BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 3, July 2024, Pages 515–524
https://doi.org/10.1090/bull/1840
Article electronically published on May 15, 2024

Regularity theory for elliptic PDE, by X. Fernández-Real and X. Ros-Oton, Zur. Lect. Adv. Math, Vol. 28, EMS Press, Berlin, 2022, viii+228 pp., ISBN 978-3-98547-028-0

1. Introduction to regularity theory

In complex analysis we learn that functions from $\mathbb{C}$ to $\mathbb{C}$ with one complex derivative must have derivatives of all orders. They are in fact analytic. This remarkable property of solutions to the Cauchy–Riemann equations is the first instance of regularity theory that many of us encounter: certain partial differential equations admit only smooth solutions, even though the equation only requires a few derivatives to make sense. The goal of regularity theory is to determine whether the same result holds for a given PDE, and if not, to what extent it fails.

Answering such questions can have deep and far-reaching consequences. It is known, for example, that the Navier–Stokes equations and the Hamilton–Ricci flow, two nonlinear systems of PDEs, admit solutions that are smooth for a short time with appropriate assumptions on the initial data. However, it is possible that after some time, the smoothness property fails and the solutions develop singularities. In the case of the Hamilton–Ricci flow it is well known that this happens. Nonetheless, it turns out that (in low dimensions) there is a good understanding of the possible singularities. This was the key to resolving the Poincaré conjecture from topology. For the Navier–Stokes equations, it remains unknown whether singularites happen. A negative answer would, e.g., imply uniqueness of solutions. This would be desirable not only mathematically, but also philosophically in view of the fact that the Navier–Stokes equations model the behavior of a huge number of particles evolving by the laws of classical physics. An answer in either direction could also deepen our understanding of turbulence and aid in the design of more reliable numerical approximations of fluid behavior.

Not all important PDEs admit only regular solutions. For example, the two-variable wave equation $u_{xy} = 0$ has a general solution of the form $f(x) + g(y)$, where f and g are arbitrary (and, in particular, need not be smooth). The issue is that this equation is *hyperbolic*, not *elliptic*. We delay stating precise definitions of ellipticity to later. For now we will mention several ways of understanding what ellipticity means intuitively, in the context of the Laplace equation

$$\Delta u = \sum_{i=1}^{n} u_{ii} = 0, \tag{1}$$

where u is a function on a domain in $\mathbb{R}^n$.

The first is based on the fact that $\Delta u(x)$ measures how much, infinitesimally, $u(x)$ deviates from its average near x:

$$\Delta u(x) = \lim_{r \to 0^+} \frac{c_n}{r^2 |B_r|} \int_{B_r(x)} (u(y) - u(x))\, dy,$$

for some dimensional constant c_n. Harmonic functions are, in fact, characterized by the property that their average in a ball agrees with the value at the center. Since averaging adds derivatives, it stands to reason that harmonic functions are smooth.

The second is variational. Harmonic functions are minimizers of the Dirichlet integral

$$\int |\nabla u|^2,$$

subject to their own boundary data. Since $|\cdot|^2$ is convex, its average value on some set is larger than its value at the center of mass. In view of this observation we do not expect ∇u to oscillate wildly, otherwise we could smooth u out (average its gradient) and get a function with smaller Dirichlet integral.

The last is the maximum principle. Equation (1) says that, near any point, the graph of u bends upwards in some direction as much as it bends downwards in another. The solution thus has no interior maxima or minima. Because the derivatives of u are also harmonic, they obey the same principle. In particular, if the solution is smooth in a neighborhood of the boundary of some region, one gets derivative bounds of all orders inside. One can, in fact, do much better (get interior derivative bounds of all orders depending only on how much u oscillates on the boundary) using more sophisticated reasoning based on the same idea.

The last two viewpoints on regularity are useful for studying nonlinear elliptic PDEs. There are two very important classes of such equations. The first class consists of the Euler–Lagrange equations of functionals of the form

$$\int F(\nabla u), \tag{2}$$

where F is convex. Examples include the minimal surface equation and its variants. This class of equations was the topic of Hilbert's 19^{th} problem, stated in 1900, which spurred tremendous advances such as the theorem of De Giorgi and Nash on the continuity of solutions to linear PDEs in divergence form with rough coefficients [10], [41]. Unsurprisingly, variational techniques are well suited to deal with such equations. The second class consists of fully nonlinear equations of the form

$$G(D^2 u) = 0, \tag{3}$$

where G is a function on the space of $n \times n$ symmetric matrices. Examples include the Monge–Ampère and special Lagrangian equations. The breakthroughs giving a satisfactory theory of such equations were due to Krylov–Safonov [28], Evans [12], and Krylov [27] in the 1980s. Fully nonlinear PDEs are studied using primarily maximum principle techniques.

Regularity theory for nonlinear elliptic PDEs (with both variational and non-variational structure) remains an extremely active field of study. It has deep connections to elasticity, differential geometry, and fluid mechanics, among other areas, and many interesting questions remain unsolved. Below we will dive a little deeper into this story, and discuss how the book *Regularity Theory for Elliptic PDE*, by Fernández-Real and Ros-Oton, fits into the picture. We stress that this is not meant to be an exhaustive overview. The topics we discuss below reflect those appearing in the book, as well as our own research directions and tastes.

2. Hilbert's 19^{TH} problem

In the statement of Hilbert's 19^{th} problem it is noted that several important PDEs have the same property as the Cauchy–Riemann equations: all solutions are analytic. It is also remarked that these PDEs tend to arise as Euler–Lagrange equations of integrals of the form (2), which can be written

$$\operatorname{div}(\nabla F(\nabla u)) = F_{ij}(\nabla u)u_{ij} = 0. \tag{4}$$

Finally, it is conjectured that solutions to all such Euler–Lagrange equations are analytic, provided F is analytic and locally uniformly convex.

Under appropriate assumptions, e.g., on boundary data, it is not hard to produce Lipschitz functions that solve the equation (4) in a weak sense. By the 1930s, work of Bernstein, Hopf, Schauder, and others reduced Hilbert's problem to proving that these solutions are in fact C^1 (see, e.g., the classical reference [34, Chapter 5], and the references therein). To understand why, it is useful to consider the nondivergence form $F_{ij}(\nabla u)u_{ij} = 0$ of the equation. When ∇u is continuous, the coefficients $F_{ij}(\nabla u)$ are nearly constant on small scales, thus the solution resembles a harmonic function.

This left the problem of filling the gap from Lipschitz to C^1 regularity. To that end it is natural to look at the equation that derivatives of u satisfy. Differentiating the equation once in direction e gives

$$\partial_i(F_{ij}(\nabla u)(u_e)_j) = 0.$$

Immediately we see why going from Lipschitz to C^1 is challenging: because we only know that u is Lipschitz, the coefficients $F_{ij}(\nabla u)$ are just bounded and measurable, so we cannot expect that u_e resembles a harmonic function at any scale. Nonetheless, it can be shown that solutions to such equations are continuous. This was accomplished by Morrey in two dimensions [36], and by De Giorgi [10] and Nash [41] in higher dimensions, completing the solution to Hilbert's problem.

Variants of Hilbert's problem continue to have a profound impact. One such variant is to consider vector-valued maps u from domains in $\mathbb{R}^n$ to $\mathbb{R}^m$ which are critical points of functionals of the form (2), where F is a smooth, uniformly convex function with bounded Hessian on the space of $m \times n$ matrices and $m \geq 2$. The components of such maps solve a *system* of PDEs. Systems are harder due to the loss of maximum principle techniques, and perhaps more fundamentally, because they model the complex behaviors we see, e.g., in fluid motion and materials science. Morrey proved that solutions are smooth if $n = 2$ and m is arbitrary [33], [35], and Uhlenbeck proved regularity for arbitrary n, m when $F(M)$ depends only on $|M|$ [56]. In general, counterexamples in [42], [52], [53], and most recently in [31] show that regularity fails when $n \geq 3$ and $m \geq 2$. The best one can get is partial regularity: solutions are smooth away from a small singular set of $n-2$-dimensional measure zero [19], [17]. The counterexamples show that the best one can hope for is that the singular set has dimension $n-3$, but this remains open.

Another important point is that when $m \geq 2$, the natural restriction on F to prove the existence of minimizers is not convexity, but a weaker condition known as quasiconvexity. In this case, minimizing (2) and being a critical point are different (they are equivalent when F is convex). Partial regularity is known for minimizers of quasiconvex functionals [14], but it turns out that critical points can be terribly behaved, e.g., Lipschitz but nowhere C^1 [37], [54]. These counterexamples were constructed using a method known as convex integration, which has its origins in the study of isometric embeddings [40] and more recently has been used to construct wild solutions to fluid equations and solve Onsager's conjecture [24].

Since counterexamples abound in the systems case, an important future direction is to identify structure conditions on F that guarantee regularity. A recent result in this direction is [23], which proves the smoothness of Lipschitz critical points of the area functional (an important quasiconvex functional) when $n = 2$ and m is arbitrary. Another interesting direction which is difficult even in the scalar case $m = 1$ is to consider convex integrands F that are not smooth and uniformly convex. A basic example is the p-Laplace energy density $F(x) = |x|^p$, $p \neq 2$. More complex examples arise, e.g., in the study of random surfaces coming from statistical mechanics [26]. Heuristics suggest that solutions are C^1 provided F is strictly convex. It is useful to assume in addition that D^2F is positive and bounded away from some small "degeneracy set" D. The C^1 regularity of solutions was confirmed in two dimensions by De Silva and Savin when D is finite [11]. In [30] this result is extended to higher dimensions provided D is finite and contained in a 2-plane, and the C^1 regularity of solutions is shown to be false for general strictly convex F (the example is in four dimensions, and D is the Clifford torus). It remains open what happens for general strictly convex F in two dimensions, or if D is finite in dimension three or higher. For the latter problem, evidence towards a negative result is presented in [30]. More precisely, a concrete counterexample is proposed in $\mathbb{R}^3$ with D consisting of four (necessarily) noncoplanar points.

3. Fully Nonlinear Elliptic PDE

Fully nonlinear elliptic equations of the form (3), which in general do not have a variational structure, are also ubiquitous in applications. Examples include the Bellman and Isaacs equations from economics, the Monge–Ampère equation from optimal transport, and the σ_k and special Lagrangian equations from differential geometry. In this context, ellipticity means that $G_{ij}(M) := \frac{\partial G}{\partial M_{ij}}$ is a positive matrix for all M. This enables the use of the maximum principle.

The first situation to attack is the one analogous to Hilbert's 19th problem. That is, assume that G is smooth and uniformly elliptic (the eigenvalues of $G_{ij}(M)$ are bounded between fixed positive constants, independent of M). It is not hard to reduce the regularity problem to proving that solutions are C^2. Indeed, differentiating the equation once gives

$$G_{ij}(D^2u)(u_k)_{ij} = 0,$$

and the coefficients are nearly constant on small scales when u is C^2. In two dimensions the regularity problem was solved by Nirenberg [43] in the 1950s. The idea is that D^2u is a quasiconformal map into the surface $\{G = 0\}$, and such maps are known to be continuous. The first major breakthrough on this problem in higher dimensions was due to Krylov–Safonov in 1980 [28]. They proved the continuity of solutions to linear elliptic equations in non-divergence form with rough coefficients (the nondivergence analogue of the De Giorgi–Nash theorem). However, this is only enough to ensure the continuity of one derivative of a solution to (3). One might hope to differentiate the equation once more and apply the Krylov–Safonov theorem again, but the resulting equation for the second derivatives of u involves a term quadratic in third derivatives that is unclear how to handle. A key point is that this term has a sign if G is convex or concave. This was leveraged by Evans [12] and Krylov [27] to make the next major breakthrough, getting full regularity of solutions when G is convex or concave and uniformly elliptic.

It remained open for a long time whether one can relax the concavity assumption on G. This was only solved in the last 10–15 years, when Nadirashvili–Vlăduţ produced a series of spectacular counterexamples in progressively smaller dimension, culminating in examples in dimension $n \geq 5$ [39]. Remarkably, the regularity question for general uniformly elliptic equations remains open in dimensions 3 and 4. There is some evidence that a positive result may hold in these cases [38].

With the counterexamples in hand, a natural question is what structural conditions, apart from concavity, will guarantee regularity. Some interesting works in this direction include positive results for the three operator Isaacs equation ([2]), the case that $\{G = 0\}$ has only one negative principal curvature [6], and the case that G is the sum of a convex and a concave operator [9]. It is also known that in dimension n, solutions to (3) are smooth away from a small closed singular set of dimension slightly smaller than n [45], [1]. The examples suggest that one might be able to prove that the singular set has dimension $n - 5$, but there is only minor progress towards closing this gap (see, e.g., [29]).

Finally, it must be noted that in most applications, the uniform ellipticity condition is not satisfied. This is the case for the Monge–Ampère (σ_n), σ_k, and special Lagrangian equations. Work on the regularity theory for these equations remains extremely active. The σ_k equation is concave, but well-known examples of Pogorelov [44] and Urbas [57] show that regularity cannot be expected when $k \geq 3$. It remains a remarkable open problem whether solutions to $\sigma_2(D^2u) = 1$ are smooth. This problem was only recently solved in dimension $n \leq 4$ [50]. The special Lagrangian equation is not concave in general, and when this is the case, very little is known. The interest in the equation stems from the fact that smooth solutions have volume-minimizing gradient graph [21]. The existence of viscosity solutions to the Dirichlet problem is known [22], and one might hope that these viscosity solutions also have minimal gradient graph. This was recently shown to be false in [31], where examples of non-C^1 viscosity solutions to the special Lagrangian equation with non-minimal gradient graph were constructed in three dimensions. A beautiful open problem for the special Lagrangian equation is to determine whether there exist solutions that are homogeneous of degree two but not quadratic (that is, whether there exist non-flat graphical special Lagrangian cones).

4. Free boundary problems

A final class of problems that is central in elliptic PDE theory consists of problems in which a function solves a PDE on some region, but the boundary of this region is not a fixed part of the problem. Typically, the solution satisfies some overdetermined condition on this “free boundary.” Among the most well-studied examples is the obstacle problem, in which an elastic membrane is stretched over fixed obstacle. The height of the membrane solves a PDE (e.g., the minimal surface equation or the Laplace equation) where the membrane lies above the obstacle, and the boundary of this region is the free boundary. The overdetermined condition on the free boundary is that the solution and the obstacle match up to order one. After subtracting the height of the obstacle, the simplest version of this problem is

$$\Delta u = \chi_{\{u>0\}} \text{ in } B_1 \subset \mathbb{R}^n, \quad u \geq 0, \tag{5}$$

and the goal is to understand the regularity of the solution u and the free boundary $\partial\{u > 0\}$.

It turns out that the techniques developed to deal with the classes of PDEs discussed above, along with ideas from the theory of minimal surfaces, are extremely useful to study the properties of the solution and its free boundary. In many ways, the regularity theory parallels that from the theory of minimal surfaces. Namely, one can "blow up" (perform a sequence of rescalings that zoom in infinitely close) at points on the free boundary. Monotonicity formulae and PDE arguments permit a classification of the possible blow-up limits, analogues of tangent cones in minimal surface theory. When a blow-up limit at a free boundary point is (up to a rotation) the function $(\max\{x_1, 0\})^2/2$ (analogous to a flat tangent cone), the point is called a "regular point." Otherwise, the point is called a "singular point."

The optimal regularity of the solution u (namely, $C^{1,1}$) was established in [16]. In Caffarelli's seminal 1977 work on the topic [4], the smoothness of the free boundary in a neighborhood of any regular point is proven. This result can be regarded as an analogue of the De Giorgi–Allard ϵ-regularity theorem for minimal surfaces. Caffarelli also showed that the collection of singular points is locally contained in a C^1 hypersurface [3]. Examples show that this cannot be improved in general [49]. However, this does not rule out the possibility that making tiny perturbations of the boundary data of u can make the free boundary smooth everywhere. In other words, that singular points are nongeneric. This was conjectured to be true by Schaeffer in 1974 [48]. Schaeffer's conjecture has been confirmed up to dimension $n = 4$ only recently [15], but it remains open in higher dimensions. This closely parallels recent developments in the theory of minimal hypersurfaces. Indeed, in [7], [8] it is shown that small boundary perturbations remove singularities in solutions to the Plateau problem in ambient dimensions up to 10.

The subject of free boundary problems continues to flourish, and to find applications in surprising areas. Examples include problems involving many interacting membranes [46], which arise naturally in min-max procedures in differential geometry [58], [59], and problems involving different operators (e.g., fully nonlinear operators) for which monotonicity formulae are not available [47]. Moreover, although it is well understood that minimal surfaces and free boundary problems can be approached in similar ways, it remains an intriguing and mysterious task to find explicit connections between the subjects. Some beautiful progress in this direction can be found in [55] and [25], in low dimensions.

5. Book review

The book *Regularity Theory for Elliptic PDE* by Ros-Oton and Fernández-Real offers a readable and self-contained introduction to the topics discussed above, beginning with harmonic functions and Schauder estimates, proceeding to Hilbert's 19^{th} problem and the De Giorgi–Nash–Moser theorem, continuing with the theory of fully nonlinear elliptic equations (including a complete discussion of the two-dimensional case and touching on the theory in higher dimensions), and concluding with a thorough treatment of the obstacle problem.

There is a welcome difference in both presentation and choice of topics than in books on the same subject such as those of Gilbarg and Trudinger [18], Han and Lin [20], and Caffarelli and Cabré [5]. We highlight some of these here. First, the Schauder estimates are treated in several ways, including via the blow-up method of Simon [51], which is a clean and conceptually compelling approach. In addition, the Schauder estimates for equations in both divergence form and non-divergence form are included, giving both variational and non-variational viewpoints a chance to shine. Second, a proof of the existence of viscosity solutions to the Dirichlet problem for fully nonlinear elliptic PDEs via Perron's method is presented. This fills a notable gap in the treatment of fully nonlinear PDEs in other references. Third, the obstacle problem is discussed at length, and complete, concise, and self-contained proofs of the most important results are presented, including Caffarelli's celebrated results on the smoothness of the free boundary near regular points. Previously, this topic was left to more specialized works. Finally, at the end of the chapters on nonlinear PDEs and on the obstacle problem, the current state of the topic and outstanding open questions (including some of those mentioned above) are presented, giving the reader a glimpse of the vitality of the subject. This is a real achievement, given that the book starts with the very basics.

Readers hoping to learn other central topics such as the Calderón-Zygmund $W^{2,p}$ estimates, the theory of fully nonlinear elliptic PDEs in higher dimensions (e.g., the Evans–Krylov estimate), or the minimal surface and Monge–Ampère equations, will have to look elsewhere (e.g., in the books of Gilbarg and Trudinger or Caffarelli and Cabré). Nonetheless, this book covers a lot of ground and can serve as the basis for a year-long graduate course on elliptic PDEs at a similar level to the book of Han and Lin. It is appropriate for students with some exposure to PDEs, e.g., from the book of Evans [13]. This book can moreover serve as a reference for researchers hoping to rapidly to get up to speed on the theory of the obstacle problem. As such, I warmly recommend it to all serious students and researchers in applied mathematics, geometry, and PDE. It will be a valuable addition to your bookshelf.

References

[1] Scott N. Armstrong, Luis E. Silvestre, and Charles K. Smart, *Partial regularity of solutions of fully nonlinear, uniformly elliptic equations*, Comm. Pure Appl. Math. **65** (2012), no. 8, 1169–1184, DOI 10.1002/cpa.21394. MR2928094

[2] Xavier Cabré and Luis A. Caffarelli, *Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations* (English, with English and French summaries), J. Math. Pures Appl. (9) **82** (2003), no. 5, 573–612, DOI 10.1016/S0021-7824(03)00029-1. MR1995493

[3] L. A. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 383–402, DOI 10.1007/BF02498216. MR1658612

[4] Luis A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), no. 3-4, 155–184, DOI 10.1007/BF02392236. MR454350

[5] Luis A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995, DOI 10.1090/coll/043. MR1351007

[6] Luis A. Caffarelli and Yu Yuan, *A priori estimates for solutions of fully nonlinear equations with convex level set*, Indiana Univ. Math. J. **49** (2000), no. 2, 681–695, DOI 10.1512/iumj.2000.49.1901. MR1793687

[7] O. Chodosh, C. Mantoulidis, and F. Schulze, *Generic regularity for minimizing hypersurfaces in dimensions 9 and 10*, Preprint, `arXiv:2302.02253`, 2023.

[8] O. Chodosh, C. Mantoulidis, and F. Schulze, *Improved generic regularity of codimension-1 minimizing integral currents*, Preprint, `arXiv:2306.13191`, 2023.

[9] Tristan C. Collins, *$C^{2,\alpha}$ estimates for nonlinear elliptic equations of twisted type*, Calc. Var. Partial Differential Equations **55** (2016), no. 1, Art. 6, 11, DOI 10.1007/s00526-015-0950-y. MR3441283

[10] Ennio De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari* (Italian), Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR93649

[11] Daniela De Silva and Ovidiu Savin, *Minimizers of convex functionals arising in random surfaces*, Duke Math. J. **151** (2010), no. 3, 487–532, DOI 10.1215/00127094-2010-004. MR2605868

[12] Lawrence C. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **35** (1982), no. 3, 333–363, DOI 10.1002/cpa.3160350303. MR649348

[13] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998, DOI 10.1090/gsm/019. MR1625845

[14] Lawrence C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rational Mech. Anal. **95** (1986), no. 3, 227–252, DOI 10.1007/BF00251360. MR853966

[15] Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra, *Generic regularity of free boundaries for the obstacle problem*, Publ. Math. Inst. Hautes Études Sci. **132** (2020), 181–292, DOI 10.1007/s10240-020-00119-9. MR4179834

[16] Jens Frehse, *On the regularity of the solution of a second order variational inequality* (English, with Italian summary), Boll. Un. Mat. Ital. (4) **6** (1972), 312–315. MR318650

[17] Mariano Giaquinta and Enrico Giusti, *Nonlinear elliptic systems with quadratic growth*, Manuscripta Math. **24** (1978), no. 3, 323–349, DOI 10.1007/BF01167835. MR481490

[18] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983, DOI 10.1007/978-3-642-61798-0. MR737190

[19] E. Giusti and M. Miranda, *Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari* (Italian), Arch. Rational Mech. Anal. **31** (1968/69), 173–184, DOI 10.1007/BF00282679. MR235264

[20] Qing Han and Fanghua Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, vol. 1, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. MR1669352

[21] Reese Harvey and H. Blaine Lawson Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157, DOI 10.1007/BF02392726. MR666108

[22] F. Reese Harvey and H. Blaine Lawson Jr., *Dirichlet duality and the nonlinear Dirichlet problem*, Comm. Pure Appl. Math. **62** (2009), no. 3, 396–443, DOI 10.1002/cpa.20265. MR2487853

[23] J. Hirsch, C. Mooney, and R. Tione, *On the Lawson-Osserman conjecture*, Preprint, `arXiv:2308.04997`, 2023.

[24] Philip Isett, *A proof of Onsager's conjecture*, Ann. of Math. (2) **188** (2018), no. 3, 871–963, DOI 10.4007/annals.2018.188.3.4. MR3866888

[25] David Jerison and Nikola Kamburov, *Structure of one-phase free boundaries in the plane*, Int. Math. Res. Not. IMRN **19** (2016), 5922–5987, DOI 10.1093/imrn/rnv339. MR3567263

[26] Richard Kenyon, Andrei Okounkov, and Scott Sheffield, *Dimers and amoebae*, Ann. of Math. (2) **163** (2006), no. 3, 1019–1056, DOI 10.4007/annals.2006.163.1019. MR2215138

[27] N. V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 1, 75–108. MR688919

[28] N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 161–175, 239. MR563790

[29] Connor Mooney, *A proof of the Krylov-Safonov theorem without localization*, Comm. Partial Differential Equations **44** (2019), no. 8, 681–690, DOI 10.1080/03605302.2019.1581807. MR3952774

[30] Connor Mooney, *Minimizers of convex functionals with small degeneracy set*, Calc. Var. Partial Differential Equations **59** (2020), no. 2, Paper No. 74, 19, DOI 10.1007/s00526-020-1723-9. MR4079759

[31] C. Mooney and O. Savin, *Non C^1 solutions to the special Lagrangian equation*, To appear in Duke Math. J., `arXiv:2303.14282`, 2023.

[32] Connor Mooney and Ovidiu Savin, *Some singular minimizers in low dimensions in the calculus of variations*, Arch. Ration. Mech. Anal. **221** (2016), no. 1, 1–22, DOI 10.1007/s00205-015-0955-x. MR3483890

[33] Charles B. Morrey Jr., *Existence and differentiability theorems for the solutions of variational problems for multiple integrals*, Bull. Amer. Math. Soc. **46** (1940), 439–458, DOI 10.1090/S0002-9904-1940-07229-5. MR2473

[34] Charles B. Morrey Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag New York, Inc., New York, 1966. MR202511

[35] Charles B. Morrey Jr., *Multiple integral problems in the calculus of variations and related topics*, Univ. California Publ. Math. (N.S.) **1** (1943), 1–130. MR0011537

[36] Charles B. Morrey Jr., *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), no. 1, 126–166, DOI 10.2307/1989904. MR1501936

[37] S. Müller and V. Šverák, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Ann. of Math. (2) **157** (2003), no. 3, 715–742, DOI 10.4007/annals.2003.157.715. MR1983780

[38] Nikolai Nadirashvili and Serge Vlăduţ, *Homogeneous solutions of fully nonlinear elliptic equations in four dimensions*, Comm. Pure Appl. Math. **66** (2013), no. 10, 1653–1662, DOI 10.1002/cpa.21456. MR3084701

[39] Nikolai Nadirashvili and Serge Vlăduţ, *Singular solutions of Hessian elliptic equations in five dimensions* (English, with English and French summaries), J. Math. Pures Appl. (9) **100** (2013), no. 6, 769–784, DOI 10.1016/j.matpur.2013.03.001. MR3125267

[40] John Nash, *C^1 isometric imbeddings*, Ann. of Math. (2) **60** (1954), 383–396, DOI 10.2307/1969840. MR65993

[41] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954, DOI 10.2307/2372841. MR100158

[42] Jindrich Nečas, *Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity*, Theory of nonlinear operators (Proc. Fourth Internat. Summer School, Acad. Sci., Berlin, 1975), Abhandlungen der Akademie der Wissenschaften der DDR, Abteilung Mathematik, Naturwissenschaften, Technik, No. 1, Akademie-Verlag, Berlin, 1977, pp. 197–206. MR509483

[43] Louis Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. Math. **6** (1953), 103–156; addendum, 395, DOI 10.1002/cpa.3160060105. MR64986

[44] A. V. Pogorelov, *The regularity of the generalized solutions of the equation* $\det(\partial^2 u/\partial x^i \partial x^j) = \varphi(x^1, x^2, \ldots, x^n) > 0$ (Russian), Dokl. Akad. Nauk SSSR **200** (1971), 534–537. MR293227

[45] Ovidiu Savin, *Small perturbation solutions for elliptic equations*, Comm. Partial Differential Equations **32** (2007), no. 4-6, 557–578, DOI 10.1080/03605300500394405. MR2334822

[46] Ovidiu Savin and Hui Yu, *On the multiple membranes problem*, J. Funct. Anal. **277** (2019), no. 6, 1581–1602, DOI 10.1016/j.jfa.2019.06.003. MR3985514

[47] Ovidiu Savin and Hui Yu, *Regularity of the singular set in the fully nonlinear obstacle problem*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 2, 571–610, DOI 10.4171/jems/1182. MR4556790

[48] David G. Schaeffer, *An example of generic regularity for a non-linear elliptic equation*, Arch. Rational Mech. Anal. **57** (1975), 134–141, DOI 10.1007/BF00248415. MR387810

[49] D. G. Schaeffer, *Some examples of singularities in a free boundary*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1977), 133–144.

[50] R. Shankar and Y. Yuan, *Hessian estimates for the sigma-2 equation in dimension four*, Preprint, `arXiv:2305.12587`, 2023.

[51] Leon Simon, *Schauder estimates by scaling*, Calc. Var. Partial Differential Equations **5** (1997), no. 5, 391–407, DOI 10.1007/s005260050072. MR1459795

[52] Vladimír Šverák and Xiaodong Yan, *A singular minimizer of a smooth strongly convex functional in three dimensions*, Calc. Var. Partial Differential Equations **10** (2000), no. 3, 213–221, DOI 10.1007/s005260050151. MR1756327

[53] Vladimír Sverák and Xiaodong Yan, *Non-Lipschitz minimizers of smooth uniformly convex functionals*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 24, 15269–15276, DOI 10.1073/pnas.222494699. MR1946762

[54] László Székelyhidi Jr., *The regularity of critical points of polyconvex functionals*, Arch. Ration. Mech. Anal. **172** (2004), no. 1, 133–152, DOI 10.1007/s00205-003-0300-7. MR2048569

[55] Martin Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane*, Geom. Funct. Anal. **24** (2014), no. 2, 690–720, DOI 10.1007/s00039-014-0268-5. MR3192039

[56] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240, DOI 10.1007/BF02392316. MR474389

[57] John I. E. Urbas, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J. **39** (1990), no. 2, 355–382, DOI 10.1512/iumj.1990.39.39020. MR1089043

[58] Z. Wang and X. Zhou, *Existence of four minimal spheres in* S^3 *with a bumpy metric*, Preprint, `arXiv:2305.08755`, 2023.

[59] Z. Wang and X. Zhou, *Improved* $C^{1,1}$ *regularity for multiple membranes problem*, Preprint, `arXiv:2308.00172`, 2023.

Connor Mooney

Deprtment of Mathematics, University of California Irvine

Email address: `mooneycr@uci.edu`