$ when $|z| > 0$ and $\arg z \in (0, 2\pi)$ (b) $\operatorname*{Res}_{z=i} \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\pi + 2i}{8}$

(c) $\operatorname*{Res}_{z=z_n} \big(z \sec z\big) = (-1)^{n+1} z_n$, where $z_n = \frac{\pi}{2} + n\pi$ and $n \in \mathbb{Z}$

3. Find the value of the integral round each circle:

$$\oint_C \frac{3z^3 + 2}{(z-1)(z^2+9)} \, dz$$

(a) $|z - 2| = 2$ (b) $|z| = 4$

4. Let $C$ be the circle $|z| = 2$. Evaluate $\oint_C \tan z \, dz$.

5. Suppose functions $f, g$ satisfy $g(z) = \frac{1}{f(z)}$. Prove:

$$\text{At } z_0, \quad f(z) \text{ has a zero of order } m \iff g(z) \text{ has a pole of order } m$$

6. (a) Evaluate $\operatorname*{Res}_{z=\frac{1}{n}} \left(z \sin \frac{\pi}{z}\right)^{-1}$ for each $n \in \mathbb{Z}$.

   (b) Why doesn't $\operatorname*{Res}_{z=0} \left(z \sin \frac{\pi}{z}\right)^{-1}$ make sense?

7. Suppose that $C$ is the rectangle whose sides are the lines $x = \pm 2, y = 0$ and $y = 1$. Prove that

$$\oint_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$$

   (*Hint: the integrand has four simple poles, only two of which lie inside* $C$)

8. (Hard) Suppose $f(z)$ and the contour $C$ satisfy the hypotheses of Cauchy's Argument Principle (Theorem 6.19).

   (a) Explain why we can be sure that there are only *finitely many* poles/zeros inside $C$ and that all zeros are isolated.

   (b) Complete the proof of the argument principle by performing a suitable integral on a small circle round each pole and applying Cauchy–Goursat and/or the Residue Theorem.

   (c) (Winding number) Consider the curve $\gamma = f(C)$. By substituting $w = f(z)$, explain why $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz$ counts the number times $\gamma$ orbits the origin counter-clockwise.

   (d) i. (Rouché's Theorem) Suppose $f$ is analytic on/inside C and that $|g(z)| < |f(z)|$ for all $z \in C$. Prove that the number of zeros of $f + g$ inside C equals that of $f$.
   (*Hints: Apply the argument principle to* $f + g = f \cdot (1 + \frac{g}{f})$ *and consider why* $1 + \frac{g}{f}$ *has winding number zero—why is* $\operatorname{Re}(1 + \frac{g}{f}) > 0$*?*)

   ii. How many solutions (up to multiplicity) are there to the equation $z^{22} + 4z^3 + 2i = 0$ on the domain $|z| < 1$?

9. (Hard) Prove both parts of Theorem 6.22; for simplicity, assume $z_0 = 0$.

   (*Hint 1:* $f(z)$ *has a pole if and only if* $\frac{1}{f(z)}$ *has a zero.*

   *Hint 2: If no such sequence exists, show that* $g(z) := \frac{1}{f(z)-w}$ *is analytic and bounded.*)

### 6.4 Improper Integrals

The theory of residues may be applied to help evaluate certain *real* improper integrals. We start with an alternative definition of improper integral suited to our purposes.

**Definition 6.25.** Suppose $f:\mathbb{R}\to\mathbb{R}$ is integrable. Provided the limit exists, the *Cauchy principal value* of the improper integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is the limit

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x$$

This definition is potentially misleading. In standard calculus such an improper integral requires *two* limits, *both* of which must exist for the integral to converge:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,\mathrm{d}x + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,\mathrm{d}x \qquad (*)$$

If the standard integral $(*)$ converges, then it equals its Cauchy principal value. However, the converse doesn't necessarily hold.

**Example 6.26.** If $f(x)$ is *any* odd function $\big(f(-x)=-f(x)\big)$, then

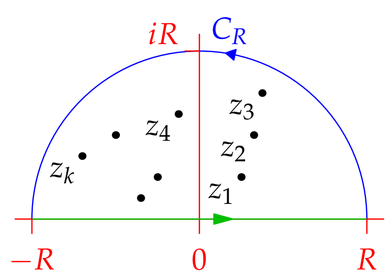
$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x = \lim_{R\to\infty} 0 = 0$$

However, if one (necessarily both) of the 1-sided improper integrals diverges, then the full integral also diverges: for instance,

$$\text{P.V.}\int_{-\infty}^{\infty} x^3\,\mathrm{d}x = 0 \quad\text{but}\quad \int_{0}^{\infty} x^3\,\mathrm{d}x \text{ diverges} \implies \int_{-\infty}^{\infty} x^3\,\mathrm{d}x \text{ diverges}$$

Residue theory supplies a neat trick for computing Cauchy principal values:

1. Suppose $f(x)$ is the restriction to the real line of a *complex function* $f(z)$ which is analytic on the upper half-plane ($\operatorname{Im} z\geq 0$) except at finitely many poles $z_1,\ldots,z_n$, none of which lie on the real axis.



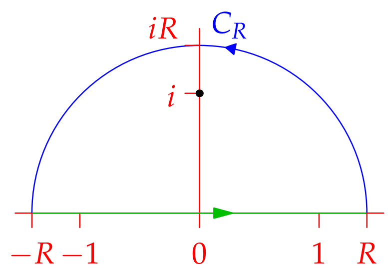

2. Choose $R>0$ so that all poles $z_k$ lie inside the curve formed by the real axis and the semi-circle $C_R$ with radius $R$. By Cauchy's Residue Theorem,

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)$$

3. If $\lim_{R\to\infty}\int_{C_R} f(z)\,\mathrm{d}z = 0$, then $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)$

The method is easiest when applied to rational functions $f(z)=\frac{p(z)}{q(z)}$, though it works more generally. Beyond the relative ease of residue calculation, one reason to stick to rational functions is that $\deg q\geq \deg p+2$ is enough to guarantee convergence in step 3 (Exercise 5).

**Examples 6.27.** 1. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$ has simple poles at $\pm i$. Provided $|z| = R > 1$,

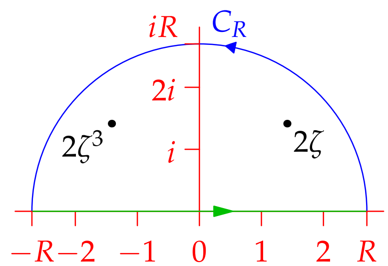



$$\left|z^2+1\right| \geq \left||z|^2 - 1\right| = R^2 - 1 \implies \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$
$$\implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0$$
$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{2\pi i}{2i} = \pi$$

Compare with the usual calculus method:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1} x\Big|_{-R_1\to-\infty}^{R_2\to\infty} = \pi$$

2. $f(z) = \frac{4(z^2-1)}{z^4+16}$ has simple poles at $\pm 2\zeta, \pm 2\zeta^3$ where $\zeta = e^{\frac{\pi i}{4}}$.
Let $p(z) = 16(z^2-1)$ and $q(z) = z^4+16$, so that

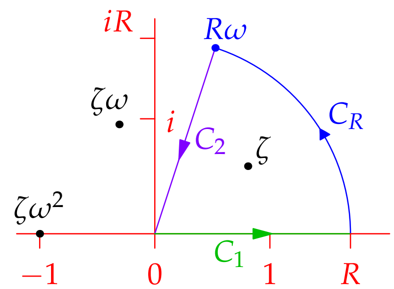



$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^2-1}{z_0^3}$$

When $|z| = R > 2$, we see that

$$\left|z^4+16\right| \geq R^4 - 16 \implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{4\pi R(R^2+1)}{R^4-16} \to 0$$
$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\left(\operatorname*{Res}_{z=2\zeta} f(z) + \operatorname*{Res}_{z=2\zeta^3} f(z)\right) = 2\pi i\left(\frac{4\zeta^2-1}{8\zeta^3} + \frac{4\zeta^6-1}{8\zeta^9}\right) = \frac{3\pi}{2\sqrt{2}}$$

3. Variations are possible, for instance by taking only part of a semi-circular arc. The function $f(z) = \frac{1}{z^5+1}$ has five simple poles: the fifth-roots of $-1$.
Since the pole $\zeta\omega^2 = -1$ lies on the negative real axis, the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ diverges. Instead consider the arcs in the picture when $R > 1$. Parametrizing $C_2$ via $z(t) = t\omega$,




$$\int_{C_2} \frac{1}{z^5+1}\,\mathrm{d}z = \int_R^0 \frac{\omega}{t^5+1}\,\mathrm{d}t = -\omega\int_0^R \frac{1}{t^5+1}\,\mathrm{d}t$$
$$= -\omega\int_{C_1} \frac{1}{z^5+1}\,\mathrm{d}z$$

$$\implies (1-\omega)\int_0^R \frac{1}{x^5+1}\,\mathrm{d}x + \int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=\zeta} \frac{1}{z^5+1} = \frac{2\pi i}{5\zeta^4} = \frac{2\pi i}{5\omega^2}$$

When $|z| = R > 1$, we see that $\left|z^5+1\right| \geq R^5 - 1 \implies \left|\int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z\right| \leq \frac{2\pi R}{5(R^5-1)} \xrightarrow[R\to\infty]{} 0$, and we conclude

$$\int_0^{\infty} \frac{1}{x^5+1}\,\mathrm{d}x = \frac{2\pi i}{5(\omega^2-\omega^3)} = \frac{2\pi i}{5\zeta\omega^2(\zeta^{-1}-\zeta)} = \frac{2\pi i}{5(2i\sin\frac{\pi}{5})} = \frac{\pi}{5}\csc\frac{\pi}{5}$$

**Jordan's Lemma**

It is often desired, particularly when computing Fourier transforms,[18] to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = \int_{-\infty}^{\infty} f(x)\cos ax\,dx + i\int_{-\infty}^{\infty} f(x)\sin ax\,dx$$

where $a > 0$ is a real constant and $f : \mathbb{R} \to \mathbb{C}$ is a given function. If $f(x)$ is real-valued, then the above breaks the integral into real and imaginary parts. Given reasonable conditions on $f(x)$, the principal-value method can often be employed.

**Example 6.28.** The function $f(z) = \frac{1}{z^2+4}$ is analytic on the upper half-plane except at the simple pole $z = 2i$. With $R > 2$ and $C_R$ the usual semi-circle, we see that

$$\left|e^{3iz}\right| = e^{-3y} \le 1 \implies \left|\int_{C_R} f(z)e^{3iz}\,dz\right| \le \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{3ix}}{x^2+4}\,dx = 2\pi i \operatorname*{Res}_{z=2i}\frac{e^{3iz}}{z^2+4} = 2\pi i\frac{e^{-6}}{4i} = \frac{1}{2}\pi e^{-6}$$

Since this is real, we see that the result is in fact the integral $\int_{-\infty}^{\infty}\frac{\cos 3x}{x^2+4}\,dx$. We don't need the Cauchy principal value here since the full improper integral converges. The corresponding imaginary integral is trivially zero since $\frac{\sin 3x}{x^2+1}$ is an odd function.

The following result, which we state without proof, assists with such computations; as usual, for any $R > 0$, $C_R = \{z : |z| = R, \operatorname{Im} z \ge 0\}$ is the counter-clockwise semi-circle in the upper half-plane.

**Theorem 6.29 (Jordan's Lemma).** *Let $a, R_0 > 0$ be constants and $f(z)$ a function. Suppose:*

1. *$f(z)$ is analytic on the upper half-plane outside the semi-circle $C_{R_0}$.*
2. *Whenever $|z| = R > R_0$, $f(z)$ is bounded by some function $M(R)$ with $\lim_{R\to\infty} M(R) = 0$.*

*Then $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0$. If $f(z)$ moreover satisfies the hypotheses of our method, then*

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)e^{iaz}$$

**Example 6.30.** If $f(x) = \frac{x+2}{x^2+9}$ and $R > 3$, then

$$|f(z)| = \frac{|z+2|}{|z^2+9|} \le \frac{R+2}{R^2-9} = M_R \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty}\frac{(x+2)e^{iax}}{x^2+9}\,dx = 2\pi i\operatorname*{Res}_{z=3i}\frac{(z+2)e^{iaz}}{z^2+9} = \frac{2\pi i(2+3i)e^{-3a}}{6i} = \frac{\pi(2+3i)}{3}e^{-3a}$$

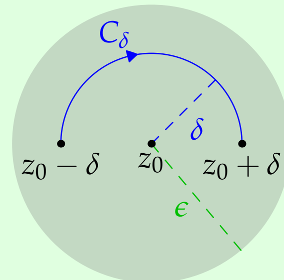
By considering even and odd functions, etc., we can rewrite this as

$$\int_0^{\infty}\frac{\cos ax}{x^2+9}\,dx = \frac{\pi}{6}e^{-3a} \qquad \int_0^{\infty}\frac{x\sin ax}{x^2+9}\,dx = \frac{\pi}{2}e^{-3a}$$

[18] The Fourier transform of $f(x)$ is the function $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx$.

**Indented paths** Another modification allows $f(z)$ to have a simple pole on the real axis.

**Lemma 6.31.** *Let $D$ be the disk $|z-z_0| \le \epsilon$, let $\delta < \epsilon$, and let $C_\delta$ be the clockwise semi-circle pictured.*




1. *If $\phi(z)$ is analytic on $D$, then $\lim_{\delta \to 0} \int_{C_\delta} \phi(z)\,\mathrm{d}z = 0$.*
2. *If $f(z)$ is analytic on $D \setminus \{z_0\}$ with a simple pole at $z_0$, then*

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -\pi i \operatorname*{Res}_{z=z_0} f(z)$$

More generally, if $C_\delta$ spans $\theta$ radians clockwise round $z_0$, then $\lim_{\delta \to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -i\theta \operatorname*{Res}_{z=z_0} f(z)$.

*Proof.* 1. $\phi$ is continuous on $D$ and is thus bounded by some $M$; but then $\left|\int_{C_\delta} \phi(z)\,\mathrm{d}z\right| \le M\pi\delta$.

2. The Laurent series expansion of $f(z)$ on $D \setminus \{z_0\}$ is

$$f(z) = \frac{a_{-1}}{z-z_0} + \phi(z)$$

where $a_{-1} = \operatorname*{Res}_{z=z_0} f(z)$ and $\phi(z)$ is analytic on $D$. Now observe that

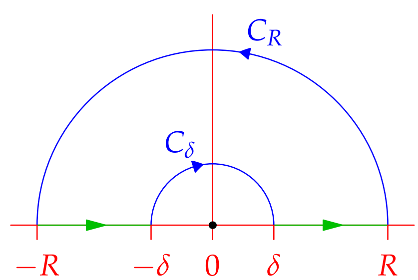
$$\int_{C_\delta} \frac{a_{-1}}{z-z_0}\,\mathrm{d}z = a_{-1} \int_\pi^0 \frac{1}{\delta e^{i\theta}} i\delta e^{i\theta}\,\mathrm{d}\theta = -ia_{-1} \int_0^\pi \mathrm{d}\theta = -\pi i a_{-1}$$

is independent of $\delta$. By part (a), we conclude that $\lim_{\delta \to 0} \int_{C_\delta} f(z)\,\mathrm{d}z = -\pi i \operatorname*{Res}_{z=z_0} f(z)$. ∎

**Example 6.32.** Consider $f(z) = \frac{e^{iz}}{z}$, which has a simple pole at $z_0 = 0$. If $0 < \delta < R$, then

$$\left(\int_{-R}^{-\delta} + \int_\delta^R\right) f(x)\,\mathrm{d}x = \left(\int_{-R}^{-\delta} + \int_\delta^R\right) \frac{\cos x + i \sin x}{x}\,\mathrm{d}x = 2i \int_\delta^R \frac{\sin x}{x}\,\mathrm{d}x$$

by even/oddness. Moreover, by Lemma 6.31,




$$\lim_{\delta \to 0} \int_{C_\delta} \frac{e^{iz}}{z}\,\mathrm{d}z = -i\pi \operatorname*{Res}_{z=0} f(z) = -i\pi$$

Since $|f(z)| = \frac{e^{-y}}{R} \le \frac{1}{R}$ on $C_R$, Jordan's lemma tells us that

$$0 = 2i \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - i\pi \implies \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

The example relied on the evenness of $\frac{\sin x}{x}$ and on the fact that the region of the half-plane between $C_R$ and $C_\delta$ contains no poles of $f(z)$. We essentially evaluated $\int_0^R \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2}\int_{-R}^R \frac{\sin x}{x}\,\mathrm{d}x$ using an *indented path* lying on the $x$-axis but dodging round the simple pole at zero. Many other versions of this trick are possible!

**Exercises 6.4.** Many of these problems require extensive calculations using residues. Take your time and use them as an excuse to practice.

1. Use residues to verify each improper integral:

   (a) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4}$ (b) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^6+1} = \frac{\pi}{6}$

   (c) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$ (d) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$

2. Find the Cauchy principal value of each integral:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^2+2x+2}$ (b) $\displaystyle\int_{-\infty}^\infty \frac{x\,\mathrm{d}x}{(x^2+1)(x^2+2x+2)}$

3. Let $m, n$ be integers where $0 \le m \le n-2$. By mimicking Example 6.27.3, prove that

   $$\int_0^\infty \frac{x^m}{x^n+1}\,\mathrm{d}x = \frac{\pi}{n}\csc\frac{(m+1)\pi}{n}$$

4. (a) If $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges, prove that it equals its Cauchy principal value.

   (b) Suppose $f(x)$ is an even function and that P.V. $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists. Prove that $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists and has the same value.

5. Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function where $q(x)$ has no (real) zeros and $2 + \deg p \le \deg q$. Prove that $\int_0^\infty f(x)\,\mathrm{d}x$ converges.

   (*Hint: let $p, q$ be monic and recall the comparison test for improper integrals*)

6. Prove the identities:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$ if $a > b > 0$

   (b) $\displaystyle\int_0^\infty \frac{\cos ax\,\mathrm{d}x}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab}$ if $a, b > 0$

7. Evaluate the integrals:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{x\sin x\,\mathrm{d}x}{(x^2+1)(x^2+4)}$ (b) $\displaystyle\int_0^\infty \frac{x^3\sin x\,\mathrm{d}x}{(x^2+1)(x^2+9)}$

8. If $a$ is any real number and $b > 0$, find the Cauchy principal value of $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x+a)^2+b^2}$

9. Use $f(z) = z^{-2}(e^{iaz} - e^{ibz})$ and an indented contour around $z_0 = 0$ to prove that

   $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} = \frac{\pi}{2}(b-a) \quad \text{whenever} \quad a, b \ge 0$$

10. By integrating $f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp(-\frac{1}{2}\log z)}{z^2+1}$ where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ along an indented contour, prove that

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

11. What happens to part 2 of Lemma 6.31 if $f(z)$ is analytic on $D \setminus \{z_0\}$ but has a pole of order $m \geq 2$ at $z_0$.

12. (Hard) A similar trick can be applied with sequences of boundary curves $C_N$. For instance, for each $N \in \mathbb{N}$, let $C_N$ denote the positively oriented boundary of the square whose edges lie along the lines $x, y = \pm\left(N + \frac{1}{2}\right)\pi$. Prove that

$$\oint_{C_N} \frac{\mathrm{d}z}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

Hence conclude that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.