DUAL LYAPUNOV EXPONENTS AND OPERATORS OF TYPE I

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Dedicated to the memory of my mother, Valentina Borok

Abstract. The theory of one-dimensional one-frequency quasiperiodic operators with analytic potentials has long been centered around several very specific models, with solutions of several famous open problems not extending even to small analytic perturbations of those. We describe a duality-based approach to Avila's global theory of such operators that has both uncovered its fundamental mysteries and allowed to solve the robust versions of key spectral problems. This article is related to the lecture delivered at ICM22/v(WM)^2, July 2022 (with the proceedings, [48] disseminated in January 2022), where this approach was first announced. The talk and the proceedings [48] consisted of two parts, one devoted to the overview of already published almost Mathieu results [51, 52] and related developments, and the other to the announcement of this approach and some of its consequencies. We expand here only on the second part, that has since seen some remarkable developments, including powerful corollaries, first announced in September 2022.

As explained in more detail in [48], one-dimensional discrete one-frequency Schrödinger operators

$$(H_{V,\alpha,x}u)_n := u_{n-1}+u_{n+1}+V(x+n\alpha)u_n,\ u \in \ell^2(\mathbb{Z}), \alpha \in \mathbb{R}\backslash\mathbb{Q},\ x \in \mathbb{T} := \mathbb{R}/\mathbb{Z},\ V : \mathbb{T} \to \mathbb{R}, \tag{0.1}$$

and related questions of the dynamics of analytic quasiperiodic cocycles have a strong background in physics, largely centered around the Harper's model of two-dimensional Bloch electron in a uniform perpendicular magnetic field, and the related operator family $H_{2\lambda\cos,\alpha,x}$ called, following Barry Simon [69], the almost Mathieu operator. The model remains highly popular and relevant in physics.

Originally approached in mathematics in a perturbative way, going back to the work of Dinaburg-Sinai [23] [1], several remarkable symmetries of the family $H_{2\lambda\cos,\alpha,x}$ were noticed in the early 80s by the physicists Aubry and Andre [1]. This, coupled with heuristics and numerics by M. Azbel and D. Hofstadter [17, 42], as well as the emerging understanding that the arithmetics of parameters plays an unexpected role in this physics-based problem [16, 35], led to several beautiful and precise conjectures about this family, heavily popularized [2] by B. Simon in his lists of problems in mathematical physics [70, 71]. Based on partial advances [44, 56], one conjecture took a sharp arithmetic turn in 1994 [45], so the key list[3] became

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[1] the first KAM success in the spectral theory

[2] originally, in one case, in a wrong form

[3] This is leaving out two other famous but not quite fitting in the later narrative of this note problems: measure of the spectrum problem, finally solved in [13, 49], and purely singular continuity of the spectrum for **all** α, x and $\lambda = 1$ problem, finally solved in [47]

(1) The[4] spectrum of $H_{2\lambda\cos,\alpha,x}$ is a Cantor set for **all** $\alpha \notin \mathbb{Q}$, $\lambda \neq 0$ (aka the *ten martini problem*)
(2) the integrated density of states is absolutely continuous for **all** $\lambda \neq 1$ and **all** α
(3) (a) For a.e. (in fact, Diophantine) α there is a sharp transition from purely absolutely continuous for $\lambda < 1$ to (arithmetically defined) a.e, pure point spectrum for $\lambda > 1$ (aka the **Aubry-Andre or AA** conjecture)
(b) For $\lambda \geq 1$ there is a sharp transition between singular continuous and (arithmetically defined) a.e. pure point spectrum at $\lambda = e^{\beta(\alpha)}$ (aka the *Aubry-Andre-Jitomirskaya, or AAJ* conjecture)
(4) The spectrum is purely absolutely continuous for **all** α, x for $\lambda < 1$.

Here,

$$\beta(\alpha) := \limsup_{k\to\infty} -\frac{\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{0.2}$$

where $||x||_{\mathbb{R}/\mathbb{Z}} = \mathrm{dist}(x, \mathbb{Z})$, - an arithmetic parameter originally introduced in [45]. It is equal to zero for Diophantine (thus a.e.) α.

Remarkably, after various significant developments (see, e.g. [54, 55] for the history), **all** these problems have been fully solved: the ten martini problem (1) in [8] , the absolute continuity of the $\lambda \neq 1$ IDS (2) in [6], the AA conjecture (3a) in [46], the AAJ conjecture (3b) in [14, 51], and pure absolute continuity for $\lambda < 1$ (4) in [2, 8]. [5]

It should be noted that problem (4) has reappeared on Simon's list [71] already after having been solved for the Diophantine (thus a.e.) α in [46], underscoring the particular importance given to proving the sharp arithmetic results, as formulated.[6] Problems (1), (2) were also solved for (arithmetically) almost all parameters [46, 67] before their final celebrated all α solutions. Problems (1)-(4) therefore can be viewed in some sense as number theory problems, with no room whatsoever for any approximation or parameter perturbation.

Let $L(E)$ be the (top) Lyapunov exponent of the transfer-matrix cocycle of operator (0.1) at energy $E : L(E) := \lim_{n\to\infty} \int \frac{1}{n} \ln \|A_n(x, E)\| \, dx$, where

$$A_n(x, E) := \prod_{J=n-1}^{0} A(x + j\alpha, E). \tag{0.3}$$

and

$$A(x, E) := \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}$$

(see e.g. [48] for the discussion).

The distinction between the regimes $\lambda < 1$ and $\lambda > 1$ is based on the behavior of the Lyapunov exponents $L(E)$ of corresponding transfer-matrix cocycles $(\alpha, A(x, E))$ of $H_{2\lambda\cos,\alpha,x}$ for energies E in its spectrum: $L(E) = 0$ for $\lambda \leq 1$ and $L(E) > 0$ for $\lambda > 1$ [1, 20], with $\lambda = 1$ being the transition point aka critical value.

[4] x-independent

[5] Clearly, for the almost Mathieu, due to the self-duality (2) follows from (4) and (3a) also follows from a combination of (3b) and (4), but historically those proofs came later.

[6] the same holds for the measure of the spectrum problem, that was already solved for arithmetically a.e. α in [61] before being reiterated in [71]

The Lyapunov exponent based methods for the proofs of localization for operators (0.1) were developed (for the almost Mathieu) in [46], making it natural to expect the universality of the supercritical Diophantine localization (3a) already then. The measure theoretical such universality was proved by Bourgain and Goldstein [19] leading to various significant developments by Bourgain and collaborators (see [18] and also [53] and [66] for more recent extensions of some of the most sophisticated Bourgain's results to more general toral dimensions). The arithmetic (Diophantine (3a) or the sharp (3b)) global version of the localization result however remains open.

Avila's global theory of analytic quasiperiodic cocycles [3] (later extended to the high-dimensional cocycles in [12]) is based on the structure of complexified Lyapunov exponent $L_\epsilon := \lim_{n\to\infty} \int \frac{1}{n} \ln \| \prod_{j=n-1}^{0} A_j(x + j\alpha + i\epsilon)\| \, d\mu$, where $A \in C^\omega(\mathbb{T}, SL(2,\mathbb{C}))$. Avila observed that, for a given cocycle, L_ϵ is a convex function of ϵ, and proved that it has quantized derivative in ϵ.

Theorem 1. *[3] For any complex analytic one-frequency cocycle,*

$$\omega(A) = \lim_{\varepsilon\to 0^+} \frac{L_\varepsilon(A) - L_0(A)}{2\pi\varepsilon} \in \mathbb{Z}.$$

The integer-valued $\omega(A)$ is called the acceleration of the cocycle. This has led Avila to distinguish three cases of behavior on the spectrum, with the terminology inspired by the almost Mathieu family:

Subcritical: $L_\epsilon = 0, \epsilon < \delta, \ \delta > 0$, or, alternatively, $L_0 = \omega(A) = 0$.
Critical: $L_0 = 0, L_\epsilon > 0, \epsilon > 0$, or, alternatively, $L_0 = 0, \omega(A) > 0$.
Supercritical: $L_0 > 0, \omega(A) > 0$,

underscoring the prototypical nature of the almost Mathieu family.

According to the λ (non)dependence of the celebrated almost Mathieu results (1)-(4), it is natural then to ask the following questions:

(1) For what non-trivial analytic $H_{V,\alpha,x}$ does the ten martini (Cantor spectrum for **all** $\alpha \notin \mathbb{Q}$) hold?
(2) For what non-trivial non-critical $H_{V,\alpha,x}$ does the absolute continuity of the IDS hold?[7]
(3) For what non-trivial super-critical $H_{V,\alpha,x}$ does sharp aritmetic transition AAJ hold?
(4) For what non-trivial sub-critical $H_{V,\alpha,x}$ does absolutely continuous spectrum for **all** α, θ hold?

Problem (4) has quickly morphed into something much bigger: the **almost reducibility conjecture (ARC)** (first formulated in [10]), stating that subcritical analytic cocycles are almost reducible, that is can be analytically conjugated arbitrarily close to constants. This property (also modeled on the almost Mathieu operator [2, 10], and previously known, more generally, for small couplings [10, 21], implies that subcritical cocycles can be conjugated into the Eliasson's regime [26] leading to a host of spectral consequences in the Schrödinger case, including the absolute continuity of the spectral

[7] slightly abusing the language we say H is non-critical if its transfer-matrix cocycles for E in the spectrum are non-critical, a prevalent property [3]. We also say $H_{V,\alpha,x}$ is sub/super critical if its transfer-matrix cocycles for E in the spectrum are.

measures (4) and also various more delicate properties, such as, e.g. uniform Hölder regularity of the absolutely continuous spectral measures [11].

The ARC was proved by Avila [4, 5], thus, in particular, taking care of the universality of (4) for subcritical operators in the entire class of operators (0.1). For problems (1)-(3), however, not only there are no global universality results, but by the time of the $(\mathrm{WM})^2$-ICM22 talk, there were not even known arithmetic results for problems, as formulated, for **any** V other than the cos.

Indeed, the proofs of (1)-(3) in, correspondingly [2, 6, 8, 14, 46, 51] are all very specific for $V = \cos$, utilizing a number of its very special features, making extending these results particularly challenging. However, the strong physics origin of the almost Mathieu operator suggests that these results should be at least **robust** with respect to small changes of V, so should hold for an open set of analytic V (corresponding to allowing small probabilities of not-nearest-neighbor hopping on the original two-dimensional lattice [64], a very physically plausible scenario).

Additionally, Avila's quantizatization of the acceleration implies that as a function of $\varepsilon > 0$, L_ε is convex, piecewise affine, and thus is fully characterized by $L = L_0$ and monotone increasing sequences of turning points b_i and slopes $n_i \in 2\pi\mathbb{Z}_+$, where the slope of L_ε between b_i and b_{i+1} is n_i. It was unclear however what information about the cocycle is revealed by the sequences b_i and n_i.

The breakthrough in [32], as reported in [48] is the approach to Avila's global theory through Aubry duality that has revealed the above mystery and became a foundation for not only the first non-cos but also the **robust** versions of (1)-(3), as well as the new proof of ARC.

The Aubry dual of the one-frequency Schrödinger operator (0.1) is defined as

$$(\hat{H}_{V,\alpha,\theta}u)_n = \sum_{k=-\infty}^{\infty} V_k u_{n+k} + 2\cos 2\pi(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}. \tag{0.4}$$

where V_k is the k-th Fourier coefficient of V viewed as a transformation of the entire family indexed by x for fixed V, α, namely a unitary conjugation on $\mathcal{H} = L^2(\mathbb{T}\times\mathbb{Z})$, via

$$U\psi(x,n) = \hat{\psi}(n, x + \alpha n), \tag{0.5}$$

where $\hat{\psi} : L^2(\mathbb{Z}\times\mathbb{T}) \to L^2(\mathbb{T}\times\mathbb{Z})$ is the Fourier transform. The almost Mathieu family is self-dual with respect to this transformation: $\hat{H}_{2\lambda\cos,\alpha,x} = H_{\frac{2}{\lambda}\cos,\alpha,\theta}$, and, in particular, $H_{2\cos,\alpha,x}$, that is, $H_{2\lambda\cos,\alpha,x}$ with $\lambda = 1$, is the fixed (self-dual) point.

It can be explained by the magnetic nature and corresponding gauge invariance of two-dimensional magnetic Laplacians that lead to $H_{V,\alpha,x}$ [64], so, in particular, spectra and integrated densities of states of $H_{V,\alpha,x}$ and $\hat{H}_{V,\alpha,x}$ coincide. A form of Aubry duality was first formulated in [1], leading, in particular, to the AA conjecture. Aubry duality has been formulated and explored on different levels, e.g. [64], [36], [10].

In general, operator (0.4) is long-range. If V is a trigonometric polynomial of degree d, the transfer-matrix $A(x,E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ gives rise to a $2d$-dimensional cocycle, which has a complex-symplectic structure [39], so can be viewed as a $Sp(2d,\mathbb{C})$ cocycle $(\alpha, A), A \in Sp(2d,\mathbb{C})$, a linear skew product:

$$(\alpha, A)\colon \left\{ \begin{array}{ccc} \mathbb{T}\times\mathbb{C}^{2d} & \to & \mathbb{T}\times\mathbb{C}^{2d} \\ (x,v) & \mapsto & (x+\alpha, A(x)\cdot v) \end{array} \right\}.$$

The Lyapunov exponents $L_1(\alpha, A) \geq L_2(\alpha, A) \geq ... \geq L_{2d}(\alpha, A)$, repeated according to their multiplicity, are defined by

$$L_k(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln(\sigma_k(A_n(x)))dx,$$

where for a matrix $B \in M_m(\mathbb{C})$, $\sigma_1(B) \geq ... \geq \sigma_m(B)$ are its singular values (eigenvalues of $\sqrt{B^*B}$). Since for real E the transfer-matrix $A(x, E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ is symplectic, its Lyapunov exponents come in the opposite pairs $\{\pm L_i(\alpha, A)\}_{i=1}^d$. Let

$$\hat{L}_i := L_{d-i}(\alpha, A), \tag{0.6}$$

so that $0 \leq \hat{L}_1 \leq \hat{L}_2 \leq ... \leq \hat{L}_d$. It turns out, the dual Lyapunov exponents $\hat{L}_i$ are precisely the turning points b_i of the piecewise-linear complexified Lyapunov exponent L_ε of the transfer-matrix cocycle of $H_{V,\alpha}$ at E, with the number of $\hat{L}_i < \varepsilon$ (counting multiplicity) being the slopes n_i after a 2π normalization. For V with infinitely many non-zero Fourier coefficients, there is no dual cocycle, but dual Lyapunov exponents can still be defined as appropriate limits of dual Lyapunov exponents constructed for the Fourier cutoffs of V, and the above still holds. This is represented by the following theorem that can be viewed as the multiplicative version of the classical Jensen's formula

Theorem 2. *[32, 48] For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $V \in C^\omega(\mathbb{T}, \mathbb{R})$, there exist non-negative $\{\hat{L}_i(E)\}$ such that for any $E \in \mathbb{R}$*

$$\hat{L}_i(E) = \lim_{d\to\infty} \hat{L}_i^d(E),$$

where $\hat{L}_i^d(E), i = 1, \cdots, d$, are the Lyapunov exponents, as defined in (0.6), of the $Sp(2d, \mathbb{C})$ transfer-matrix cocycle of the dual eigenvalue equation $\hat{H}_{V^d,\alpha,x}\Psi = E\Psi$, with $V^d(x) = D_d \star V$, D_d being the Dirichlet kernel. Moreover,

$$L_\varepsilon(E) = L_0(E) - \sum_{\{i:\hat{L}_i(E)<2\pi|\varepsilon|\}} \hat{L}_i(E) + 2\pi(\#\{i : \hat{L}_i(E) < 2\pi|\varepsilon|\})|\varepsilon|$$

In particular, dual Lyapunov exponents are the analogue of $\ln|z_i|$ where z_i are zeros of an analytic function in the Jensen's formula (and, in fact, the classical Jensen's formula can also be explained through spectral theory and dual Lyapunov exponents for pure diagonal quasiperiodic operators [32]. Also, the acceleration $\omega(E)$ is precisely the number of vanishing dual Lyapunov exponents (an analogue of the winding number for an analytic function on $\mathbb{T}$.) In fact, one can say more

Theorem 3. *[32] The acceleration $\omega(E) > 0$ if and only if the dual $\mathrm{Sp}_{2d}(\mathbb{C})$ cocycle $(\alpha, \widehat{A}_E)$ is partially hyperbolic with the dimension of the center equal to $2\omega(E)$ and zero center Lyapunov exponents.*

This duality based approach led, in particular, to a new proof of ARC [27] for Schrödinger cocycles. While not as strong as Avila's result [5] in several aspects, this provided a novel elegant vision of ARC and several important technical tools.

Ge's proof [27] was first announced in the review [48] without detail, and first presented, along with the entire strategy and various details, in Ge's talk at QMath 15, September 2022. The authors of [38] developed another duality proof, using Ge's strategy, but different from Ge's in some technical aspects (along with a technically different from [32] proof of the multiplicative Jensen's formula of [32, 48]).

We now depart from the talk as given at $(\mathrm{WM})^2/\mathrm{ICM22}$ and the material presented in [48], and list several exciting developments for which it has been the foundation. They were first presented in the above mentioned Qmath 15 talk by L. Ge.

It turns out that the partially hyperbolic structure of the dual cocycle described in Theorem 3 helps approach various questions that previously seemed intractable. One property of the almost Mathieu operators is that its acceleration is always either zero or one. However (as easily demonstarted by the almost Mathieu itself), acceleration is not a stable quantity. To this end, in [30] we introduced the generalized acceleration:

$$\omega'(E) = \lim_{\varepsilon \to 0^+} \frac{L_\varepsilon(E) - L_{\varepsilon_1}(E)}{2\pi(\varepsilon - \varepsilon_1)}.$$

where $\varepsilon_1 \geq 0$ is the first turning point of the piecewise affine function $L_\varepsilon(E)$. Then we define operators of type 1 as operators with $\omega'(E) = 1$ on the spectrum.[8]. It turns out to be a set open in C^ω_h for each $h > 0$, that contains the entire almost Mathieu family, along with several other popular families, and therefore their analytic neighborhoods. In [29–31] we present full solutions of problems (1)-(3) for all operators of type I. Namely, we have

Theorem 4. *[30, 31][Robust Problem 1]. The ten martini holds for all operators of type I: the spectrum of $H_{V,\alpha,\theta}$ is a Cantor set for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.*

Theorem 5. *[29][Robust Problem 2]. The IDS of all non-critical operators $H_{V,\alpha,\theta}$ of type I is absolutely continuous*

Theorem 6. *[29][Robust Problem 3]. Supercritical operators $H_{V,\alpha,\theta}$ of type I have pure point spectrum for a.e. θ for E with $L(E) > \beta(\alpha)$ and singular continuous spectrum for all θ for E with $L(E) < \beta(\alpha)$.*

Each of these theorems is not only the solution of the corresponding robust problem, but also presents the first corresponding **arithmetic** result for **any** operator $H_{V,\alpha,\theta}$ with V other than the $\lambda \cos$.

The proof of the original ten martini problem in [8] has crucially depended in several places on V being a $\lambda \cos$. The original Puig's argument, establishing Cantor spectrum for the almost Mathieu for Diophantine α, was based on the fact that localization for $H_{2\lambda \cos,\alpha,\mathbf{0}}$ proved in [46, 50] (with the argument requiring $V = \lambda \cos$) implies reducibility at each eigenvalue E for the dual $SL(2, \mathbb{R})$ cocycle to

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

. Puig showed that $c \neq 0$ implies by a Moser-Pöshel argument that E is an edge of an open gap, while $c = 0$ is impossible by the simplicity of the eigenvalues for second-difference operators. This therefore requires $V = \lambda \cos$ in three different places; to ensure localization at $x = 0$, to ensure that the dual operator is second-difference, and, at the last step, that the one where we need to argue through the simplicity of the eigenvalues, is also second-difference. Since localization for $x = 0$ is only expected to hold up to the $\beta(\alpha) = L(E)/2$ threshold [8, 9, 63], this had to be replaced in [8] by a more sophisticated argument, that still required $V = \lambda \cos$ in three different places,

[8]A subtle point is that the operator may be of type 1 in C^ω_h but not in ${C_{h'}}^\omega$ for some $h' < h$ if the first turning point $h' < e_1 < h$

already for the "Diophantine" side of the argument. Indeed, for the Kotani theory for ergodic Schrödinger operators and (fictitious) improved regularity of the m-function [8] the second-differenceness of the dual operator is needed; the elements of Puig's argument that were left required both the operator and its dual to be second-difference, and fixed frequency localization for the dual model also required $V = \lambda\cos$. Even more importantly, the Liouville side of the proof, which was an extension of a C^*-algebraic argument of [22] requires $V = \lambda\cos$ in a rather dramatic way. It is mainly the last consideration that has prompted the authors of [8] to state that they viewed the fact that the Diohantine and Liouville approaches met at the middle as some sort of a miracle.

To get rid of this cos spell, the general theory of finite difference operators whose complex-symplectic transfer-matrix cocycles are partially hyperbolic with two dimensional center (so called **PH2** operators) was developed in [30].

The two key results are

- the Kotani theory: for PH2 $Sp(2d, C)$ cocycles over minimal dynamics T that have zero center Lyapunov exponents, there is L^2 reducibility in the center (see [30] for details).
- the simplicity of point spectrum: ergodic finite difference operators with minimal underlying dynamics, whose transfer matrix cocycles are PH2 have simple point spectrum.

The Kotani theory part, in particular, makes the first progress in the solution of the Kotani-Simon problem [60]: to find the conditions when vanishing of **some** Lyapunov exponents of the transfer-matrix cocycle imply the potential is deterministic[9]. Similar theory for partially hyperbolic cocycles with center of arbitrary dimension will appear in [28], thus solving the Kotani-Simon problem whenever the cocycle with zero Lyapunov exponents in the center is partially hyperbolic.

The simplicity of point spectrum part provides a condition for effective *second-differenceness* for higher difference linear operators, thus having also many other possible consequences.

Finally, to develop an all-frequency argument, it was most important to not rely on reducibility to the identity, since it has Diophantine obstructions. Avila, Fayad, and Krikorian [7], using the "cheap trick" of [24], proved that if one replaces reducibility to *constants* by the reducibility to *rotations*, then the a.e. energy reducibility result (in the regime of zero Lyapunov exponents) holds for every α. [10] In the same spirit, the "all $\alpha \notin \mathbb{Q}$" nature of the proof of [30] is largely enabled by replacing the impossibility to reduce to the identity, used in the proofs of [8, 67] with the impossibility to reduce to a rotation with zero rotation number, with, in turn, the simplicity of eigevalues for operators with PH2 cocycles with the impossibility to have two linearly independent almost localized solutions.

These arguments constitute the core of [30], that therefore present the first non-cos ten martini result. [30] still requires though V to be a trigonometric polynomial (so that there is a dual cocycle) and to be even. The robust version of the ten martini statement, as presented in Theorem 4 will appear in [31]. It is based on the tools developed in [29] to

[9] It was proved in [60] for the case when **all** Lyapunov exponents vanish.

[10] We note also a recent *all* $\alpha \notin \mathbb{Q}$ argument, where Avila-You-Zhou [15] replace reducibility to the identity by quantitative almost reducibility to the identity.

(1) pass from trigonometric polynomials to analytic functions in the duality based arguments
(2) remove the need for the **evenness** of V.

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The requirement for V to be even first appeared in [25], and was significantly used in [46] and the follow-up arithmetic results [8, 51, 52, 63], exploiting the fact that reflection-based resonances appearing for even V can be handled efficiently. The fact that the approach of [46] can be extended to the case of acceleration 1 for **even** V has been clear ever since the introduction of acceleration in [3] where Avila's proof essentially showed that, for energies with acceleration 1, traces of transfer-matrices (i.e., determinants of block-restrictions with periodic boundary conditions) of size q_n effectively behave like trigonometric polynomials of degree q_n, which they are for the almost Mathieu operator, and therefore have no more than q_n zeros, a feature that enables arithmetic localization proofs in [8, 46, 51, 52, 62, 63] as well as many other recent localization results that fundamentally go back to the same zero-counting idea, e.g. [43, 49, 57, 59, 68, 73]. The details have been recently implemented in [37].

However, if V is not even, the zero count, even for the case of acceleration 1, becomes $2q_n$ making the approaches that go back to [46] break down. The approach of [29] is based instead on the structure of dual cocycle, and allows also to uncover the deep dynamical meaning of evenness. It is shown in [29] that the complex-symplectic structure in the dual center is analytically conjugated to a real-symplectic one modulated by a diagonal rotation, with the angle of the latter being zero if and only if V is even. This allows to introduce the rotation numbers for the complex-symplectic case, which, while important in its own right, leads also to the solution of problems (2) and (3) for all trigonometric polynomial V without the need for V to be even.

While the above structures are defined for the dual cocycles associated with trigonometric polynomial V, for general analytic V one can consider a sequence of cocycles corresponding to its trigonometric polynomial cutoffs. While the cocycles itself change dramatically, with no limit in any of their components, it was shown in [32] that when the size of the cutoffs increases, their appropriately ordered Lyapunov exponents converge in the limit to so-called dual Lyapunov exponents (that exist in absence of a cocycle). A key result of [29] is that the corresponding complex-symplectic structures in the center and therefore the related rotation numbers also converge, leading to the existence of well-structured center (and the rotation numbers) for the dual operator, still in absence of an underlying cocycle.

The final details of the proof of Theorem 4 will appear in [31], completing therefore the program of showing robustness of all of (1)-(4).

Personal remarks

Other than doubling as the plenary virtual ICM 22 and $(\mathrm{WM})^2$ talk, that I was given a special opportunity to deliver in-person in front of a great audience at the Probability and Mathematical Physics conference in Helsinki on June 30 2022, my presentation has had yet another role as the talk for being honored with the inaugural Olga Ladyzhenskaya medal in Mathematical Physics. I am very grateful for the encouragement by the editors and the opportunity to present here what I remember of some of the personal remarks I made then in this regard.

Ladyzhenskaya was a giant of a mathematician. She was a truly great role model to men and women mathematicians, through her lifetime of dedication to mathematics, dedication that has never stopped. Receiving an award honoring her is not only a tremendous honor, but also a huge inspiration.

It is additionally special to me, because, while most mathematicians, even if restricting to the PDE people, probably learn about Ladyzhenskaya only sometime in their 20s, I've been regularly hearing her name literally right from the time I was born. My mom, Valentina Borok, and my dad, Yakov Zhitomirskiy, were both mathematicians working on PDEs, and very much in the orbit of Ladyzhenskaya, especially my mom. In particular, Ladyzhenskaya, was an opponent at both my mom's Candidate (PhD) and Doctor (a significantly more demanding version of habilitation) defenses. Despite the sound of the word "opponent", this role essentially meant a well-known arm-length person in a close field who reads the thesis in detail and gives a presentation on it at the defense, so it should rather have been called "a supporter". Her name (usually, as a patronimic "Olga Alexandrovna") was mentioned a lot by my parents, with a great deal of reverence. She was arguably the strongest woman mathematicians of the second half of the last century. Moreover, after the deaths of Petrovskij and Shilov in the early 70s, Ladyzhenskaya was definitely the best PDE expert in the USSR - not a small feat! The school she created includes people such as Buslaev, L. Faddev, and counts over 300 descendants on math genealogy. Possibly for political reasons in the USSR [11] and for the iron curtain reasons outside it, she hasn't had many formal accolades up until the 90s when she was already in her 70s, but the respect to her was deep and universal.

I may or may not have met Ladyzhenskaya at age four, when she came to Kharkiv as an opponent at my mom's doctoral (habilitation+) defense. After that, I did have multiple opportunities to meet Ladyzhenskaya, but lost all of them. My mom would travel to St. Petersburg about once a year to speak at Ladyzhenskaya's seminar and/or meet with her. She would quite often take me along, mostly with a purpose to expose me to more culture. So she would drop me off at one of the great St. Petersburg musea, while she visited Ladyzhenskaya. I liked that.

In fact, my mom, didn't really want me to become a mathematician and even discouraged me quite a bit. "This job is not good for a girl", "You want to come home at five and only have your family in mind; You cannot do it as a mathematician", "The most important thing for a girl is family" are things I heard growing up. It is particularly interesting because she was actually a very successful mathematician herself. Succeeding professionally against the odds; for over 20 years she was the only female professor of mathematics in the entire country of Ukraine (over 30 mln people). Reaching the Professor level there has been prohibitively difficult for all; the rate among faculty of professors to more junior titles was 1/20 if not less, and the title brought a lot of prestige, for anyone. With no affirmative action, everybody knew that all mom's achievements were hers, and the respect to her - from everyone around - was palpable. When I was five, there was a long article about her, in the city's central newspaper. It started with "Little Svitlana has big blue eyes" and featured a picture of me helping mom at a subbotnik. I was enormously proud.

So why would she discourage me? Larger-than-life, universally loved and admired, sharp and multi-talented, she had only one goal for me: happiness. My theory for a

[11] She was involved with the dissident circles, and was a friend of Ahmatova and Solzhenitzyn

long time was that she just didn't want to traumatize me by being my unachievable role model. Yet I well knew I was not as brilliant as her and never thought I could achieve nearly as much, anyway. Also, my parents didn't discourage my brother; to the contrary it was almost assumed from the early on that he would be a mathematician. When a few years ago, I finally asked my father why, he told me that it was mom's idea because she was afraid... that, being a bit obsessive, I would try to go into the footsteps of Ladyzhenskaya. Because to be a really great mathematician, you had to be fully dedicated, like her. Mathematics was her life. Indeed, at least in the Soviet Union, at that time it was essentially incompatible with family life for women because of various cultural issues, see e.g. [58]. My mom was one of the handful of women in the entire Soviet Union who were able to combine high level math with family, but she didn't achieve the greatness at a really high level.

I only followed about 50% of my mom's advice, but I am lucky to be living in different time and circumstances. I think that the times when, as my mom said, mathematician was a bad job for a girl are long gone. One needs a lot of luck to do something that gets a lot of recognition, and I am very humbled by and grateful for the recognition I've been getting lately. But whether or not it happens is not what is really important. It is a great job for all who are lucky enough to develop their interest in math. Setting your level of dedication and your balance to whatever suits you best, ranging from a quest to the true greatness, like Ladyzhenskaya, all the way to mainly focusing on the other things in life when outside the office, but doing the work that you love, much of it in the environment that you love, what can be better?

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