Hurwitz spaces

1. In a 1891 paper, Hurwitz explains how the set of degree d simple covers (*i.e.*, such that all fibers consist of at least $d-1$ points) of $\mathbb{P}^1$ can be endowed with a structure of complex manifold. Nowadays Hurwitz spaces refer more generally to moduli spaces of covers with specified automorphism group and with certain constraints on the ramification.

Moduli spaces of covers constitute an appropriate tool for studying arithmetic questions on covers of the line; for example, the Regular Inverse Galois Problem amounts to finding rational points on these spaces. The general idea is to look at the constraints a given question imposes on the intrinsic data of the covers in question, as the automorphism group and the ramification, and then to investigate whether there exist possible solutions on the associated moduli space, first over $\mathbb{C}$, and then over the ground field. The diophantine nature of the problem remains, but this approach somewhat classifies the equations by abstracting their structural properties.

Recent developments consider modular towers. The motivating example is the tower of modular curves of levels p^n. A modular tower is a tower of Hurwitz spaces, whose levels p^n come from a universal Frattini construction starting with a given finite group and a prime p. The general question is to extend the rich arithmetic contents of modular curve towers to general modular towers.

2. Hurwitz spaces are moduli spaces of covers of $\mathbb{P}^1$ with fixed monodromy group[1] G (given as a subgroup of the symmetric group S_d with d the degree of the cover) and with a fixed number $r \geq 3$ of branch points. The basic notation for it is $\mathcal{H}_{r,G}$ and a point representing a cover f, or more exactly its equivalence class, is denoted by $[f]$. There are several variants of Hurwitz spaces, depending

first, on whether one is interested in

- *the mere cover situation:* the covers are not necessarily Galois, or
- *the G-cover situation:* the covers are Galois covers given with an isomorphism between their automorphism group and the group G,

and, second, on which cover equivalence is used:

- *the original equivalence*: two covers $f : X \to \mathbb{P}^1$ and $g : Y \to \mathbb{P}^1$ are equivalent if there exists an isomorphism $\chi : X \to Y$ such that $g \circ \chi = f$, or
- *the* PGL_2*-reduced equivalence*: $f : X \to \mathbb{P}^1$ and $g : Y \to \mathbb{P}^1$ are equivalent if there exist two isomorphisms $\chi : X \to Y$ and $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ such that $g \circ \chi = \alpha \circ f$,

[1] equivalently, the monodromy group is the Galois group of the Galois closure of the cover.

with for both equivalences the extra condition that $\chi : X \to Y$ be compatible with the actions of G in the G-cover situation. For simplicity, we will not distinguish here between these different variants.

3. At the beginning stage, covers are considered over the complex field $\mathbb{C}$. The corresponding moduli space is then a complex smooth quasi-projective variety, denoted by $\mathcal{H}^{\infty}_{r,G}$. Given an (unordered) r-tuple $\mathbf{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, we let $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ be the union of all components[2] of $\mathcal{H}^{\infty}_{r,G}$ whose points correspond to covers with ramification type equal to $\mathbf{C}$ [3]. Fixing the ramification type can be regarded analogous to fixing the genus in the theory of curves and their moduli spaces.

For each $\tau \in \mathrm{Aut}(\mathbb{C})$, the conjugate space $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})^{\tau}$ is still a Hurwitz space, which only depends on the restriction $\tau|_{\mathbb{Q}^{\mathrm{ab}}} \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$; namely it is $\mathcal{H}^{\infty}_{r,G}(\mathbf{C}^{\chi(\tau)})$ (where χ is the cyclotomic character and $\mathbf{C}^{\chi(\tau)} = (C_1^{\chi(\tau)}, \ldots, C_r^{\chi(\tau)})$). Thus the (generally reducible) varieties $\mathcal{H}^{\infty}_{r,G}$ and $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ can be defined over $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{ab}}$ respectively, in the sense that their (geometric) components are permuted transitively by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\mathrm{ab}})$ respectively. Furthermore, the Hurwitz space $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ is itself defined over $\mathbb{Q}$ if $\mathbf{C}$ is a *rational union of conjugacy classes* of G, *i.e.*, if for every integer m prime to $|G|$, there exists $\sigma \in S_r$ such that $C_i^m = C_{\sigma(i)}$. More generally, given a field $k \subset \mathbb{Q}^{\mathrm{ab}}$, the tuple $\mathbf{C}$ is said to be a k-rational union of conjugacy classes of G if the same property holds for all integers $m \equiv \chi(\tau)$ modulo $|G|$ with $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k)$. Under this condition, the Hurwitz space $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ is defined over k. For example, the field generated by all roots of unity of order $|G|$ is a rationality field for $\mathbf{C}$.

4. Denote the configuration space for finite subsets of $\mathbb{P}^1$ of cardinality r by $\mathcal{U}_r$ and then by $\Psi_r : \mathcal{H}^{\infty}_{r,G} \to \mathcal{U}_r \otimes_{\mathbb{Z}} \mathbb{C}$ the map sending each point $[f] \in \mathcal{H}^{\infty}_{r,G}(\mathbb{C})$ to the branch point set $\mathbf{t} = \{t_1, \ldots, t_r\} \in \mathcal{U}_r(\mathbb{C})$ of the cover f. A key to the theory, based on Riemann's existence theorem, is that this map is an etale cover (of algebraic varieties); furthermore there is a one-one correspondence between the fiber $\Psi_r^{-1}(\mathbf{t})$ and the set

[2] By component we mean irreducible or connected components; due to smoothness of $\mathcal{H}^{\infty}_{r,G}$, these coincide.

[3] The ramification type is the r-tuple of conjugacy classes (in the monodromy group G) of the monodromy branch cycles associated to sample loops revolving about the r branch points. It is locally constant on $\mathcal{H}_G(\mathbb{C})$, thus is constant on each connected component of $\mathcal{H}_G(\mathbb{C})$. More arithmetically, the ramification type corresponds to the inertia canonical invariant. This invariant is the collection $(C_t)_t$ of conjugacy classes C_t (in the Galois group of the Galois closure) of distinguished generators of inertia groups above t as t ranges over the branch points of the cover. The distinguished generator of some inertia group I, say of order e, is the generator that corresponds to $e^{2i\pi/e}$ in the natural isomorphism between I and the group μ_e of e-th roots of 1.

$$\text{ni}(\mathbf{C})^\bullet = \left\{ (g_1,\ldots,g_r)\in G^r \;\middle|\; \begin{array}{l} g_1\cdots g_r = 1 \\ < g_1,\ldots,g_r > = G \\ g_i\in C_{\sigma(i)}, i=1,\ldots,r \text{ for some } \sigma\in S_r \end{array} \right\} \;/\; \sim$$

Here by "$/\ \sim$", we mean that the tuples $(g_1,\ldots,g_r)$ are regarded up to componentwise conjugation by elements of a certain subgroup of S_d (depending on the situation: for example it is G for G-covers with the original equivalence, it is the normalizer $\text{Nor}_{S_d}(G)$ for mere covers, etc.). The identification $\Psi_r^{-1}(\mathbf{t}) \simeq \text{ni}(\mathbf{C})^\bullet$ is given by the monodromy representation: for any choice of a topological bouquet[4] $\underline{\Gamma} = (\Gamma_1,\ldots,\Gamma_r)$ for $\mathbb{P}^1\setminus(\mathbf{t}\}$ based at some point $t_0\notin \mathbf{t}$, the map $\text{BCD}_{\underline{\Gamma}}$ [5], sending each complex cover $f: X\to\mathbb{P}^1$ to the r-tuple with entries the monodromy permutations of $f^{-1}(t_0)$ associated with $\Gamma_1,\ldots,\Gamma_r$, provides the correspondence $\Psi_r^{-1}(\mathbf{t}) \to \text{ni}(\mathbf{C})^\bullet$. There is a classical outer action of the Hurwitz braid group[6] $H_r = \pi_1^{\text{top}}(\mathcal{U}_r,\mathbf{t})$ on $\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0)$, which induces an action on the fiber $\Psi_r^{-1}(\mathbf{t})$, and on $\text{ni}(\mathbf{C})^\bullet$ *via* maps $\text{BCD}_{\underline{\Gamma}}$. This induced action on $\Psi_r^{-1}(\mathbf{t})$ is the monodromy action corresponding to the topological cover $\Psi_r : \mathcal{H}_{r,G}^\infty(\mathbb{C}) \to \mathcal{U}_r(\mathbb{C})$. It can be explicitly determined: $\pi_1(\mathcal{U}_r,\mathbf{t})^{\text{top}}$ has generators $Q_1,\ldots,Q_{r-1}$ whose action on $\Psi_r^{-1}(\mathbf{t})$, when computed relative to some suitable topological bouquet $\underline{\Gamma}$, corresponds to the following action on $\text{ni}(\mathbf{C})^\bullet$:

$$(g_1,\ldots,g_r)\xrightarrow{Q_i}(g_1,\ldots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+2},\ldots,g_r),\qquad i=1,\ldots,r-1.$$

Components of $\mathcal{H}_{r,G}^\infty(\mathbf{C})$ correspond to orbits of the Hurwitz braid group action.

5. S. Wewers has given a general construction of Hurwitz spaces, which leads to a definition of $\mathcal{H}_{r,G}$ and of some compactification $\overline{\mathcal{H}_{r,G}}$ as schemes over $\text{Spec}(\mathbb{Z}[1/|G|])$. For each prime p not dividing $|G|$, the corresponding fibers above p are denoted by $\mathcal{H}_{r,G}^p$ and $\overline{\mathcal{H}_{r,G}^p}$. This includes the case $p=\infty$ for which one recovers the space $\mathcal{H}_{r,G}^\infty$.

There is good reduction of $\mathcal{H}_{r,G}$ at those primes $p \not| |G|$: the fiber $\mathcal{H}_{r,G}^p$ is a (reducible) smooth variety defined over $\overline{\mathbb{F}}_p$ and its components correspond to those of $\mathcal{H}_{r,G}^\infty$ through the reduction process. Furthermore, each $\mathcal{H}_{r,G}^p$ is a moduli space, for covers of $\mathbb{P}^1$ with r branch points and monodromy group G, over algebraically closed fields of characteristic p.

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[4] i.e., a r-tuple $\underline{\Gamma}=(\Gamma_1,\ldots,\Gamma_r)$ of homotopy classes of sample loops based at some point $t_o\notin\mathbf{t}$ generating the topological fundamental group $\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0)$ with the unique relation $\Gamma_1\cdots\Gamma_r=1$ (plus some other technical conditions).

[5] where BCD stands for "branch cycle description".

[6] The Hurwitz braid group H_r has a classical presentation: it is the group on $r-1$ generators $Q_1,\ldots,Q_{r-1}$ with relations $Q_iQ_j=Q_jQ_i$ for $|i-j|>1$, $Q_{i+1}Q_iQ_{i+1}=Q_iQ_{i+1}Q_i$ for $1\le i\le r-2$ and $Q_1\cdots Q_{r-1}Q_{r-1}\cdots Q_1=1$.

The compactification $\overline{\mathcal{H}_{r,G}}$ is locally the quotient of a smooth variety by a finite group Components in $\overline{\mathcal{H}_{r,G}}$ are closures of components in $\mathcal{H}_{r,G}$. The natural étale morphism $\Psi_r : \mathcal{H}_{r,G} \to \mathcal{U}_r$ extends to a ramified cover $\overline{\mathcal{H}_{r,G}} \to \overline{\mathcal{U}}_r$. Points on the boundary $\overline{\mathcal{U}}_r \setminus \mathcal{U}_r$ represent degenerations of tuples $\mathbf{t} = (t_1, \ldots, t_r)$ when two or more of the t_i "coalesce" (*i.e.* become equal). More formally they correspond to stable marked curves of genus 0 with a root, *i.e.* trees of curves of genus 0 with a distinguished component T_0 — the *root* — equipped with an isomorphism $\mathbb{P}^1 \simeq T_0$ and at least three marked points (including the double points) on any component but the root. Points on the boundary $\overline{\mathcal{H}_{r,G}} \setminus \mathcal{H}_{r,G}$ represent *admissible covers* (in a certain sense) of stable marked curves of genus 0.