

7.1 Integration

7.1.1 The Darboux Integral

The German mathematician Georg Riemann successfully captured the notion of “area under the curve $y = f(x)$ ” when he introduced his concept of the integral of a function f . We follow a development which was given later by Darboux which simplifies his idea. At the end of our treatment of the “Darboux Integral” we introduce the “Riemann Integral” and show that they are the same.

We want to capture the notion of the “area under the curve”¹ for some function $f(x)$, defined on an interval $[a, b]$. We suppose initially, for simplicity, that f is continuous over this interval. Actually, we need only assume that f is *bounded* over $[a, b]$, for that is all we shall need in our development of Darboux integrals.

Partitions

Define a **partition** P of the interval $[a, b]$ to be a finite set of real numbers, given in increasing order, such that the first one is a and the last one is b :

$$a = x_0 < x_1 < \dots < x_n = b.$$

Note that this partition of $[a, b]$ divides $[a, b]$ into (not necessarily equal) sub-intervals:

$$[a, x_1], [x_1, x_2], \dots [x_{n-1}, b].$$

We shall often write

$$P = \{x_0, x_1, \dots, x_n\}$$

to identify the partition we are speaking about.

¹It may be helpful to think of f as a positive function, so that “area under the curve” makes sense. But our definition of the Darboux integral will not depend upon this assumption.

Upper and Lower Sums

Darboux's idea was to approximate the area under the curve $y = f(x)$ first by rectangles inscribed below the curve, and then by rectangles which were circumscribed above the curve. In either case, the area of rectangles is easy to compute, so Darboux obtained **lower bounds** for the area under the curve, via the inscribed rectangles, and **upper bounds** for the area under the curve, via the rectangles which circumscribed the curve. We shall call the sum of the areas of rectangles below the curve a **lower sum**, and the sum of the area of the rectangles which cover the curve an **upper sum**.

Then Darboux proved, for certain kinds of functions $f(x)$, that when we consider all possible ways to partition the interval $[a, b]$, the *supremum* of the lower sums so obtained is equal to the *infimum* of the upper sums. Since, intuitively, the area under the curve is always between the lower sums and the upper sums, it then makes sense to define the area under the curve to be this common value, *when it exists*.

We can consider “inscribed rectangles” as being the largest rectangles that can be inscribed “underneath the curve” $y = f(x)$, using the partition P :

(*diagram here*)

If the function $f(x)$ is continuous (or merely bounded), the largest rectangle that can be inscribed “underneath the curve” on the sub-interval $[x_{i-1}, x_i]$ is the rectangle whose base is $[x_{i-1}, x_i]$ and whose height is “the smallest value of $f(x)$ on $[x_{i-1}, x_i]$ ”:

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = \inf_{[x_{i-1}, x_i]} f(x)$$

[We shall also occasionally write $m_{[x_{i-1}, x_i]}(f)$ for $m_i(f)$ when we want to indicate the set $[x_{i-1}, x_i]$ over which the *infimum* of $f(x)$ is taken:

$$m_{[x_{i-1}, x_i]}(f) = \inf_{[x_{i-1}, x_i]} f(x)$$

and more generally,

$$m_A(f) = \inf\{f(x) : x \in A\} = \inf_A f(x)$$

for any set A .

In order to simplify the notation, when the discussion is about a single, fixed function f we shall often drop the “ f ” from

$$m_i(f), \quad m_{[x_{i-1}, x_i]}(f), \quad \text{and } m_A(f)$$

and write just

$$m_i, \quad m_{[x_{i-1}, x_i]}, \quad \text{and } m_A$$

In all that follows we shall not be assuming that f is continuous, but as we said, that f is at least **bounded**, so that talking about sup’s and inf’s makes sense.

The *area* of a rectangle whose base is $[x_{i-1}, x_i]$ and whose height is m_i is

$$m_i(x_i - x_{i-1})$$

and the sum of the areas of the n inscribed rectangles is

$$s(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}).$$

We shall call such a sum a **lower sum** for the obvious reason.

What we just did for “inscribed rectangles” can be done for “circumscribed rectangles” as well. We get the smallest rectangle that can be circumscribed “above the curve” on the sub-interval $[x_{i-1}, x_i]$ as the rectangle whose base is $[x_{i-1}, x_i]$ and whose height is

$$M_i = \sup_{[x_{i-1}, x_i]} f(x) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

The area of such a rectangle will be

$$M_i(x_i - x_{i-1})$$

and the sum of the areas of the n circumscribed rectangles will be

$$S(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}).$$

We shall call such a sum an **upper sum**, again for the obvious reason.

Since the inscribed rectangles are “below the curve $y = f(x)$ ” and the circumscribed rectangles are “above the curve”, we expect the following should hold, for partitions P and Q :

$$s(f, P) \leq \text{area under the curve} \leq S(f, Q) \quad (7.1)$$

regardless of the choice of the partition P .

Since (7.1) is supposed to hold for every choice of partition, it might also hold² if we take sup’s of the lower sums over all partitions P , and inf’s of the upper sums over all partitions Q :

$$\sup_P s(f, P) \leq \text{area under the curve} \leq \inf_Q S(f, Q)$$

In fact, we can use this idea to make a *definition* of “area under the curve $y = f(x)$ ”, in the following way: If

$$\sup_P s(f, P) = \inf_Q S(f, Q)$$

then we take as the definition of area under the curve $y = f(x)$ over the interval $[a, b]$, denoted $A(f, [a, b])$,

$$A(f, [a, b]) = \sup_P s(f, P) = \inf_Q S(f, Q).$$

If

$$\sup_P s(f, P) < \inf_Q S(f, Q)$$

we say that $A(f, [a, b])$ does not exist.

For reasons that will become clear later on, we shall call $A(f, [a, b])$ the **integral** of $f(x)$ over the interval from a to b .

Two things now need to be proved: first, we must prove that for any bounded f , and any partitions P and Q , $s(f, P) \leq S(f, Q)$. Then we must prove the basic result due to Riemann, that a continuous function on a closed, bounded interval $[a, b]$ always has an integral.

²We have to prove this. Read on.

Definition 7.1 Suppose $P = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a, b]$. Suppose $c \in (x_{i-1}, x_i)$. Denote by P' the partition obtained by adjoining c to P , to obtain the partition

$$P' = \{x_0, x_1, \dots, x_{i-1}, c, x_i, \dots, x_n\}.$$

Then P' is called a **refinement** of P . Furthermore, if P is a partition of $[a, b]$, and Q is any finite set such that

$$P \subset Q \subset [a, b],$$

then Q is also called a **refinement** of P . Note that any Q which is a refinement of P can be obtained by a succession of adjunctions of single elements to P :

$$P \subset P' \subset P'' \subset \dots \subset Q.$$

Lemma 7.1 *If P is a partition of $[a, b]$ and Q is any refinement of P , then*

$$s(f, P) \leq s(f, Q) \leq S(f, Q) \leq S(f, P).$$

(In words, when you refine a partition, lower sums increase, and upper sums decrease.)

Proof: In light of the remarks immediately preceding the Lemma, it suffices³ to consider the case when Q is a refinement of P obtained by

$$Q = P \cup \{c\}.$$

We have three inequalities to prove. The first of these is Claim 1:

$$s(f, P) \leq s(f, Q).$$

Proof of Claim 1: Suppose $c \in (x_{k-1}, x_k)$. If we recognize that $s(f, P)$ is the sum of n terms

$$s(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$$

³What do you need to do to complete this idea?

while $s(f, Q)$ is the sum of $n + 1$ terms

$$s(f, Q) = \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_{[x_{k-1}, c]}(c - x_{k-1}) + m_{[c, x_k]}(x_k - c) + \sum_{i=k+1}^n m_i(x_i - x_{i-1})$$

then we observe that

$$\begin{aligned} s(f, Q) - s(f, P) &= m_{[x_{k-1}, c]}(c - x_{k-1}) + m_{[c, x_k]}(x_k - c) - m_{[x_{k-1}, x_k]}(x_k - x_{k-1}) \\ &= m_{[x_{k-1}, c]}(c - x_{k-1}) + m_{[c, x_k]}(x_k - c) - m_{[x_{k-1}, x_k]}(x_k - c + c - x_{k-1}) \\ &= m_{[x_{k-1}, c]}(c - x_{k-1}) + m_{[c, x_k]}(x_k - c) - m_{[x_{k-1}, x_k]}((x_k - c) + (c - x_{k-1})) \\ &= (m_{[x_{k-1}, c]} - m_{[x_{k-1}, x_k]})(c - x_{k-1}) + (m_{[c, x_k]} - m_{[x_{k-1}, x_k]})(x_k - c) \\ &\geq 0 \end{aligned}$$

since ⁴

$$m_A \geq m_B$$

if $A \subset B$.

The second inequality is Claim 2:

$$s(f, Q) \leq S(f, Q)$$

which follows immediately from the definitions of the two quantities, and the recognition that

$$m_A \leq M_A$$

for any set A .

The third inequality we need to prove is

$$S(f, Q) \leq S(f, P).$$

To prove this, we could repeat the argument we gave to prove the analogous result for lower sums: $s(f, P) \leq s(f, Q)$. But it is easier to observe⁵ that

$$s(f, P) = -S(-f, P)$$

⁴Why is this true?

⁵See Exercise 1.

for any function f and any partition P . Then

$$S(f, Q) = -s(-f, Q) \leq -s(-f, P) = S(f, P)$$

where the middle inequality follows from Claim 1. This completes the proof of Lemma 7.1.

Exercise 1 Prove that for any f which is bounded on an interval $[a, b]$, and any partition P , that

$$s(f, P) = -S(-f, P).$$

Exercise 2 Let $f(x) = x$, $a = 0$, $b = 1$.

a. Let $P = \{0, 1/4, 1/2, 3/4, 1\}$. Compute $s(f, P)$ and $S(f, P)$.

b. Let $P_n = \{0, 1/n, 2/n, \dots, 1\}$. Compute $s(f, P_n)$ and $S(f, P_n)$. Hint: you may wish to use the identity

$$\sum_{i=1}^n i = n(n+1)/2,$$

which you may assume.

Exercise 3 Let $f(x) = c$ for all $x \in [a, b]$. Prove that

$$s(f, P) = S(f, P) = c(b - a)$$

for any partition P of $[a, b]$.

The following theorem is what allows us to define *area*.

Theorem 7.2 *Let f be a bounded function on the interval $[a, b]$. If P and Q are any two partitions of $[a, b]$, then*

$$s(f, P) \leq S(f, Q).$$

(In words: any lower sum is less than or equal to any other upper sum.)

Proof: Since P and Q are partitions of $[a, b]$, they are each finite sets containing a and b . It follows that their union, $P \cup Q$, is also a partition of $[a, b]$. Applying our previous Lemma,

$$s(f, P) \leq s(f, P \cup Q) \leq S(f, P \cup Q) \leq S(f, Q)$$

which establishes the result.

Area Under a Curve

Now we are in a position to define area: For a fixed f , and fixed interval $[a, b]$, pick *any* partition Q , e.g. $Q = \{a, b\}$. Then by the previous theorem, since $s(f, P) \leq S(f, Q)$, the set of lower sums is bounded above, as P varies over all partitions of $[a, b]$. Hence

$$\sup_P s(f, P) \leq S(f, Q).$$

But now note that in the above, the inequality holds regardless of the choice of Q , which was arbitrary. Hence⁶

$$\sup_P s(f, P) \leq \inf_Q S(f, Q).$$

Recall from (7.1) that whatever the area under the curve

$$s(f, P) \leq \text{area under the curve} \leq S(f, Q)$$

which then implies that

$$\sup_P s(f, P) \leq \text{area under the curve} \leq \inf_Q S(f, Q).$$

Intuitively, when $\sup_P s(f, P) = \inf_Q S(f, Q)$, we make the natural definition that this value is the area under the curve.

Definition 7.2 Suppose $f(x)$ is bounded over the interval $[a, b]$. Let

$$L(f) = \sup_P s(f, P)$$

which we shall call the **lower Darboux integral of f** , and let

$$U(f) = \inf_Q S(f, Q)$$

which we shall call the **upper Darboux integral of f** .

⁶Provide the reasoning to justify this. This is a delicate issue and needs to be considered carefully.

Restating what we had above,

$$L(f) \leq \text{area under the curve} \leq U(f).$$

Another notation for the lower Darboux integral of f is

$$L(f) = \int_a^b f,$$

and another notation for the upper Darboux integral of f is

$$U(f) = \overline{\int}_a^b f.$$

Now, finally, we are in a position to define area under the curve $y = f(x)$ over the interval $[a, b]$, for a bounded function f :

If $L(f) = U(f)$, then define the area⁷ under the curve $y = f(x)$ over the interval $[a, b]$, which we will denote $A(f, [a, b])$, by

$$L(f) = A(f, [a, b]) = U(f).$$

The standard notation for $A(f, [a, b])$ is as the **Darboux Integral of f** :

$$\int_a^b f, \text{ or as } \int_a^b f(t)dt$$

to indicate the dependence of the function f on a particular variable (t), and the “ dt ” at the end of the expression is placed there to indicate what the variable of integration is.

If $L(f) < U(f)$, we say $\int_a^b f$ is **undefined**.

Note also that existence of the Darboux Integral of f does not depend upon the assumption that f is a non-negative function. The definition makes perfect sense even if f is negative, or sometimes positive and sometimes negative. The interpretation of “area under the curve” has to be changed, of course.

Definition 7.3 If f is bounded on $[a, b]$, and $\int_a^b f$ exists, then we say f is **integrable** on $[a, b]$.

⁷Even if the function f is not non-negative on the interval $[a, b]$, the definition makes sense, and we shall still call it the Darboux Integral of f .

Example 1 A function which is **not integrable**.

Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{Q}^1 \\ 1 & \text{if } x \notin \mathcal{Q}^1 \end{cases}$$

Then f is not integrable on $[0,1]$! To see this, let us compute upper and lower sums:

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \equiv 0$$

while

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \equiv 1$$

regardless of what sub-interval we are considering. Therefore, for any partition of $[0, 1]$,

$$s(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = 0$$

while

$$S(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = \sum_{i=1}^n 1(x_i - x_{i-1}) = x_n - x_0 = 1.$$

Hence

$$L(f) = \sup_P s(f, P) = 0$$

while

$$U(f) = \inf_Q S(f, Q) = 1$$

from which it follows that $f(x)$ is not integrable over $[0, 1]$.

Lemma 7.3 *Let f be bounded on $[a, b]$. Then*

$$\sup_P s(f, P) = \inf_Q S(f, Q)$$

if and only if

for any $\epsilon > 0$, there exist partitions P' and Q' such that

$$S(f, Q') - s(f, P') < \epsilon.$$

Proof: Suppose $\sup_P s(f, P) = \inf_Q S(f, Q)$. The Approximation Theorem for sup's and inf's implies there exist P' and Q' such that

$$s(f, P') > \sup_P s(f, P) - \epsilon/2$$

and

$$S(f, Q') < \inf_Q S(f, Q) + \epsilon/2.$$

Then

$$S(f, Q') - s(f, P') < \inf_Q S(f, Q) + \epsilon/2 - (\sup_P s(f, P) - \epsilon/2) = \epsilon/2 + \epsilon/2 = \epsilon.$$

Conversely, suppose for every $\epsilon > 0$ there exist P' and Q' such that

$$S(f, Q') - s(f, P') < \epsilon.$$

Then

$$s(f, P') \leq \sup_P s(f, P) \leq \inf_Q S(f, Q) \leq S(f, Q') < s(f, P') + \epsilon$$

which implies that

$$0 \leq \inf_Q S(f, Q) - \sup_P s(f, P) < \epsilon$$

which in turn implies that

$$\sup_P s(f, P) = \inf_Q S(f, Q).$$

Lemma 7.4 *Let f be bounded on $[a, b]$. Then for any $\epsilon > 0$, there exist partitions P and Q such that*

$$S(f, Q) - s(f, P) < \epsilon$$

if and only if

for any $\epsilon > 0$, there exist one partition P^ such that*

$$S(f, P^*) - s(f, P^*) < \epsilon.$$

Proof: Let $\epsilon > 0$. Suppose there exist partitions P and Q such that

$$S(f, Q) - s(f, P) < \epsilon.$$

By Lemma 7.1,

$$s(f, P) \leq s(f, P \cup Q) \leq S(f, P \cup Q) \leq S(f, Q)$$

which then implies

$$S(f, P \cup Q) - s(f, P \cup Q) < \epsilon.$$

Letting $P^* = P \cup Q$ then proves the result in one direction. The reverse implication is trivial, because the existence of P^* implies the existence of P and Q , namely, choose $P = P^*$ and $Q = P^*$.

Combining Lemmas 7.3 and 7.4, we have proved

Theorem 7.5 *Let f be bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that*

$$S(f, P) - s(f, P) < \epsilon.$$

Theorem 7.6 *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. I.e. “continuous functions on closed, bounded intervals are integrable”.*

Proof: In light of the preceding theorem, it suffices to prove that for any $\epsilon > 0$ there is a partition P such that

$$S(f, P) - s(f, P) < \epsilon.$$

Recall from the Uniform Continuity Theorem, that a continuous function on a closed, bounded interval is uniformly continuous. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon/(b - a).$$

If we now choose a partition P of $[a, b]$ which divides $[a, b]$ into n equal size sub-intervals, each of length $(b - a)/n$,

$$P = \{x_0 < x_1 < \dots < x_n\}$$

and choose n so large that

$$(b - a)/n < \delta,$$

then in any sub-interval $I = [x_{k-1}, x_k]$, if $x, y \in I$ then $|x - y| < \delta$, which implies that $|f(x) - f(y)| < \epsilon/(b - a)$. Then

$$\begin{aligned} S(f, P) - s(f, P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n (M_i - m_i)(b - a)/n \\ &< \sum_{i=1}^n (\epsilon/(b - a))(b - a)/n \\ &= \epsilon. \end{aligned}$$

Thus we produced the partition P , which proves the Theorem.

Exercise 4 Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{Q}^1 \\ x & \text{if } x \notin \mathcal{Q}^1 \end{cases}$$

Prove that for $[a, b] = [0, 1]$, $L(f) = 0$, and $U(f) = 1/2$. Hint: you may wish to use the identity

$$\sum_{i=1}^n i = n(n + 1)/2,$$

which you may assume. How would you prove this identity (a one- word answer is sufficient)?

Exercise 5 [Part of this Exercise appeared earlier, in Exercise 2.] Let $f(x) = x$, $a = 0$, $b = 1$.

- a. Let $P = \{0, 1/4, 1/2, 3/4, 1\}$. Compute $s(f, P)$ and $S(f, P)$.
- b. Let $P_n = \{0, 1/n, 2/n, \dots, 1\}$. Compute $s(f, P_n)$ and $S(f, P_n)$. Hint: use the identity from the previous problem.

- c. Compute $L(f)$ and $U(f)$. Is $f(x)$ integrable on $[0, 1]$? Hint: show that $1/2$ is a lower bound for $U(f)$ and an upper bound for $L(f)$. The identity $(x_i^2 - x_{i-1}^2) = (x_i + x_{i-1})(x_i - x_{i-1})$ and the inequality $x_i \geq (x_i + x_{i-1})/2$ may be helpful.

Exercise 6 Let $f(x) = x^2$, $a = 0$, $b = 1$.

- a. Is f integrable on $[0, 1]$? Why?
- b. Compute $L(f)$ and $U(f)$. Hint: you may need to use the identity

$$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6,$$

which you may assume. How would you prove this identity (a one-word answer is sufficient)? You may find using the methods of the preceding problem, part (c), useful as well.

Exercise 7 Prove: if f is integrable on $[a, b]$, and $[c, d] \subset [a, b]$, then f is integrable on $[c, d]$.

Exercise 8 Prove: if f is integrable on $[a, b]$, then

$$\lim_{h \rightarrow 0^+} \int_c^{c+h} f = 0$$

for any $c \in (a, b)$. Hint: f is integrable means that f is *bounded*.

Exercise 9 Prove: if f is integrable on $[c, d]$ and is continuous at c , then

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_c^{c+h} f = f(c).$$

[Hint: prove that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < h < \delta$,

$$f(c) - \epsilon \leq m(f, [c, c+h]) \leq \frac{1}{h} \int_c^{c+h} f \leq M(f, [c, c+h]) \leq f(c) + \epsilon.]$$

This result is an essential part of the proof of the Fundamental Theorem of Calculus, which we shall prove in the next chapter.

The Riemann Integral

What we have introduced so far is the Darboux integral. The Riemann integral is equivalent to the Darboux integral, in that for any function f , the Darboux Integral of f exists if and only if the Riemann Integral of f exists, and the two integrals are equal. We shall not prove this here. What we prove is that under the assumption that the function f is continuous, the Riemann integral exists, and its value is the same as that of the Darboux integral.

Definition 7.4 Suppose $f(x)$ is a function defined on $[a, b]$, with the following properties: Suppose $P = \{x_0, x_1, \dots, x_n\}$ is a partition of the interval $[a, b]$, and x_i^* is chosen in any fashion from the i -th subinterval: $x_i^* \in [x_{i-1}, x_i]$. Let $\|P\|$ denote the length of the largest subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$. We say that “ f is **Riemann integrable on the interval $[a, b]$** ” and “the **Riemann integral of f over $[a, b]$ is I** ” if I is a real number such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, x_1, \dots, x_n\}$ is any partition of $[a, b]$ and $\|P\| < \delta$, then $|\sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) - I| < \epsilon$, or what is the same,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) = I.$$

Comments: a few remarks about the Riemann integral. What makes the definition a bit unwieldy is the large number of quantifiers. Recall that as soon as we speak about it limits, we are making an $\epsilon - \delta$ statement. Then we have the partitions, then the choice of x_i^* 's in the i -th sub-interval. It's all quite messy. That is why we are restricting our attention to this special case.

Theorem 7.7 *Suppose $f(x)$ is continuous on $[a, b]$. Then the Riemann integral of f over $[a, b]$ exists, and equals $L(f)$.*

Proof: We already know from Theorem 7.6 that for continuous functions, $L(f) = U(f)$. Let $I = U(f) = L(f)$. To prove

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) = I,$$

let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Then for any partition P such that $\|P\| < \delta$, $P = \{a = x_0 < x_1 < \dots < x_n = b\}$,

$$\begin{aligned} S(f, P) - s(f, P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &< \sum_{i=1}^n (\epsilon/(b - a))(x_i - x_{i-1}) \\ &= (\epsilon/(b - a)) \sum_{i=1}^n (x_i - x_{i-1}) \\ &= (\epsilon/(b - a))(b - a) \\ &= \epsilon. \end{aligned}$$

But then for any choice of $x_i^* \in [x_{i-1}, x_i]$,

$$m_i \leq f(x_i^*) \leq M_i.$$

But this implies

$$s(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \leq \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) \leq \sum_{i=1}^n M_i(x_i - x_{i-1}) = S(f, P).$$

Now recall that

$$s(f, P) \leq L(f) = I = U(f) \leq S(f, P) < s(f, P) + \epsilon,$$

and the proof is almost completed⁸

Exercise 10 Complete the proof of Theorem 7.7.

Exercise 11 Let $f(x) = [x]$, the greatest integer in x . Prove f is integrable over the interval $[1/2, 3/2]$.

⁸See Exercise 10.

Properties of the Darboux Integral

We now develop the properties of the (Darboux) integral, namely that the integral is a **positive, linear functional**:

Definition 7.5 A function \mathcal{L} whose domain is a set of functions, and whose range is \mathcal{R}^1 is called a **linear functional** if

$$\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$$

and

$$\mathcal{L}(cf) = c\mathcal{L}(f)$$

for all functions f and g , and all constants c . A linear functional with the further property that

$$\mathcal{L}(f) \geq 0$$

for any f such that $f(x) \geq 0$ will be called a **positive linear functional**

Theorem 7.8 (The Darboux Integral is a positive linear functional.) *If f and g are both integrable on the interval $[a, b]$, then so are $f + g$ and cf , for any real number c . When f and g are both integrable,*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c \int_a^b f.$$

If $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f \geq 0.$$

First, we prove a lemma, which we shall need in the course of the proof of the theorem:

Lemma 7.9 *If f is integrable on $[a, b]$, then $-f$ is integrable on $[a, b]$, and*

$$\int_a^b (-f) = - \int_a^b f.$$

Proof: Since f is integrable, let $\epsilon > 0$ and P a partition of $[a, b]$ for which

$$S(f, P) - s(f, P) < \epsilon.$$

Then

$$S(-f, P) - s(-f, P) = -s(f, P) + S(f, P) < \epsilon,$$

which shows $-f$ is integrable on P . Then $\int_a^b (-f) = U(-f) = \inf_P S(-f, P) = {}^9 \inf_P \{-s(f, P)\} = -\sup_P s(f, P) = -L(f) = -\int_a^b f$.

Proof of the Theorem: We prove this in several parts:

1. If f and g are integrable on $[a, b]$ then so is $f + g$.

2. In this case

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

3. If f is integrable, $\int_a^b cf = c \int_a^b f$.

4. The (Darboux) Integral is a positive functional.

To prove (1), suppose $\epsilon > 0$. Theorem 7.5 implies (after a little work) there exists a partition P such that¹⁰ both

$$S(f, P) - s(f, P) < \epsilon/2$$

and

$$S(g, P) - s(g, P) < \epsilon/2.$$

Then

$$\begin{aligned} S(f + g, P) &= \sum_{i=1}^n M_i(f + g)(x_i - x_{i-1}) \leq \sum_{i=1}^n M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^n M_i(g)(x_i - x_{i-1}) \\ &= S(f, P) + S(g, P) \end{aligned}$$

⁹Why?

¹⁰See Exercise 12. You need to construct this partition.

and

$$\begin{aligned} s(f+g, P) &= \sum_{i=1}^n m_i(f+g)(x_i - x_{i-1}) \geq \sum_{i=1}^n m_i(f)(x_i - x_{i-1}) + \sum_{i=1}^n m_i(g)(x_i - x_{i-1}) \\ &= s(f, P) + s(g, P) \end{aligned}$$

But then

$$\begin{aligned} S(f+g, P) - s(f+g, P) &\leq S(f, P) + S(g, P) - (s(f, P) + s(g, P)) \\ &= S(f, P) - s(f, P) + S(g, P) - s(g, P) < \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

which, by Theorem 7.5, proves that $f+g$ is integrable on $[a, b]$. To prove (2) we evaluate $U(f+g)$. Consider the following (for the partition P we used above):

$$\begin{aligned} L(f+g) = U(f+g) &\leq S(f+g, P) \leq S(f, P) + S(g, P) \\ &\leq s(f, P) + s(g, P) + \epsilon \\ &\leq L(f) + L(g) + \epsilon. \end{aligned}$$

In a similar¹¹ way,

$$U(f+g) \geq U(f) + U(g) - \epsilon.$$

Combining these two results, we obtain

$$U(f) + U(g) - \epsilon \leq L(f+g) = U(f+g) \leq L(f) + L(g) + \epsilon = U(f) + U(g) + \epsilon.$$

Thus $U(f+g) = U(f) + U(g)$.

To prove (3), if $c > 0$, then

$$U(cf) = \inf_P S(cf, P) = \inf_P cS(f, P) = c \inf_P S(f, P) = cU(f).$$

If $c < 0$, then

$$\int_a^b cf = \int_a^b (-c)(-f) = -c \int_a^b (-f) = (-c) \left(- \int_a^b f \right) = c \int_a^b f.$$

¹¹Provide the reasoning.

Finally, to prove (4), assume $f(x) \geq 0$ and f integrable. For any partition $\|P\|$, $M_i(f) \geq 0$ since $f(x) \geq 0$. Therefore $S(f, P) \geq 0$, and hence (why?) $\inf_P S(f, P) \geq 0$. Then $U(f) \geq 0$. But since f is integrable,

$$\int_a^b f = U(f) \geq 0.$$

Corollary 7.10 *If f and g are integrable on $[a, b]$, and $f(x) \leq g(x)$, for all $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Proof: By Theorem 7.8, $g - f$ is integrable, and $g - f \geq 0$. So $\int_a^b g - \int_a^b f = \int_a^b (g - f) \geq 0$.

Exercise 12 Prove that if f and g are integrable on $[a, b]$, then for every $\epsilon > 0$ there exists a partition P such that $S(f, P) - s(f, P) < \epsilon/2$ and $S(g, P) - s(g, P) < \epsilon/2$.

Exercise 13 Suppose we consider only those functions which are differentiable at 0. Using your basic knowledge from introductory calculus, verify that differentiation and then evaluation at 0 is a positive linear functional (that is, the function Φ defined by

$$\Phi(f) = f'(0) \quad \text{for every differentiable } f$$

is a positive linear functional.)

Theorem 7.11 (Triangle Inequality for Integrals) *If f is integrable on $[a, b]$, then*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof: Recall that for any real numbers a and b ,

$$|a| = |a - b + b| \leq |a - b| + |b|$$

from which it follows that

$$|a| - |b| \leq |a - b|.$$

Therefore¹², for any partition P of $[a, b]$,

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f) \tag{7.2}$$

¹²Provide the details. See Exercise 14.

from which it follows that

$$S(|f|, P) - s(|f|, P) \leq S(f, P) - s(f, P).$$

Since we assumed f was integrable, for any $\epsilon > 0$ there is a partition P such that $S(f, P) - s(f, P) < \epsilon$. It then follows that $S(|f|, P) - s(|f|, P) < \epsilon$, which shows $|f|$ is integrable.

Now recall that for any x ,

$$-|f(x)| \leq f(x) \leq |f(x)|$$

so by Corollary 7.10,

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|,$$

i.e.,

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Exercise 14 Prove Equation 7.2.

Definition 7.6 A function f is said to be **piecewise continuous** on the interval $[a, b]$ if there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that $f(x)$ is continuous on the open subintervals (x_{i-1}, x_i) , $1 \leq i \leq n$.

Note: informally, a piecewise continuous function is one which is continuous, except at a finite number of discontinuities (which, in our definition above, are x_1, \dots, x_{n-1} .)

Example 2 $f(x) = [x]$ restricted to the interval $[0, 10]$ has discontinuities at $\{1, 2, \dots, 10\}$.

Theorem 7.12 A bounded, piecewise continuous function on $[a, b]$ is integrable on $[a, b]$.

Proof: First we need a helpful Lemma:

Lemma 7.13 If f is continuous and bounded on the open interval (a, b) then f is integrable on $[a, b]$.

Proof of the Lemma:

Let $\epsilon > 0$. Suppose $|f(x)| \leq M$ on (a, b) . Choose $\delta = \min\{\epsilon/8M, (b-a)/2\}$ ¹³. f is continuous on $[a + \delta, b - \delta]$, so f is uniformly continuous there, hence integrable on $[a + \delta, b - \delta]$. Hence there exists a partition P of $[a + \delta, b - \delta]$ such that

$$S(f, P) - s(f, P) < \epsilon/2.$$

Now,

$$P^* = P \cup \{a, b\}$$

is a partition of $[a, b]$, and

$$\begin{aligned} S(f, P^*) - s(f, P^*) &= S(f, P) - s(f, P) + 2M(a + \delta - a) + 2M(b - (b - \delta)) \\ &< \frac{\epsilon}{2} + 2M(2\delta) \\ &\leq \epsilon. \end{aligned}$$

Hence f is integrable on (a, b) by Theorem 7.5.

Proof of Theorem 7.12: Suppose f is bounded on (a, b) and continuous on (a, b) except perhaps at the points $\{x_1, \dots, x_n\}$. Then by Lemma 7.13 f is integrable on each subinterval $(x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$. Let $\epsilon > 0$ and choose partitions P_i on (x_{i-1}, x_i) such that $S(f, P_i) - s(f, P_i) < \epsilon/n$. Then let $P = P_1 \cup \dots \cup P_n$, and P is then a partition of (a, b) for which $S(f, P) - s(f, P) < \epsilon$. Hence f is integrable on (a, b) .

Definition: A function $f(x)$ is said to be **monotonic** on the interval I if $x < y \implies f(x) \leq f(y)$ for all $x, y \in I$.

Theorem 7.14 *A monotonic function f defined on $[a, b]$ is integrable on $[a, b]$.*

Proof: Assume that f is monotone increasing. Let $\epsilon > 0$. Let $\delta = \epsilon/(f(b) - f(a))$. Then let $P = \{x_0, x_1, \dots, x_n\}$ be any partition for which $\|P\| < \delta$. For this partition,

$$\begin{aligned} S(f, P) &= \sum_{i=1}^n M_i(f)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) \end{aligned}$$

¹³Why must $\delta \leq (b-a)/2$?

while

$$\begin{aligned} s(f, P) &= \sum_{i=1}^n m_i(f)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}) \end{aligned}$$

so that

$$\begin{aligned} S(f, P) - s(f, P) &= \sum_{i=1}^n (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n (f(x_i) - f(x_{i-1}))\delta \\ &\leq \delta \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\ &\leq \delta(f(x_n) - f(x_0)) \\ &= \epsilon / (f(b) - f(a)) \cdot (f(b) - f(a)) \\ &= \epsilon. \end{aligned}$$

Thus, by Theorem 7.5, f is integrable on $[a, b]$. If f is monotone decreasing, consider $-f$ instead.

Theorem 7.15 *If f is integrable on $[a, b]$, and also on $[b, c]$, then f is integrable on $[a, c]$ and*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof: See Exercise 15.

Exercise 15 Prove Theorem 7.15.

Definition 7.7 We shall also need to be able to compute the value of $\int_a^b f$ in the case where $b < a$, and for this reason we define

$$\int_a^b f = -\int_b^a f \quad \text{if } a < b.$$

Theorem 7.16 (Mean Value Theorem for Integrals) Suppose f is continuous on $[a, b]$.

Then there exists a c , $a < c < b$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f.$$

Proof: Since f is continuous, $m = \inf_{[a,b]} f(x)$ and $M = \sup_{[a,b]} f(x)$ both exist, and therefore

$$\int_a^b m \leq \int_a^b f \leq \int_a^b M,$$

from which it follows (why?) that

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

But $f(x)$ takes the values m and M , by the Extreme Value Theorem, and therefore¹⁴ there is a $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f.$$

Exercise 16 Give an example of a function $f(x) \geq 0$, $f(x) \not\equiv 0$ on the interval $[0, 1]$ which is integrable, but for which $U(f) = L(f) = 0$. Hint: this function cannot be continuous on $[0, 1]$.

Exercise 17 Prove: Suppose $f(x)$ is continuous on $[0, 1]$ and $f(x) \geq 0$. If $f(x_0) > 0$ for some $x_0 \in [0, 1]$, then $\int_0^1 f > 0$.

Exercise 18 Prove that if $f(x)$ is integrable on $[a, b]$ and $g(x) = f(x)$, except at a finite number of points, $x_1, \dots, x_n \in [a, b]$, then $g(x)$ is integrable on $[a, b]$ and

$$\int_a^b g = \int_a^b f.$$

7.1.2 Improper Integrals

We can extend our abilities to integrate functions in two different ways: First, we can, under certain circumstances, allow either or both of the limits of integration to be infinite. Second, we can allow the function f to be unbounded, in some circumstances. We shall deal separately with these two issues:

¹⁴Why? Quote the appropriate theorem to justify this.

Improper integrals of type $\int_a^\infty f$ or $\int_{-\infty}^\infty f$.

If we wish to compute the integral of some function f , say over $[0, \infty)$ there is no convenient way to speak about this integral in terms of upper and lower sums, or Riemann sums, for that matter. The problem is that the interval over which we are integrating is infinite. The way out of this difficulty is to consider the integral

$$\int_0^M f(x)dx$$

as an approximation to

$$\int_0^\infty f(x)dx$$

when M is large. More precisely, we shall define the latter as the limit of the former:

Definition 7.8 *Suppose $f(x)$ is defined on (at least) the interval $[a, \infty)$, for some real number a . Then we define*

$$\int_a^\infty f(x)dx$$

as the limit

$$\lim_{M \rightarrow \infty} \int_a^M f(x)dx.$$

If this limit does not exist, we say **the integral does not exist**. When the limit does exist, we call this **the improper Riemann integral of f** , and we say **the integral converges**. If the limit exists but equals $+\infty$, we say **the integral diverges to $+\infty$** .

Example 3 $\int_1^\infty 1/x^2 dx$. To see if this integral converges, we compute

$$\begin{aligned} \int_1^\infty \frac{1}{x^2} dx &= \lim_{M \rightarrow \infty} \int_1^M \frac{1}{x^2} dx \\ &= \lim_{M \rightarrow \infty} \left(-\frac{1}{x} \right) \Big|_1^M \\ &= \lim_{M \rightarrow \infty} 1 - \frac{1}{M} \\ &= 1. \end{aligned}$$

Definition 7.9 Suppose $f(x)$ is defined on (at least) the interval $(-\infty, a]$, for some real number a . Then we define

$$\int_{-\infty}^a f(x)dx$$

as the limit

$$\lim_{M \rightarrow -\infty} \int_M^a f(x)dx.$$

If this limit does not exist, we say the integral does not exist. When the limit does exist, we call this **the improper Riemann integral of f** , and we say the integral **converges**.

Definition 7.10 For a function $f(x)$ which is defined for all real x , we define

$$\int_{-\infty}^{\infty} f(x)dx \tag{7.3}$$

as the sum of the two integrals

$$\int_{-\infty}^a f(x)dx + \int_a^{\infty} f(x)dx \tag{7.4}$$

for an arbitrary a , and say that the integral (7.3) **converges** if and only if the two integrals (7.4) converge.

Example 4 The improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \pi.$$

According to the definition,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{1}{1+x^2}dx \\ &= \lim_{M \rightarrow -\infty} \int_M^0 \frac{1}{1+x^2}dx + \lim_{M \rightarrow \infty} \int_0^M \frac{1}{1+x^2}dx \\ &= \lim_{M \rightarrow -\infty} -\arctan M + \lim_{M \rightarrow \infty} \arctan M \\ &= \pi/2 + \pi/2 = \pi \end{aligned}$$

Exercise 19 Explain why, in Example 4, it wouldn't have made any difference if we had chosen $a = 1$, instead of $a = 0$.

Exercise 20 Prove that the improper Riemann integral

$$\int_1^{\infty} \frac{1}{x^p} dx$$

converges if and only if $p > 1$.

Exercise 21 Prove the following “Comparison Test for Integrals”: Suppose $0 \leq f(x) \leq g(x)$ for all $x \geq 0$. If $\int_0^{\infty} g(x) dx$ converges, then $\int_0^{\infty} f(x) dx$ converges. Then use this result to show that

$$\int_0^{\infty} \frac{1}{1+x^4} dx \text{ converges.}$$

Exercise 22 Does the improper Riemann integral $\int_0^{\infty} \sin x dx$ converge? Explain.

Infinite Integrands: Improper integrals of type $\int_0^1 \frac{1}{x^{1/2}} dx$.

Just as with improper integrals of type $\int_a^{\infty} f(x) dx$, where we reduced the integral to a proper integral and then took limits, we now study integrals which are *improper because they fail to be bounded*.

Definition 7.11 Suppose that $f(x)$ is defined on the half-open interval $(a, b]$ and for every $\epsilon > 0$ is bounded on the closed subinterval $[a + \epsilon, b]$. Then if

$$\lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b f(x) dx$$

exists, we say **the improper Riemann integral**

$$\int_a^b f(x) dx$$

converges and take its value to be the value of the limit.

Example 5 $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges.

We apply the definition:

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{a \rightarrow 0^+} 2\sqrt{x} \Big|_a^1 \\ &= \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) \\ &= 2. \end{aligned}$$

Exercise 23 Prove that the improper Riemann integral

$$\int_0^1 \frac{1}{x^p} dx$$

converges if and only if $p < 1$.

If $f(x)$ is unbounded at b rather than at a , the definition is similar:

Definition 7.12 Suppose that $f(x)$ is defined on the half-open interval $[a, b)$ and for every $\epsilon > 0$ is bounded on the closed subinterval $[a, b - \epsilon]$. Then if

$$\lim_{\epsilon \rightarrow 0^+} \int_a^{b-\epsilon} f(x) dx$$

exists, we say **the improper Riemann integral**

$$\int_a^b f(x) dx$$

converges and take its value to be the value of the limit.