EXPONENTIAL DYNAMICAL LOCALIZATION FOR RANDOM WORD MODELS

1. Abstract

We give a new proof of spectral localization for one-dimensional Schrödinger operators whose potentials arise by randomly concatenating words from an underlying set. We then show that once one has the existence of a complete orthonormal basis of eigenfunctions (with probability one), the same estimates used to prove it naturally lead to a proof of exponential dynamical localization in expectation (EDL) on any compact set which trivially intersects a finite set of critical energies.

2. Introduction

We consider the random word models on $l^2(\mathbb{Z})$ given by

$$H_\omega\psi(n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

The potential is a family of random variables defined on a probability space Ω. To construct the potential V above, we consider words $\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots$ which are vectors in $\mathbb{R}^n$ with $1 \leq n \leq m$, so that $V_\omega(0)$ corresponds to the kth entry in ω_0. A precise construction of the probability space Ω and the random variables $V_\omega(n)$ is carried out in the next section.

One well-known example of random word models is the random dimer model. In this situation, ω_i takes on values (λ, λ) or $(-\lambda, -\lambda)$ with Bernoulli probability. First introduced in [6], the random dimer model is of interest to both physicists and mathematicians because it exhibits a delocalization-localization phenomenon. It is known that the spectrum of the operator H_ω is almost surely pure point with exponentially decaying eigenfunctions. On the other hand, when $0 < \lambda \leq 1$ (with $\lambda \neq \frac{1}{\sqrt{2}}$), there are critical energies at $E = \pm\lambda$ where the Lyapunov exponent vanishes [5]. The vanishing Lyapunov exponent at these energies can be exploited to prove lower bounds on quantum transport resulting in almost sure overdiffusive behavior [13]. The authors in [13] show that for almost every ω and for every $\alpha > 0$ there is a positive constant C_α such that

$$\frac{1}{T}\int_0^T \langle \delta_0, e^{iH_\omega t}|X|^q e^{-iH_\omega t}\delta_0\rangle dt \geq C_\alpha T^{q-\frac{1}{2}-\alpha}.$$

This was later extended to a sharp estimate in [16].

Again, the over-diffusive behavior above is to be contrasted with the fact that not only does the random dimer model display spectral localization, but also dynamical localization on any compact set I not containing the critical energies $\pm\lambda$ [5]:

$$\sup_t \langle P_I(H_\omega)e^{-iH_\omega t}\psi, |X|^q P_I(H_\omega)e^{-iH_\omega t}\psi\rangle < \infty, \tag{1}$$

Here $P_I(H_\omega)$ is the spectral projection of H_ω onto the set I.

We strengthen this last result by showing that there is exponential dynamical localization in expectation (EDL) on any compact set I with $\pm\lambda \notin I$. That is, there are $C, \alpha > 0$ such that for any $p, q \in \mathbb{Z}$,

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} |\langle \delta_p, P_I(H_\omega)e^{-itH_\omega}\delta_q\rangle|\right] \leq Ce^{-\alpha|p-q|}. \tag{2}$$

Another well-known example of random word models is the Anderson model. One of the central challenges in dealing with this model in full generality arises when there is a lack of the regularity of the single-site distribution. The absence of regularity was previously overcome using multi-scale analysis; first, in the Anderson setting [2], then in the dimer case [5, 10], and finally in random word models [4]. The multi-scale approach leads to weaker dynamical localization results than those where sufficient regularity of the single-site distribution allows one to instead appeal to the fractional moment method (e.g. [8], [1]). In particular, EDL always follows in the framework of the fractional moment method, but of course regularity is required.

Loosely speaking, the multi-scale analysis shows that the complement of the event where one has exponential decay of the Green's function has small probability. One of the consequences of this method is that while this event does have small probability, it can only be made sub-exponentially small.

A recent new proof of spectral and dynamical localization for the one-dimensional Anderson model for arbitrary single-site distributions [14] uses positivity and large deviations of the Lyapunov exponent to replace parts of the multi-scale analysis. The major improvement in this regard (aside from a shortening of the length and complexity of localization proofs in one-dimension) is that the complement of the event where the Green's function decays exponentially can be shown to have exponentially (rather than sub-exponentially) small probability. These estimates were implicit in the proofs of spectral and dynamical localization given in [14] and were made explicit in [9]. The authors in [9] then used these estimates to prove EDL for the Anderson model and we extend those techniques to the random word case.

There are, however, several issues one encounters when adapting the techniques developed for the Anderson model in [9, 14] to the random word case. Firstly, in the Anderson setting, a uniform large deviation estimate is immediately available using a theorem in [17]. Since random word models exhibit local correlations, there are additional steps that need to be taken in order to obtain suitable analogs of large deviation estimates used in [9, 14]. Secondly, random word models may have a finite set of energies where the Lyapunov exponent vanishes and this phenomena demands some care in obtaining estimates on the Green's functions analogous to those in [9, 14].

As above, with $P_I(H_\omega)$ denoting the spectral projection of H_ω onto the interval I, we have the following result:

Theorem 1. *The spectrum of H_ω is almost surely pure point with exponentially decaying eigenfunctions.*

Our main result is:

Theorem 2. *There is a finite $D \subset \mathbb{R}$ such that if I is a compact set and $D \cap I = \emptyset$ then there are $C > 0$ and $\alpha > 0$ such that $\mathbb{E}[\sup_{t\in\mathbb{R}} |\langle \delta_p, P_I(H_\omega)e^{-itH_\omega}\delta_q\rangle|] \le Ce^{-\alpha|p-q|}$ for any $p, q \in \mathbb{Z}$.*

The remainder of the paper is organized as follows: section 3 contains preliminaries needed to discuss the large deviation estimates found in section 4 and the lemmas (which are established in section 6) needed for localization found in section 5. Finally, section 7 contains the proof of Theorem 1 and section 8 contains the proof of Theorem 2.

3. Preliminaries

We begin by providing details on construction of Ω and $V_\omega(n)$ by following [4].

Fix $m \in \mathbb{N}$ and $M > 0$. Set $\mathcal{W} = \bigcup_{j=1}^m \mathcal{W}_j$ where $\mathcal{W}_j = [-M, M]^j$ and ν_j are finite Borel measures on $\mathcal{W}_j$ so that $\sum_{j=1}^m \nu_j(\mathcal{W}_j) = 1$.

Additionally, we assume that $(\mathcal{W}, \nu)$ has two words which do not commute. That is,

$$\begin{cases} \text{For } i = 0, 1 \text{ there exist } w_i \in \mathcal{W}_{j_i} \in supp(\nu) \text{ such that} \\ (w_0(1), w_0(2), ..., w_0(j_0), w_1(1), w_1(2), ..., w_1(j_1)) \\ \text{and } (w_1(1), w_1(2), ..., w_1(j_1), w_0(1), w_0(2), ..., w_0(j_0)) \text{ are distinct.} \end{cases} \tag{3}$$

Set $\Omega_0 = \mathcal{W}^{\mathbb{Z}}$ and $\mathbb{P}_0 = \otimes_{\mathbb{Z}}\nu$ on the σ-algebra generated by the cylinder sets in Ω_0.

The average length of a word is defined by $\langle L\rangle = \sum_{j=1}^m j\nu(\mathcal{W}_j)$ and if $w \in \mathcal{W} \cap \mathcal{W}_j$, we say w has length j and write $|w| = j$.

We define $\Omega = \bigcup_{j=1}^m \Omega_j \subset \Omega_0 \times \{1, ..., m\}$ where $\Omega_j = \{\omega \in \Omega_0 : |\omega_0| = j\} \times \{1, ..., j\}$. We define the probability measure $\mathbb{P}$ on the σ-algebra generated by the sets $A \times \{k\}$ where $A \subset \Omega_0$ such that for all $\omega \in A$, $|\omega_0| = j$ and $1 \le k \le j$.

For such sets we set $\mathbb{P}[A \times \{k\}] = \frac{\mathbb{P}_0(A)}{\langle L\rangle}$.

The shifts T_0 and T on Ω_0 and Ω (respectively) are given by:

$(T_0\omega)_n = \omega_{n+1}$ and

$$T(\omega, k) = \begin{cases} (\omega, k+1) & \text{if } k < |\omega_0| \\ (T_0(\omega), 1) & \text{if } k = |\omega_0|. \end{cases} \tag{4}$$

With this set-up, the shift T is ergodic and the potential $V_{\omega,k}$ is obtained through $..., \omega_{-1}, \omega_0, \omega_1, ...$ so that $V_{\omega,k}(0) = \omega_0(k)$.

Thus, the results from [3] can be used to show the spectrum of H_ω is almost surely a non-random set.

Definition. We call $\psi_{\omega,E}$ a *generalized eigenfunction* with *generalized eigenvalue* E if $H_\omega\psi_{\omega,E} = E\psi_{\omega,E}$ and $|\psi_{\omega,E}(n)| \le (1 + |n|)$.

We denote H_ω restricted to the interval $[a, b]$ by $H_{\omega,[a,b]}$ and for $E \notin \sigma(H_{\omega,[a,b]})$ the corresponding Green's function by

$$G_{[a,b],E,\omega} = (H_{\omega,[a,b]} - E)^{-1}.$$

Additionally, we let

$$P_{[a,b],\omega,E} = \det(H_{\omega,[a,b]} - E).$$

We also let $E_{j,\omega,[a,b]}$ denote the jth eigenvalue of the operator $H_{\omega,[a,b]}$ and note that there are $b - a + 1$ many of them (counting multiplicity).

Definition. $x \in \mathbb{Z}$ is called (c, n, E, ω)-*regular* if there is a $c > 0$ so that:

(1) $|G_{[x-n,x+n],E,\omega}(x, x-n)| \leq e^{-cn}$ and
(2) $|G_{[x-n,x+n],E,\omega}(x, x+n)| \leq e^{-cn}$.

It is well-known that for any generalized eigenfunction $\psi_{\omega,E}$ and any $x \in [a, b]$,

$$\psi(x) = -G_{[a,b],E,\omega}(x, a)\psi(a-1) - G_{[a,b],E,\omega}(x, b)\psi(b+1), \tag{5}$$

and using Cramer's rule on the $b-a+1$-dimensional matrix $H_{\omega,[a,b]}$, one obtains:

$$|G_{[a,b],E,\omega}(x, y)| = \frac{|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|}. \tag{6}$$

For $w \in \mathcal{W}$, with $w = (w_1, ..., w_j)$, we define word transfer matrices by $T_{w,E} = T_{w_j,E} \dots T_{w_1,E}$ where $T_{v,E} = \begin{pmatrix} E-v & -1 \\ 1 & 0 \end{pmatrix}$.

The transfer matrices over several words are given by

$$T_{\omega,E}(m, k) = \begin{cases} T_{\omega_k,E} \dots T_{\omega_m,E} & \text{if } k > m, \\ \mathbb{I} & \text{if } k = m, \\ T_{\omega,E}(m, k)^{-1} & \text{if } k < m. \end{cases} \tag{7}$$

and $T_{[a,b],E,\omega}$ denotes the product of the transfer matrices so that:

$$T_{[a,b],E,\omega}\begin{pmatrix} \psi(a) \\ \psi(a-1) \end{pmatrix} = \begin{pmatrix} \psi(b+1) \\ \psi(b) \end{pmatrix}. \tag{8}$$

By induction, we have:

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}. \tag{9}$$

4. The Lyapunov Exponent

Kingman's subadditive ergodic theorem allows us to define the Lyapunov exponent:

$$\gamma(E) := \lim_{k\to\infty} \frac{1}{k\langle L\rangle} \log ||T_{\omega,E}(1, k)||. \tag{10}$$

Note that for a fixed E, the above limit exists for $a.e.$ ω.

Let μ_E denote the smallest closed subgroup of $SL(2, \mathbb{R})$ generated by the 'word'-step transfer matrices. It is shown in [4] that μ_E is strongly irreducible and contracting for all E outside of a finite set $D \subset \mathbb{R}$ and hence, Furstenberg's theorem implies $\gamma(E) > 0$ for all such E. Since the Lyapunov exponent is defined as a product of i.i.d. matrices, γ is continuous. So, if I is a compact interval such that $D \cap I = \emptyset$ and

$$v := \inf\{\gamma(E) : E \in I\},$$

then $v > 0$.

Motivated by (9) above and large deviation theorems, we define:

$$B^+{}_{[a,b],\varepsilon} = \left\{(E, \omega) : E \in I, |P_{[a,b],E,\omega}| \geq e^{(\gamma(E)+\varepsilon))(b-a+1)}\right\}, \tag{11}$$

and

$$B^-{}_{[a,b],\varepsilon} = \left\{ (E,\omega) : E \in I, |P_{[a,b],E,\omega}| \le e^{(\gamma(E)-\varepsilon))(b-a+1)} \right\}. \tag{12}$$

and the corresponding sections:

$$B^\pm{}_{[a,b],\varepsilon,\omega} = \left\{ E : (E,\omega) \in B^\pm_{[a,b],\varepsilon} \right\}, \tag{13}$$

and

$$B^\pm{}_{[a,b],\varepsilon,E} = \left\{ \omega : (E,\omega) \in B^\pm_{[a,b],\varepsilon} \right\}. \tag{14}$$

5. Large Deviations

The goal of this section is to obtain a uniform large deviation estimate for $P_{[a,b],E}$. In the Anderson model, a direct application of Tsai's theorem for matrix elements of products of i.i.d. matrices results in both an upper and lower bound for the above determinants. In the general random word case, there are two issues. Firstly, the one-step transfer matrices are not independent. This issue is naturally resolved by considering ω_k-step transfer matrices and treating products over each word as a single step. However, in this case, both the randomness in the length of the chain as well as products involving partial words need to be accounted for. Since matrix elements are majorized by the norm of the matrix and all matrices in question are uniformly bounded, we can obtain an upper bound identical to the one obtained in the Anderson case. Lower bounds on the matrix elements are more delicate and require the introduction of random scales. For the reader's convenience, we first recall Tsai's theorem and then give the precise statements and proofs of the results alluded to above.

As remarked above, μ_E is strongly irreducible and contracting for $E \in I$. In addition, the 'word'-step transfer matrices (defined in (5)) are bounded, independent, and identically distributed. These conditions are sufficient to apply Tsai's theorem on large deviations of matrix elements for products of i.i.d. matrices.

Theorem 3 ([17])**.** *Suppose I is a compact interval and for each $E \in I$, $Z_1^E, ..., Z_n^E, ...$ are bounded i.i.d random matrices such that the smallest closed subgroup of $SL(2,\mathbb{R})$ generated by the matrices is strongly irreducible and contracting. Then for any $\varepsilon > 0$, there is an $\eta > 0$ and an N such that for any $E \in I$, any unit vectors u, v and $n > N$,*

$$\mathbb{P}\left[e^{(\gamma(E)-\varepsilon)n} \le |\langle Z_n^E \dots Z_1^E u, v\rangle| \le e^{(\gamma(E)+\varepsilon)n} \right] \ge 1 - e^{-\eta n}.$$

Lemma 1. *For any $\varepsilon > 0$ there is an $\eta > 0$ and an N such that*

$$\mathbb{P}[B^+_{[a,b],\varepsilon,E}] \le e^{\eta(b-a+1)}$$

for all a,b $\in \mathbb{Z}$ such that $b - a + 1 > N$ and all $E \in I$.

Proof. Let $Y_i = |\omega_i|$, so Y_i is the length of the ith word and let $S_n = Y_1 + ... + Y_n$.

Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $P_{E,(\omega_1,\omega_n)} = \det(H_{(\omega_1,\omega_n)} - E)$ where $H_{(\omega_1,\omega_2)}$ denotes H_ω restricted to the interval where V takes values determined by ω_1 through ω_n. By formula (9) from the previous section, $P_{E,(\omega_1,\omega_n)} = \langle T_{\omega,E}(1,n)u, u\rangle$.

Letting $\varepsilon > 0$ and applying Theorem 3 to the random products $T_{\omega,E}(1,k)$, we obtain an $\eta_1 > 0$ and an N_1 such that for $n > N_1$, $\mathbb{P}[|P_{(\omega_1,\omega_n),E}| \le e^{(\gamma(E)+\varepsilon)n\langle L\rangle}] \ge 1 - e^{-\eta_1 n}$.

Now let $\varepsilon_1 > 0$ so that $\varepsilon_1 \sup\{\gamma(E) : E \in I\} < \varepsilon$. We apply large deviation estimates (e.g. [7]) to the real, bounded, i.i.d. random variables Y_i to obtain an N_2 and an η_2 such that for $n > N_2$, $\mathbb{P}[S_n - n\varepsilon_1 < n\langle L\rangle < S_n + n\varepsilon_1] \geq 1 - e^{-\eta_2 n}$.

Denoting the intersection of the above events by A_n, we have,

$$\begin{aligned}
|P_{E,(\omega_1,\omega_n)}| &\leq e^{(\gamma(E)+\varepsilon)n\langle L\rangle} \\
&\leq e^{(\gamma(E)+\varepsilon)(S_n+n\varepsilon_1)} \\
&= e^{(\gamma(E)+\varepsilon)S_n+\gamma(E)\varepsilon_1 n+n\varepsilon\varepsilon_1} \\
&\leq e^{(\gamma(E)+\varepsilon)S_n+\gamma(E)\varepsilon_1 S_n+S_n\varepsilon\varepsilon_1} \\
&\leq e^{(\gamma(E)+3\varepsilon)S_n}.
\end{aligned}$$

Thus, we have an estimate where the Lyapunov behavior is a true reflection of the length of the interval.

We are now in a position to get an estimate in between two words. For any $1 \leq k \leq S_{n+1} - S_n$, let $P_{E,(\omega_1,\omega_n+k)} = \det(H_{(\omega_1,\omega_n+k)} - E)$ where $H_{(\omega_1,\omega_n+k)}$ denotes H_ω restricted to the interval where V_ω takes values determined by ω_1 through the kth letter of ω_{n+1}.

Since the one-step transfer matrices are uniformly bounded, formula (9) and the last inequality imply for any $1 \leq k \leq S_{n+1}-S_n$, $|P_{\omega,E,[1,S_n+k]}| \leq Ce^{(\gamma(E)+3\varepsilon)S_n} \leq e^{(\gamma(E)+4\varepsilon)S_n}$ on A_n.

Let $\eta_3 = \min\{\eta_1, \eta_2\}$, and choose $0 < \eta_4 < (2\eta_3)/m$). Then for sufficiently large n,

$$\begin{aligned}
\mathbb{P}[A_n] &\geq 1 - e^{-2\eta_3 n} \\
&\geq 1 - e^{-\eta_4(S_n+k)}.
\end{aligned}$$

We have an estimate where the Lyapunov behavior and the probability of the event reflect the true length of the interval, so we can apply the shift T to conclude that for any sufficiently large n,

$$\mathbb{P}[\{|P_{\omega,E,[1,n]}| \leq e^{(\gamma(E)+4\varepsilon)n}\}] \geq 1 - e^{-\eta_4 n}.$$

The result now follows for any interval $[a, b]$ with $b - a + 1 > N$ since T preserves the probability of events.

□

We now deal with the lower bound.

As in the proof of Lemma 1, let $Y_i = |\omega_i|$, so Y_i is the length of the ith word and let $S_n = Y_1 + ... + Y_n$.

Lemma 2. *For any $\varepsilon > 0$ there is an $\eta > 0$, sets $A_{n,l} \subset \Omega$, a sequence of random scales R_n and Q_n, and an N such that for $n > N$ and every $l \in \mathbb{Z}$, on $A_{n,l}$*

(1) $S_n - n\varepsilon_1 < n\langle L\rangle < S_n + n\varepsilon_1$,
(2) $\frac{n(\langle L\rangle-\varepsilon)}{2} - m \leq R_n \leq \frac{n(\langle L\rangle+\varepsilon)}{2} - m$,
(3) $n(\langle L\rangle - \varepsilon) + 2m \leq Q_n \leq n(\langle L\rangle + \varepsilon) + 2m$,
(4) $|P_{\omega,[-R_n,R_n],E}| \geq e^{(\gamma(E)-\varepsilon)2R_n}$,
(5) for any $S_n + 2m \leq k < S_{n+1} + 2m$, $|P_{\omega,[k-R_n,k+R_n],E}| \geq e^{(\gamma(E)-\varepsilon)2R_n}$,

(6) $\mathbb{P}[A_{n,l}] \geq 1 - e^{-\eta 2R_n}$.

Proof. Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and let $P_{E,(\omega_1,\omega_n)}$ denote $\det(H_{(\omega_1,\omega_n)} - E)$ (where $H_{(\omega_1,\omega_n)}$ denotes H_ω restricted to the interval where the values of V_ω are given by $\omega_1, ..., \omega_n$.

Let $\varepsilon > 0$ and apply Theorem 3 to the random products $T_{\omega,E}(1,n)$ with vectors u, u and u, v. By formula (9), we obtain an $\eta_1 > 0$ and an N_1 such that for $n > N_1$, $\mathbb{P}[|P_{(\omega_1,\omega_n),E}| \geq e^{(\gamma(E)-\varepsilon)(S_n(\omega))}] \geq 1 - e^{-\eta_1 n}$.

Now let $\varepsilon_1 > 0$ so that $\varepsilon_1 \sup\{\gamma(E) : E \in I\} < \varepsilon$. We now apply large deviation estimates (e.g. [7]) to the real random variables Y_i, to obtain an N and an η_2 such that for $n > N$,

$$\mathbb{P}[S_n - n\varepsilon_1 < n\langle L\rangle < S_n + n\varepsilon_1] \geq 1 - e^{-\eta_2 n}. \tag{15}$$

On the intersection of the above events, we have,

$$\begin{aligned}
|P_{E,(\omega_1,\omega_n)}| &\geq e^{(\gamma(E)-\varepsilon)n\langle L\rangle} \\
&\geq e^{(\gamma(E)-\varepsilon)(S_n - n\varepsilon_1)} \\
&= e^{(\gamma(E)-\varepsilon)S_n - (\gamma(E)\varepsilon_1)n + n\varepsilon\varepsilon_1} \\
&\geq e^{(\gamma(E)-\varepsilon)S_n - (\gamma(E)\varepsilon_1)S_n + n\varepsilon\varepsilon_1} \\
&\geq e^{(\gamma(E)-2\varepsilon)S_n}.
\end{aligned}$$

As in Lemma 1, we now have Lyapunov behavior that reflects the actual length of the interval that H_ω is restricted to. We now adjust the estimate so the interval is centered at 0.

For any $p \in \mathbb{Z}$, $\frac{(\langle L\rangle-\varepsilon)n-j}{2} \leq p \leq \frac{(\langle L\rangle+\varepsilon)n-j}{2}$ $(j = 1, 2, ..., m)$, apply T^{-p} to the set $\{|P_{(\omega_1,\omega_n),E}| \geq e^{(\gamma(E)-\varepsilon)S_n}\}$ and consider $\bigcap T^{-p}\{|P_{(\omega_1,\omega_n),E}| \geq e^{(\gamma(E)-\varepsilon)S_n}\}$.

With $\eta_3 = \min\{\eta_1, \eta_2\}$ for sufficiently large n we have,

$$\begin{aligned}
\mathbb{P}\left[\left(\bigcap T^{-p}\{|P_{(\omega_1,\omega_n),E}|\right.\right. &\leq \left.\left. e^{(\gamma(E)-2\varepsilon)S_n}\}\right)^c\right] \\
&\leq n\varepsilon e^{-\eta_3 n} \\
&\leq n e^{-\eta_3 n} \\
&\leq e^{-\frac{1}{2}\eta_3 n}.
\end{aligned}$$

In particular, $\mathbb{P}\left[\bigcap T^{-p}|P_{(\omega_1,\omega_n),E}| \geq e^{(\gamma(E)-\varepsilon)S_n}\}\right] \geq 1 - e^{-\frac{1}{2}\eta_3 n}$.

For $(\omega, k) \in \Omega$, we define random variables $R_n(\omega, k)$ by

$$R_n(\omega, k) = \begin{cases} \frac{S_n}{2} - Y_0 - k + 1 & \text{if } S_n \text{ is even,} \\ \frac{S_n - 1}{2} - Y_0 - k + 1 & \text{if } S_n \text{ is odd.} \end{cases} \tag{16}$$

Note that the desired bounds on R_n hold by (15).

Thus, on $\bigcap T^{-p}\{|P_{(\omega_1,\omega_n),E}|\} \geq e^{(\gamma(E)-\varepsilon)S_n}\}$,

$$|P_{\omega,[-R_n,R_n],E}| \geq e^{(\gamma(E)-\varepsilon)S_n}$$

$$\geq e^{(\gamma(E)-\varepsilon)2R_n}.$$

Denote this last event by C_n.

We want a similar estimate for intervals centered at points sufficiently far from 0.

Define random variables Q_n, by $Q_n = S_n + 2m$ and note that the desired bounds on Q_n again hold by (15).

Applying the shift T^p to C_n for $p \in \mathbb{Z}$, $(\langle L\rangle - \varepsilon)n + 2m \leq p \leq (\langle L\rangle + \varepsilon)n + 2m$, and considering the event $\bigcap T^p C_n$, we have

$$\begin{aligned}\mathbb{P}\left[\left(\bigcap T^p(C_n)\right)^c\right] &\leq 2n\varepsilon\mathbb{P}[(C_n)^c] \\ &\leq 2n\mathbb{P}[(C_n)^c] \\ &\leq e^{-\eta_4 n}.\end{aligned}$$

where $\eta_4 > 0$ is chosen so $2ne^{-\frac{1}{2}\eta_3 n} \leq e^{-\eta_4 n}$ for sufficiently large n.

We conclude $\mathbb{P}[\bigcap T^p C_n] \geq 1 - e^{-\eta_4 n}$.

Additionally, on $\bigcap T^p C_n$, for any $k \in \mathbb{Z}$, $S_n + 2m \leq k < S_{n+1} + 2m$,

$$|P_{\omega,[k-R_n,k+R_n],E}| \geq e^{(\gamma(E)-\varepsilon)2R_n}.$$

Finally, we set $A_{n,0} = \bigcap(T^k C_n) \cap C_n$ and all that remains is to adjust the estimate on the probability of $A_{n,0}$. With $\eta_5 = \frac{1}{2}\min\{\frac{1}{2}\eta_3, \eta_4\}$, we have

$$\begin{aligned}\mathbb{P}[A_n] &\geq 1 - e^{-\eta_5 n} \\ &\geq 1 - e^{-\eta 2R_n}\end{aligned}$$

where $\eta > 0$ is chosen so that $\eta_3 2R_n < \eta_2 n$ for all sufficiently large n. Note that this can be done given the bounds on R_n established by (15).

Since we have the desired result for intervals centered at 0, we obtain the result for intervals centered at any $l \in \mathbb{Z}$ by applying T^l to $A_{n,0}$ since T preserves the probability of events. $\square$

Remark. Given an $\varepsilon > 0$, taking the minimum of the two η's furnished by Lemma 1 and Lemma 2, we have an η which we call the large deviation parameter.

6. Lemmas

We prove localization results on a compact interval I where $D \cap I = \emptyset$. In order to do so, we fix a larger interval $\tilde{I}$ such that I is properly contained in $\tilde{I}$ and $D \cap \tilde{I} = \emptyset$ and then apply the large deviation theorems from the previous section to $\tilde{I}$.

The following lemmas involve parameters $\varepsilon_0, \varepsilon, \eta_0, \delta_0, \eta_\varepsilon, K$, and the intervals $I, \tilde{I}$. The lemmas hold for any values satisfying the constraints below:

(1) Take $0 < \varepsilon_0 < v/8$ and let η_0 denote the large deviation parameter corresponding to ε_0. Choose any $0 < \delta_0 < \eta_0$ and let $0 < \varepsilon < \min\{(\eta_0 - \delta_0)/3, \varepsilon_0/4\}$. Lastly, choose K so that $\tilde{M}^{1/K} < e^{v/2}$ and let η_ε, $\eta_{\frac{\varepsilon}{4}}$ denote the large deviation parameters corresponding to ε and $\frac{\varepsilon}{4}$ respectively. Here $\tilde{M} > 0$ is chosen so that $|P_{[a,b],E,\omega}| \leq \tilde{M}^{b-a+1}$ for all intervals $[a,b]$, $E \in \tilde{I}$, and $\omega \in \Omega$.

(2) Any N's and constants furnished by the lemmas below depend only on the parameters above (i.e. are independent of $l \in \mathbb{Z}$ and ω).

Thus, for the remainder of the paper, $\varepsilon_0, \varepsilon, \eta_0, \delta_0, \eta_\varepsilon$, and K will be treated as fixed parameters chosen in the manner outlined above.

Following [14] and [9], we define subsets of Ω below on which we have will have regularity of the Green's functions. This is the key to the proof of all the localization results. As mentioned in the introduction, the proofs of spectral and dynamical localization given in [14] show that event formed by the complement of the sets below has exponentially small probability. These estimates were exploited in [9] to provide a proof of exponential dynamical localization for the one-dimensional Anderson model. We follow the example set in these two papers with appropriate modifications needed to handle the presence of critical energies and the varying length of words.

Let m_L denote Lebesgue measure on $\mathbb{R}$.

Lemma 3. *If $n \geq 2$ and x is $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-singular, then*

$$(E, \omega) \in {B^-}_{[x-n,x+n],\varepsilon_0} \cup {B^+}_{[x-n,x-1],\varepsilon_0} \cup {B^+}_{[x+1,x+n],\varepsilon_0}.$$

Let R_n and Q_n be the random variables from Lemma 2 and for $l \in \mathbb{Z}$ set

$$\begin{aligned} F^1_{l,n,\varepsilon_0} = \{\omega : &\max\{m_L(B^-_{[l-R_n,l+R_n],\varepsilon_0,\omega}), m_L(B^-_{[l+k-R_n,l+k+R_n],\varepsilon_0,\omega})\} \\ &\leq e^{-(\eta_0-\delta_0)(2n+1)} \text{ for all } k \text{ with } Q_n \leq k < Q_{n+1}\}. \end{aligned}$$

Lemma 4. *There is an N such that for $n > N$ and any $l \in \mathbb{Z}$,*

$$\mathbb{P}[F^1_{l,n,\varepsilon_0}] \geq 1 - m_L(\tilde{I}) e^{-\delta_0(2R_n)}.$$

Proof. With $0 < \varepsilon_0 < 8v$ as above, choose N such that the conclusion of Lemma 2 holds, then for $n > N$,

$$\begin{aligned} m_L \times \mathbb{P}(B^-_{[-R_n,R_n],\varepsilon_0}) &= \mathbb{E}(m_L(B^-_{[-R_n,R_n],\varepsilon_0,\omega})) \\ &= \int_{\mathbb{R}} \mathbb{P}(B^-_{[-R_n,R_n],\varepsilon_0,E})\, dm_L(E) \\ &\leq m_L(\tilde{I}) e^{-\eta_0(2R_n)}. \end{aligned}$$

By the estimate above and Chebyshev's inequality,

$$e^{-(\eta_0-\delta_0)(2R_n)} \mathbb{P}(F^1_{l,n,\varepsilon_0}) \leq m_L(\tilde{I}) e^{-\eta_0(2R_n)}.$$

The result follows by multiplying both sides of the last inequality by $e^{(\eta_0-\delta_0)(2R_n)}$. ∎

Set

$$F^{2+}_{l,n,\varepsilon} = \bigcap_{j=1}^{2n+1} \left\{\omega : E_{j,\omega,[l+n+1,l+3n+1]} \notin B^+_{[y,l+n],\varepsilon,\omega} \text{ for all } y \in \left[l-n, l+n-\frac{n}{K}\right]\right\}, \tag{17}$$

$$F^{2-}_{l,n,\varepsilon} = \bigcap_{j=1}^{2n+1} \left\{\omega : E_{j,\omega,[l+n+1,l+3n+1]} \notin B^{+}_{[l-n,y],\varepsilon,\omega} \text{ for all } y \in \left[l-n-\frac{n}{K}, l+n\right]\right\}, \tag{18}$$

and

$$F^{2}_{l,n,\varepsilon} = F^{2+}_{l,n,\varepsilon} \cap F^{2-}_{l,n,\varepsilon}. \tag{19}$$

Lemma 5. *There is N such that for $n > N$ and any $l \in \mathbb{Z}$,*

$$\mathbb{P}[F^{2}_{l,n,\varepsilon}] \geq 1 - 2(2n+1)^2 e^{-\eta_\varepsilon(\frac{n}{K})}.$$

Proof. With $\varepsilon > 0$ as above, choose N such that the conclusions of Lemma 1 hold.

Fix n, y, and j with $l - n + \frac{n}{K} \leq y \leq l+n$, $1 \leq j \leq 2n+1$ and put

$$C_{n,y,j} = \{\omega : E_{j,[l+n+1,l+3n+1],\omega} \in B^{+}_{[l-n,y]}\}.$$

Since $[l+n+1, l+3n+1] \cap [l-n, y] = \emptyset$, by independence and Lemma 1, for $n > KN$,

$$\mathbb{P}(B^{+}_{[-n,y],E_{j,[l+n+1,l+3n+1],\omega}}) = \mathbb{P}_{[-n,y]}(B^{+}_{[-n,y],E_{j,[l+n+1,l+3n+1],\omega}}) \leq e^{-\eta_0|-n-y|}.$$

For each n, we may write

$$F^{2-}_{l,n,\varepsilon} = \bigcup_{y\in[l-n,l+n],\ |l-n-y|\geq \frac{n}{K},\ 1\leq j\leq 2n+1} C_{n,y,j}.$$

By the above, we have $\mathbb{P}[F^{2-}_{l,n,\varepsilon}] \leq (2n+1)^2 e^{-\eta_0 \frac{n}{K}}$. The result follows by applying the same argument for those y such that $l - n \leq y \leq l + n - \frac{n}{K}$. □

With the same notation as in Lemma 5 and R_n,Q_n as in Lemma 2, set

$$F^{3}_{l,n,\varepsilon} = \bigcap_{j} \left\{\omega : E_{j,\omega,[l+k-R_n,l+k+R_n]} \notin B^{-}_{[l-R_n,l+R_n],\varepsilon,\omega} \text{ for all } k \text{ with } Q_n \leq k < Q_{n+1}\right\}. \tag{20}$$

By the same proof as the previous lemma, we have,

Lemma 6. *There is N such that for $n > N$ and any $l \in \mathbb{Z}$,*

$$\mathbb{P}[F^{3}_{l,n,\varepsilon}] \geq 1 - 2(2R_n)^2 e^{-\eta_\varepsilon(2R_n)}.$$

Remark. Lemma 7 is proved in [14] and used there to give a uniform (and quantitative) Craig-Simon estimate similar to the one in Lemma 8.

Lemma 7. *Let $Q(x)$ be a polynomial of degree $n-1$. Let $x_i = \cos\frac{2\pi(i+\theta)}{n}$, for $0 < \theta < \frac{1}{2}$, $i = 1, 2, \ldots, n$. If $Q(x_i) \leq a^n$, for all i, then $Q(x) \leq Cna^n$, for all $x \in [-1, 1]$, where $C = C(\theta)$ is a constant.*

Set

$$F_{[a,b],\varepsilon} = \{\omega : |P_{[a,b],E,\omega}| \leq e^{(\gamma(E)+4\varepsilon)(b-a+1)} \text{ for all } E \in \tilde{I}\}. \tag{21}$$

Lemma 8. *There are $C > 0$ and N such that for $b - a + 1 > N$,*

$$\mathbb{P}[F_{[a,b],\varepsilon}] \geq 1 - C(b-a+2)e^{-\eta_{\frac{\varepsilon}{4}}(b-a+1)}.$$

Proof. Since $\tilde{I}$ compact and $\tilde{I} \cap D = \emptyset$, $\tilde{I}$ is contained in the union of finitely many compact intervals which all intersect D trivially. Hence, it suffices to prove the result for all E in one of these intervals. So fix one of these intervals, call it I_1. By continuity of γ and compactness of I_1, if $\varepsilon > 0$, there is $\delta > 0$ such that if $E,E^{'} \in I_1$ with $|E - E^{'}| < \delta$, then $|\gamma(E) - \gamma(E^{'})| < \frac{1}{4}\varepsilon$. Divide I_1 into sub-intervals of size δ, denoted by $J_k = [E_k^n, E_{k+1}^n]$ where $k = 1, 2, ...C$. Additionally, let $E_{k,i}^n = E_k^n + (x_i + 1)\frac{\delta}{2}$.

By lemma 1, there is an N such that for $b - a + 1 > N$,

$$\mathbb{P}[\{\omega : |P_{[a,b],E_{k,i}^n,\omega}| \geq e^{(\gamma(E_{k,i}^n)+\frac{1}{4}\varepsilon)(b-a+1)}\}] \leq e^{-\eta_{\frac{\varepsilon}{4}}(b-a+1)}.$$

Put $F_{[a,b],k,\varepsilon} = \bigcup_{i=1}^{n} \{\omega : |P_{[a,b],E_{k,i}^n,\omega}| \geq e^{(\gamma(E_{k,i}^n)+\frac{1}{4}\varepsilon)(b-a+1)}\}$ and $\gamma_k = \inf_{E\in J_k} \gamma(E)$. For $\omega \notin F_{[a,b],k,\varepsilon}$, $|P_{[a,b],E_{k,i}^n,\omega}| \leq e^{(\gamma_k+\frac{1}{2}\varepsilon)(b-a+1)}$. Thus, an application of the above lemma yields, for any such ω, $|P_{[a,b],E,\omega}| \leq e^{(\gamma(E)+\frac{3}{4}\varepsilon)(b-a+1)}$.

We have

$$\mathbb{P}[\bigcup_{k=1}^{C} F_{[a,b],k,\varepsilon}] \geq 1 - C(b - a + 2)e^{\eta_{\frac{\varepsilon}{4}}(b-a+2)}.$$

Thus, since

$$\bigcup_{k=1}^{C} F_{[a,b],k,\varepsilon} \subset F_{[a,b],\varepsilon},$$

the result follows. □

Let $n_{r,\omega}$ denotes the center of localization (if it exists) for $\psi_{\omega,E}$ (i.e. $|\psi_{\omega,E}(n)| \leq |\psi_{\omega,E}(n_{r,\omega})|$). Note that by the results in [11] $n_{r,\omega}$ can be chosen as a measurable function of ω.

Moreover, let $l \in \mathbb{Z}$ and let R_n and Q_n be the scales from Lemma 2.

Set

$$J_{l,n,\varepsilon} = F^1_{l,n,\varepsilon_0} \cap F^2_{l,n,\varepsilon} \cap F^3_{l,n,\varepsilon_0} \cap (\bigcup_{Q_n \leq k < Q_{n+1}} (F_{[l+k-R_n,l+k-1],\varepsilon} \cap F_{[l+k+1,l+k+R_n],\varepsilon}). \quad (22)$$

Lemma 9. *There is N such that if $n > N$, $\omega \in J_{l,n}$, with a generalized eigenfunction $\psi_{\omega,E}$ satisfying either*

(1) *$n_{r,\omega} = l$, or*
(2) *$|\psi_\omega(l)| \geq \frac{1}{2}$*

then if $l + k$ is $(\gamma(E) - 8\varepsilon_0, R_n, E, \omega)$-singular, there exist

$$l - R_n \leq y_1 \leq y_2 \leq l + R_n$$

and $E_j = E_{j,\omega,[l+k-R_n,l+k+R_n]}$ such that

$$|P_{[l-R_n,y_1],E_j,\tilde{\omega}}P_{[y_2,l+R_n],E_j,\omega}| \geq \frac{1}{2m_L(\tilde{I})\sqrt{2R_n+1}} e^{(\gamma(E_j)-\varepsilon)+(\eta_0-\delta_0))(2R_n)}. \quad (23)$$

Remark. Note that y_1 and y_2 depend on ω and l but we do not include this subscript for notational convenience. In particular, this is done when the other terms in expressions involving y_1 or y_2 have the correct subscript and indicate the appropriate dependence.

Proof. Firstly, since $|\psi_\omega(l)| \geq \frac{1}{2}$, we may choose N_1 such that l is $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-singular for $n > N_1$. In the case that $n_{r,\omega} = l$, then there is an N_2 such that l is naturally, $(v - 8\varepsilon_0, n, E, \omega)$-singular for all $n > N_2$. Choose N_3 so that $e^{\frac{v}{2}n} < \operatorname{dist}(I, \tilde{I})$ for $n > N_3$ and finally choose N to be larger than N_1, N_2, N_3 and the N's from lemma 4, lemma 5, and lemma 8.

Suppose that for some $n > N$, $l+k$ is $(\gamma(E)-8\varepsilon_0, n, E, \omega)$-singular. By Lemma 3 and Lemma 8, $E \in B^-_{[l+k-R_n, l+k+R_n], \varepsilon_0, \omega}$. Note that all eigenvalues of $H_{[l+k-R_n, l+k+R_n], \omega}$ belong to $B^-_{[l+k-R_n, l+k+R_n], \varepsilon_0, \omega}$. Since $P_{[l+k-R_n, l+k+R_n], \tilde{E}, \omega}$ is a polynomial in $\tilde{E}$, it follows that $B^-_{[l+k-R_n, l+k+R_n], \omega, \varepsilon_0}$ is contained in the union of sufficiently small intervals centered at the eigenvalues of $H_{\omega, [l+k-R_n, l+k+R_n]}$. Moreover, Lemma 4 gives

$$m(B_{[l+k-R_n, l+k+R_n], \omega, \varepsilon_0}) \leq m_L(\tilde{I}) e^{-(\eta_0 - \delta_0)(2R_n)},$$

so we have the existence of $E_j = E_{j, [l+k-R_n, l+k+R_n], \omega}$ so that $|E - E_j| \leq e^{-(\eta_0 - \delta_0)(2R_n)}$.

Applying the above argument with l in place of $l + k$ yields an eigenvalue $E_i = E_{i, \omega, [l-R_n, l+R_n]}$ such that $E_i \in B^-_{[l-R_n, l+R_n], \varepsilon_0, \omega}$ and $|E - E_i| \leq m_L(\tilde{I}) e^{-(\eta_0 - \delta_0)(2R_n)}$. Hence, $|E_i - E_j| \leq 2m_L(\tilde{I}) e^{-(\eta_0 - \delta_0)(2R_n)}$. By the previous line and the fact that $E_j \notin B^-_{[l-R_n, l+Rn], \varepsilon, \omega}$, we see that $||G_{[l-R_n, l+Rn], E_j, \omega}|| \geq e^{(\eta_0 - \delta_0)(2R_n)}$ so that for some y_1, y_2 with $l - R_n \leq y_1 \leq y_2 \leq l + R_n$,

$$|G_{[l-R_n, l+Rn], E_j, \omega}(y_1, y_2)| \geq \frac{1}{2m_L(\tilde{I})\sqrt{2R_n + 1}} e^{(\eta_0 - \delta_0)(2R_n)}.$$

Additionally, another application of Lemma 6 yields, $|P_{[l-R_n, l+R_n], E_j, \omega}| \geq e^{(\gamma(E_j) - \varepsilon)(2R_n)}$. Thus, by (6) from section 2 we obtain

$$|P_{[l-R_n, y_1], E_j, \omega} P_{[y_2, l+R_n], E_j, \omega}| \geq \frac{1}{2m_L(\tilde{I})\sqrt{2R_n + 1}} e^{(\gamma(E_j - \varepsilon) + (\eta_0 - \delta_0))(2R_n)}.$$

□

Lemma 10. *There is a $\tilde{\eta} > 0$ and N such that $n > N$ implies $\mathbb{P}(J_{l,n,\varepsilon}) \geq 1 - e^{-\tilde{\eta}n}$.*

Proof. Let $\mathcal{A}_1$ denote the possible values taken by $l+k-R_n$ and $\mathcal{A}_2$ denote the possible values taken by $l+k+1$. Using the bounds on R_n and Q_n established in Lemma 2, for a sufficiently large N, the number of elements in $\mathcal{A}_1$ and $\mathcal{A}_2$ are at most $3n\varepsilon$ and $4n\varepsilon$ respectively. Moreover, $\inf \mathcal{A}_2 - \sup \mathcal{A}_1 \geq \frac{n}{2}(\langle L \rangle - 5\varepsilon)$.

Thus, $\displaystyle\bigcup_{j_1 \in \mathcal{A}_1, j_2 \in \mathcal{A}_2} F_{[j_1, j_2], \varepsilon} \subset \bigcup_{Q_n \leq k < Q_{n+1}} (F_{[l+k-R_n, l+k-1], \varepsilon}$ and

$$\mathbb{P}\left[\bigcup_{j_1 \in \mathcal{A}_1, j_2 \in \mathcal{A}_2} F_{[j_1, j_2], \varepsilon}\right] \geq 1 - n^2 e^{-\eta \frac{n}{2}(\langle L \rangle - 5\varepsilon)}.$$

Choose N as in Lemma 9, and note that by Lemmas 4-6 and 8 and the argument above, for $n > N$,

$$\mathbb{P}[J_{l,n,\varepsilon}] \geq 1 - m_L(\tilde{I}) e^{-\delta_0(2R_n)} - 2(2n+1)^2 e^{-\eta_\varepsilon(\frac{n}{K})} - 2(2R_n)^2 e^{-\eta_\varepsilon(2R_n)} - n^2 e^{-\eta \frac{\varepsilon}{4} \frac{n}{2}(\langle L \rangle - 5\varepsilon)}.$$

Again, using the bounds on R_n from Lemma 2, we may choose $\tilde{\eta}$ sufficiently close to 0 and increase N such that for $n > N$, we have

$$m_L(\tilde{I})e^{-\delta_0(2R_n)} + 2(2n+1)^2 e^{-\eta_\varepsilon(\frac{n}{K})} + 2(2R_n)^2 e^{-\eta_\varepsilon(2R_n)} + n^2 e^{-\eta_{\frac{\varepsilon}{4}} \frac{n}{2}(\langle L\rangle - 5\varepsilon)} \leq e^{-\tilde{\eta}n},$$

and the result follows.

□

Lemma 11. *There is N such that for $n > N$, any $\omega \in J_{l,n,\varepsilon}$, any y_1, y_2 with $l - R_n \leq y_1 \leq y_2 \leq l + R_n$ and any $E_j = E_{j,\omega,[l+k-R_n,l+k+R_n]}$,*

$$|P_{[l-R_n,y_1],E_j,\omega}P_{[y_2,l+R_n],E_j,\omega}| \leq e^{(\gamma(E_j)+\varepsilon)(2R_n+1)}.$$

Proof. By choosing N so that Lemma 10 holds for $n > N$, we are led to consider three cases:

(1) $l - R_n + \frac{n}{K} \leq y_1 \leq y_2 \leq l + R_n - \frac{n}{K}$,
(2) $l - R_n + \frac{n}{K} \leq y_1 \leq l + R_n$, while $l + R_n - \frac{n}{K} \leq y_2 \leq l + R_n$,
(3) $l - R_n \leq y_1 \leq l - R_n + \frac{n}{K}$ and $l + R_n - \frac{n}{K} \leq y_2 \leq l + R_n$.

In the first case, Lemma 5 immediately yields:

$$|P_{[l-R_n,y_1],E_j,\omega}P_{[y_2,l+R_n],E_j,\omega}| \leq e^{(\gamma(E_j+\varepsilon)(2R_n+1)}.$$

In the second case, we have $|P_{[y_2,l+R_n],E_j,\omega}| \leq \tilde{M}^{\frac{n}{K}}$, while Lemma 5 gives

$$|P_{[l-R_n,y_1],E_j,\omega}| \leq e^{(\gamma(E_j)+\varepsilon)(y_1-l+R_n))}.$$

By our choice of K, $\tilde{M}^{\frac{1}{K}} \leq e^{\frac{v}{2}} \leq e^{(\gamma(E_j)+\varepsilon)}$, so we again obtain the desired result.

Finally, in the third case, $|P_{[l-R_n,y_1],E_j,\omega}P_{[y_2,l+R_n],E_j,\omega}| \leq \tilde{M}^{\frac{2n}{K}} \leq e^{(\gamma(E_j+\varepsilon)(2R_n+1)}$ (again by our choice of K). □

7. Spectral Localization

Theorem 4. *There is N such that if $n > N$, $Q_n \leq k < Q_{n+1}$, and $\omega \in J_{l,n,\varepsilon}$, with a generalized eigenfunction $\psi_{\omega,E}$ satisfying either*

(1) $n_{r,\omega} = l$, *or*
(2) $|\psi_\omega(l)| \geq \frac{1}{2}$

then $l + k$ is $(\gamma(E) - 8\varepsilon_0, R_n, E, \omega)$-regular.

Proof. Choose N so that Lemma 9 and Lemma 11 hold and

$$\frac{1}{2m_L(\tilde{I})\sqrt{2R_n}} e^{(\gamma(E_j)-\varepsilon+\eta_0-\delta_0)(2R_n)} > e^{(\gamma(E_j)+\varepsilon)(2R_n)}$$

for $n > N$. This can be done since $\eta_0 - \delta_0 - \varepsilon > \varepsilon$.

For $n > N$, we obtain the conclusion of the theorem. For if $l + k$ was not $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-regular, then by Lemma 9

$$|P_{[l+k-R_n,y_1],E_j,\omega}P_{[y_2,l+k+R_n],E_j,\omega}| \geq e^{(\gamma(E_j-\varepsilon)+(\eta_0-\delta_0))(2R_n+1)}.$$

On the other hand, by Lemma 11, we have

$$|P_{[l+k-R_n,y_1],E_j,\omega}P_{[y_2,l+k+R_n],E_j,\omega}| \leq e^{(\gamma(E_j+\varepsilon)(2R_n+1)}.$$

Our choice of N in the first line of the proof yields a contradiction and completes the argument.

□

We are now ready to give the proof of Theorem 1.

Again, R_n and Q_n are the scales from Lemma 2.

Proof. By Lemma 10, $\mathbb{P}[J_{0,n,\varepsilon}$ eventually $] = 1$. Thus, we obtain $\tilde{\Omega}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ and for $\omega \in \tilde{\Omega}$, there is $N(\omega)$ such that for $n > N(\omega)$, $\omega \in J_{0,n,\varepsilon}$.

Since the spectral measures are supported by the set of generalized eigenvalues (e.g. [12]), it suffices to show for all $\omega \in \tilde{\Omega}$, every generalized eigenfunction with generalized eigenvalue $E \in I$ is in fact an $l^2(\mathbb{Z})$ eigenfunction which decays exponentially.

Fix an ω in $\tilde{\Omega}$ and let $\psi = \psi_{\omega,E}$ be a generalized eigenfunction for H_ω with generalized eigenvalue E. Using (7) from section 4 and the bounds established on R_n and Q_n from Lemma 2, it suffices to show that there is $N(\omega)$ such that for $n > N(\omega)$, if $Q_n \leq k < Q_{n+1}$, then k is $(\gamma(E) - 8\varepsilon_0, R_n, E, \omega)$-regular. We may assume $\psi(0) \neq 0$, and moreover, by rescaling ψ, $|\psi(0)| \geq \frac{1}{2}$. Choose N so that the conclusions of theorem 3 hold. Additionally, we may choose $N(\omega)$ such that for $n > N(\omega)$, $\omega \in J_{0,n,\varepsilon}$. For $n > \max\{N, N(\omega)\}$, the hypotheses of Theorem 4 are met, and hence k is $(\gamma(E) - 8\varepsilon_0, R_n, E, \omega)$-regular.

□

8. Exponential Dynamical Localization

The strategy used in this section follows [9] with appropriate modifications needed to deal with the fact that single-step transfer matrices were not used in the large deviation estimates. In particular, the randomness in the conclusion of Theorem 4 will need to be accounted for.

The following lemma was shown in [15] and we state a version below suitable for obtaining EDL on the interval I.

Let $u_{k,\omega}$ denote an orthonormal basis of eigenvectors for $Ran(P_I(H_\omega))$.

Lemma 12. [15] *Suppose there is $\tilde{C} > 0$ and $\tilde{\gamma} > 0$ such that for any s, $l \in \mathbb{Z}$,*

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}(s)|^2\right] \leq \tilde{C} e^{-\tilde{\gamma}|s-l|}.$$

Then there are $C > 0$ and $\gamma > 0$ such that for any $p, q \in \mathbb{Z}$,

$$\mathbb{E}[\sup_{t\in\mathbb{R}} |\langle \delta_p, P_I(H_\omega) e^{itH_\omega} \delta_q \rangle|] \leq C(|p-q|+1)e^{-\gamma|p-q|}.$$

By Lemma 12, Theorem 2 follows from Theorem 5.

Theorem 5. *There is $\tilde{C} > 0$ and $\tilde{\gamma} > 0$ such that for any s,l $\in \mathbb{Z}$,*

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}(s)|^2\right] \leq \tilde{C} e^{-\tilde{\gamma}|s-l|}.$$

Proof. Fix $s, l \in \mathbb{Z}$ and fix $0 < c < v - 8\varepsilon$. Choose N so that Theorem 4 holds and for $n > N$

$$e^{(v-8\varepsilon)3m} \leq e^{(v-8\varepsilon-c)n}. \tag{24}$$

There are two cases to consider:

(1) $s - l - 1 > 2(N+1)(\langle L\rangle - \varepsilon) + 2m$,
(2) $s - l - 1 \leq 2(N+1)(\langle L\rangle - \varepsilon) + 2m$.

In the first case, we assume $s - l - 1$ is even as the proof is similar when it is odd. Now suppose $n_{r,\omega} = l$ and $Q_j \leq s < Q_{j+1}$, and $\omega \in J_{l,\frac{j}{2},\varepsilon}$, then using Theorem 4 and (7) (from section 2),

$$|u_{r,\omega}(s)| \leq 2|u_{r,\omega}(l)|e^{-(\gamma(E_{r,\omega})-8\varepsilon)R_j} \tag{25}$$

$$\leq 2|u_{r,\omega}(l)|e^{-(v-8\varepsilon)R_j}. \tag{26}$$

By orthonormality,

$$\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2 \leq \sum_{n_{r,\omega}=l} |u_{r,\omega}(l)|^2 e^{-(v-8\varepsilon)2R_j} \tag{27}$$

$$\leq \sum_{n_{r,\omega}=l} e^{-(v-8\varepsilon)2R_j}. \tag{28}$$

We need to replace the randomness in the exponent above with an estimate that depends only on the point s. Note that $j > 2N$ and if $\mathcal{A}$ denotes the set of $j \in \mathbb{Z}$ so that $Q_j \leq s < Q_{j+1}$, then $\mathcal{A}$ is finite.

Using the bounds on R_n and Q_n established in Lemma 2, we have:

(1) $R_j \geq j(\langle L\rangle - \varepsilon) - m$
(2) $Q_j \geq j(\langle L\rangle - \varepsilon) + 2m$.

Thus, since $s = l + k$ with $Q_j \leq k < Q_{j+1}$, by our choice of c and N,

$$\sum_{n_{r,\omega}=l} e^{-(v-8\varepsilon)2R_j} \leq \sum_{n_{r,\omega}=l} e^{-(v-8\varepsilon)j(\langle L\rangle-\varepsilon)-m}$$

$$\leq \sum_{n_{r,\omega}=l} e^{-c(s-l-1)}.$$

Finally, and letting $J = \bigcup_{j\in\mathcal{A}} J_{l,\frac{j}{2},\varepsilon}$ and using the estimate provided by Theorem 4 on $\mathbb{P}[J_{l,\frac{j}{2},\varepsilon}]$,

$$\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{k,\omega}|^2\right] = \mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}|^2 \chi_J + \sum_{n_{r,\omega}=l} |u_{r,\omega}|^2 \chi_{J^c}\right] \tag{29}$$

$$\leq Ce^{-c|s-l-1|} + Ce^{-\tilde{\eta}|s-l-1|}. \tag{30}$$

In the second case, again by orthonormality, $\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2\right] \leq 1.$

By letting $\tilde{\gamma} = \min\{c, \tilde{\eta}\}$ and choosing $\tilde{C} > 0$ to be sufficiently large, we obtain: $\mathbb{E}\left[\sum_{n_{r,\omega}=l} |u_{r,\omega}(s)|^2\right] \le \tilde{C}e^{-\tilde{\gamma}|s-l|}$, as desired.

□

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