Symplectic Analysis of Differential Forms

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ABSTRACT. We give an expository overview of the cohomologies of differential forms on symplectic manifolds found in recent years. The linear symplectic operators that the cohomologies are associated with suggest a new approach towards symplectic analysis.

1. Introduction

A symplectic manifold (M^{2n}, ω) is a manifold of even dimensions, $2n$, equipped with a symplectic structure, that is, a differential two-form ω which is both non-degenerate (i.e. $\omega^n > 0$) and d-closed.

A noted property of symplectic manifolds is the Darboux Theorem, which states that locally around any point on the symplectic manifold, the symplectic structure, ω, is diffeomorphic to the standard symplectic structure on $\mathbb{R}^{2n}$,

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n} \,.$$

Since, locally, a symplectic manifold looks just like $\mathbb{R}^{2n}$, symplectic invariants are global in nature. Many symplectic invariants studied today are based on pseudo-holomorphic curves, for instance, Floer homology theories and Gromov-Witten invariants.

This paper will describe a search for symplectic invariants from the perspective of standard geometric analysis. That is, we will be seeking invariants that are discerned from the study of solutions of partial differential equations on manifolds. It is an interesting to consider whether the powerful analytic tools of geometric analysis (see for example [**24, 10**]) can have direct, useful applications in symplectic geometry. To make use of them, one should have a good intuition of what types of equations to analyze. The pdes in geometric analysis can be linear or non-linear and involve functions or sometimes also tensors or differential forms. Certainly the simplest-type of pdes to analyze are the linear ones on functions involving the standard Laplacian operator, Δ. For instance,

- the Laplace equation ($\Delta\phi = 0$);
- the heat equation ($\partial_t \phi = \Delta\phi$);
- eigenvalues of the Laplacian ($\Delta\phi = \lambda\phi$).

Methods for studying linear equations are often useful for non-linear analysis as well. It is thus the focus of this review to consider linear partial differential equations on symplectic manifolds. We will first describe the search for Laplacian-type operators that are dependent on the symplectic structure.

1.1. Laplacians for Symplectic Manifolds? For any smooth manifold, the standard Laplacian takes the form

$$\Delta = dd^* + d^*d$$

where d^* is the adjoint operator of d and is defined with respect to some metric. Just considering the space of solutions, we know that the $\dim\ker\Delta$ gives us the Betti numbers, a basic topological invariant. So a naive question is the following: *Can we modify $\Delta \to \Delta_\omega$ to incorporate the symplectic structure ω in some way?* If Δ_ω has sufficiently good properties, then $\dim\ker\Delta_\omega$ could perhaps result in useful symplectic invariants.

Such an idea has a long history and dates back to the work of Ehresmann and Libermann from the late 1940s [**5, 11**]. They made the following observation: The usual Laplacian on a Riemannian manifold (M, g) involves the symmetric, metric two-tensor g, while a symplectic manifold (M, ω) involves the two-form ω instead. Why not just replace $g \to \omega$ in Δ which would give a "Laplacian" operator dependent on ω?

Straightforwardly, in $\Delta = dd^* + d^*d$, the metric g appears in the definition of the adjoint operator d^*, defined by the standard inner product:

$$(A, A') = \int_M A \wedge *A' = \int_M g^{i_1j_1}g^{i_2j_2}\dots g^{i_kj_k}A_{i_1i_2\dots i_k}A'_{j_1j_2\dots j_k} \tag{1.1}$$

for $A, A' \in \Omega^k(M)$, the space of differential k-forms. With respect to this inner product, the adjoint d^* satisfies the condition, $(dA, A') = (A, d^*A')$. Therefore, to define a symplectic adjoint, we would just replace g with ω in (1.1). We end up with the following product

$$(A, A')_\omega = \int_M (\omega^{-1})^{i_1j_1}(\omega^{-1})^{i_2j_2}\dots(\omega^{-1})^{i_kj_k}A_{i_1i_2\dots i_k}A'_{j_1j_2\dots j_k} \tag{1.2}$$

This suggests defining the "symplectic adjoint" of d, which we will label by d^Λ, to satisfy $(dA, A')_\omega = (A, d^\Lambda A')_\omega$. (In the literature, such a symplectic adjoint is also commonly denoted by δ.)

Let us briefly take stock of whether this simple idea of replacing g by ω makes sense. For one, since ω is skew-symmetric, i.e. $\omega_{ij} = -\omega_{ji}$, this means that $(A, A')_\omega = -(A', A)_\omega$ when A and A' are of odd degree. The product hence can not be a standard inner product as it is neither symmetric nor positive. But regardless, the symplectic adjoint, d^Λ, is clearly a linear operator that (i) depends on ω and (ii) squares to zero, i.e. $d^\Lambda d^\Lambda = 0$.

Proceeding onward, by replacing d^* with d^Λ, we have arrived at a suggested symplectic Laplacian

$$\Delta = dd^* + d^*d \qquad \longrightarrow \qquad \Delta_\omega = dd^\Lambda + d^\Lambda d$$

As Ehresmann and Libermann noted in their original work, it is not hard to show $d^\Lambda d = -dd^\Lambda$, and this would immediately imply that $\Delta_\omega = 0$. So this first early attempt of a symplectic Laplacian unfortunately did not have an auspicious start.

However, the notion of a linear symplectic adjoint operator still has its appeal. So instead of dealing with the trivial Laplacian Δ_ω, follow-up works focused on analyzing symplectic "harmonic forms" which satisfy

$$d\,A = 0\,, \qquad d^\Lambda A = 0\,. \tag{1.3}$$

Harmonic forms are of course important objects in Hodge theory. Recall for instance that there exists a unique harmonic form in each de Rham cohomology class. Under the topic of what is sometimes called "symplectic Hodge theory," the question of interest is whether there exist solutions to the above system (1.3) in each de Rham cohomology class. In fact, there is no uniqueness or existence property in general, except for a certain of class of symplectic manifolds that satisfy the hard Lefschetz condition. (See [**3, 12, 23, 13, 8**] for work in this direction.)

1.2. Symplectic Equations from Physics. So if not the Laplacian Δ_ω, then what other equations should we study? Historically, physics has been a good source for equations that shed light to interesting mathematics. For example, the wave equation, the heat equation, Ricci flow equations, all have physical applications or originated from physics. This leads to the following question for symplectic manifolds: *Are there any physical partial differential equations that involve the symplectic structure?*

Of course, there are a number of physical equations defined on Kähler manifolds, that have been widely studied in recent years, e.g. Kähler-Einstein equations and Hermitian Yang-Mills equations. A particular useful system of equations that we will consider here is the one written down about a decade ago within the context of Type IIA string theory [**7, 17, 21**]. This system of equations is defined on a six-dimensional symplectic manifold M^6, that need not be Kähler, but must have an $SU(3)$ structure. (A higher dimensional analogue of the system can be found in [**9**].) To define the system, let (M^6, ω, Ω, g) where Ω is a a non-vanishing $(3,0)$-form which defines an almost complex structure on M^6. Also, (ω, Ω) satisfy the $SU(3)$ structure equations:

$$\omega\wedge\Omega = 0\,, \qquad \sqrt{-1}\,\Omega\wedge\overline{\Omega} = 8e^{2f}\,\frac{\omega^3}{3!}\,. \tag{1.4}$$

The differential system then consists of the following equations:

$$d\omega = 0$$

$$d\,\mathrm{Re}\;\Omega = 0\,, \tag{1.5}$$

$$(dd^\Lambda)^*(e^{-f}\,\mathrm{Re}\,\Omega) = *\rho_L\,, \tag{1.6}$$

where ρ_L is the Poincaré dual of some configuration of wrapped special Lagrangian submanifolds on M. We note that if in (1.5), the condition is instead $d\Omega = 0$, which implies that the compatible almost complex structure on M^6 given by Ω is integrable, then the system reduces to requiring that ω be compatible to the Kähler Calabi-Yau metric.

Upon closer examination, the above symplectic system, in particular equations (1.5) and (1.6), seem to share very close resemblance to Maxwell's equations. Recall that the Maxwell equations are the fundamental equations of electricity and magnetism and can be expressed simply in terms of a curvature two-form F. On a four-dimensional Riemannian manifold (M^4, g), the Maxwell equations take the form

$$d\,F = 0$$
$$d^*F = *\rho_e$$

where ρ_e is the Poincaré dual of some configuration of electric charges. Heuristically, and neglecting the factor e^{-f} in (1.5), the Type IIA symplectic system and the Maxwell equations are notably similar if we make the following identifications between the two:

$$\begin{array}{ccc} d & \longleftrightarrow & d \\ (dd^\Lambda)^* & \longleftrightarrow & d^* \\ \text{Re }\Omega & \longleftrightarrow & F \\ \rho_L & \longleftrightarrow & \rho_e \end{array} \tag{1.7}$$

Such a similarity is especially noteworthy given the close relation between Maxwell's equations and de Rham harmonic forms. For consider the solution space of the Maxwell system for a fixed charge configuration. This can be analyzed by applying the variation $F \longrightarrow F + \delta F$ while keeping the charged configuration fixed, i.e. $\delta\rho_e = 0$. This results in the following conditions:

$$d\,(\delta F) = 0\,, \qquad d^*(\delta F) = 0\,, \tag{1.8}$$

which are together the harmonicity condition associated with the degree-two de Rham cohomology

$$H^2(M^4) = \frac{\ker d \cap \Omega^2(M^4)}{\text{im }d \cap \Omega^2(M^4)}\,.$$

Hence, $\dim H^2(M^4)$, i.e. the second Betti number, gives the dimension of the solution space of F for a fixed charge configuration.

In analogy with the Maxwell equations, Tseng-Yau [**21**] observed that a subset of infinitesimal deformations of $\delta\text{Re}\,\Omega$ in the Type IIA symplectic system with fixed special Lagrangian configuration (i.e. $\delta\rho_L = 0$) satisfy the the following conditions

$$d\,(\delta\text{Re }\Omega) = 0\,, \qquad (dd^\Lambda)^*(\delta\text{Re}\,\Omega) = 0\,. \tag{1.9}$$

These equations no doubt resemble those of (1.8) especially under the identifications of (1.7). But could they also be suggestive that the $\delta\text{Re}\,\Omega$ deformations are also a type of harmonic form in some sense? By this, we mean that δF satisfying (1.8) are also the zero solutions of the de Rham Laplacian $\Delta = dd^* + d^*d$. But if (1.9) also represents a harmonic condition, then we may expect that the corresponding Laplacian (which should be self-adjoint) is given by

$$\Delta_{d+d^\Lambda} = (dd^\Lambda)(dd^\Lambda)^* + d^*d \tag{1.10}$$

This in fact is an interesting *symplectic Laplacian.* Clearly, the above fourth-order operator Δ_{d+d^Λ} is dependent on the symplectic structure ω. It is also by construction self-adjoint with respect to the standard inner product on forms of (1.1). But most importantly, we can study the symbol of the operator Δ_{d+d^Λ}. And in fact, it can be checked that the symbol is a positive operator on $[\Lambda^3T^*M]_0$. (The subindex '0' here denotes the subspace of the cotangent space of three-forms that are annihilated by the wedge product of ω. This is the relevant space to consider as we should require that $\delta\text{Re}\,\Omega$ satisfies the first algebraic condition in (1.4).) With

Riemannian (M, g)	Complex (M^{2n}, J, g)	Symplectic (M^{2n}, ω, g)
Ω^k	$\Omega^k = \oplus\, A^{p,q}$	$\Omega^k = \oplus\, \omega^r \wedge P^s$
d	$d = \partial + \overline{\partial}$	$d = \partial_+ + \omega \wedge \partial_-$
$\Delta = dd^* + d^*d$	$\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$	$\Delta_+, \Delta_-, \Delta_{++}, \Delta_{--}$

TABLE 1. A comparison of Riemannian, complex and symplectic analysis.

this symbol being positive, it is possible to further show that the solution space $\Delta_{d+d^\Lambda}\, \delta \mathrm{Re}\, \Omega = 0$ is finite-dimensional.

We can certainly take our analogy with the Maxwell equations further. Since Δ is associated with the de Rham cohomology, is the symplectic Laplacian Δ_{d+d^Λ} also associated to some cohomology? The cohomology suggested by the analogy is

$$PH(M) = \frac{\ker d \cap \Omega^3_0}{\mathrm{im}\ dd^\Lambda \cap \Omega^3_0}\,,$$

and as we shall see shortly, $PH(M)$ is in fact a finite-dimensional cohomology associated with the Laplacian Δ_{d+d^Λ} [**19, 22**].

In the above, what we have is the beginning of a novel symplectic analysis of differential forms: *We have found a distinctive Laplacian and associated cohomology, both of which are explicitly dependent on ω.*

At this point, there are many natural questions to follow up on. For instance,

- Is there a fundamental framework underlying the symplectic Laplacian and cohomology? In particular, are they related to a differential graded algebra?
- Can we use symplectic Laplacians to obtain invariants for symplectic manifolds?
- Are there other symplectic Laplacians and cohomologies of differential forms that we can write down?

In fact, we now know that there are a number of symplectic elliptic Laplacians and associated cohomologies. And the analysis of them share many similarities with standard Riemannian and complex analysis. In the following, we will describe these cohomologies and Laplacians and some of their properties. More details can be found in a series of paper [**19, 20, 18, 16, 22**].

2. Analysis of Forms on Symplectic Manifolds

Similar to the analysis on complex manifolds, one should expect that analysis of differential forms on symplectic manifolds should have its own special features. To motivate them, we will describe some of the peculiarities of symplectic analysis in analogy with Riemannian and complex analysis. As an example, see Table 1.

2.1. Form Decomposition in the Presence of ω. In complex geometry, the complex structure ensures the existence of holomorphic coordinates. Hence, a differential k-form can be standardly decomposed into holomorphic/anti-holomorphic (p, q)-components:

$$\Omega^k = \oplus_{k=p+q} A^{p,q} .$$

On a symplectic manifold, the symplectic structure, ω, is a differential two-form. So it is natural to decompose or express forms in terms of powers of ω. For example, for a generic six-form, we would write

$$A_6 = B_6 + \omega \wedge B_4 + \omega^2 \wedge B_2 + \omega^3 \wedge B_0 ,$$

where $\{B_6, B_4, B_2, B_0\}$ are *primitive* forms. Here, primitive forms are those forms which are in some sense "orthogonal" to ω, i.e. you can not extract out a symplectic form from them. Specifically, we say that a form is primitive, $B_s \in P^s(M)$ for $s = 0, 1, 2, \ldots, n$ if it vanishes under the contraction with ω^{-1}. In the symplectic context then, we have the decomposition

$$\Omega^k = \oplus_{k=2r+s} \omega^r \wedge P^s . \tag{2.1}$$

Writing forms in this way, as a polynomial of ω, is called the Lefschetz decomposition of differential forms.

Now, such a decomposition in powers of ω suggests a natural filtration of forms. For any k-form $A_k \in \Omega^k$, we can apply a projection that keeps only terms up to the ω^p order. In particular, for a differential k-form that is Lefschetz decomposed, we define the projection operator Π^p for $p = 0, 1, \ldots, n$ as follows:

$$\begin{aligned}
A_k &= B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \omega^3 \wedge B_{k-6} + \ldots , \\
\Pi^0 A_k &= B_k , \\
\Pi^1 A_k &= B_k + \omega \wedge B_{k-2} , \\
\Pi^2 A_k &= B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} , \\
&\vdots \\
\Pi^p A_k &= B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \ldots + \omega^p \wedge B_{k-2p} .
\end{aligned}$$

Such a projected space $F^p\Omega^k(M) := \Pi^p\Omega^k(M) \subset \Omega^k(M)$ is called the space of *p-filtered* forms [**18**]. The index p in $F^p\Omega^k(M) = \Pi^p\Omega^k(M)$ which ranges from 0 to n parametrizes a natural filtration

$$\mathcal{P}^* = F^0\Omega^* \subset \ldots \subset F^p\Omega^* \subset F^{p+1}\Omega^* \subset \ldots \subset F^n\Omega^* = \Omega^*$$

that interpolates from the space of primitive forms $\mathcal{P}$ to the space of all differential forms Ω.

2.2. First-Order Symplectic Differential Operators. We turn now to differential operators. The basic building blocks of differential operators on complex manifolds are the Dolbeault operators, $(\partial, \overline{\partial})$. They arise naturally when applying the exterior differential on a (p, q)-form

$$dA^{p,q} \longrightarrow A^{p+1,q} \oplus A^{p,q+1}$$

Projecting the action of d onto each of the two components on the right gives the decomposition $d = \partial + \overline{\partial}$.

For symplectic geometry, we can also find natural first-order differential operators in a similar way. With the Lefschetz decomposition (2.1), we let the exterior derivative act on an element of $\omega^r \wedge \mathcal{P}^s$. Since ω is symplectic (i.e. $d\omega = 0$), we have

$$d(w^r \wedge B_s) = \omega^r \wedge d(B_s)\,.$$

Acting on primitive forms, it can be shown that Lefschetz decomposition of a primitive form acted upon by an exterior derivative only has two components

$$dB_s = B_{s+1} + \omega \wedge B_{s-1}\,.$$

Thus, projecting onto the two components on the right, we find two natural linear operators [**20**]

$$\partial_+ : \omega^r \wedge \mathcal{P}^s \to \omega^r \mathcal{P}^{s+1}\,, \qquad \partial_- : \omega^r \wedge \mathcal{P}^s \to \omega^r \wedge \mathcal{P}^{s-1}\,,$$

resulting in the decomposition of the exterior derivative

$$d = \partial_+ + \omega \wedge \partial_-\,. \tag{2.2}$$

It further follows from $d^2 = 0$ that

$$(\partial_+)^2 = (\partial_-)^2 = 0\,, \qquad \omega \wedge \partial_+\partial_- = -\omega \wedge \partial_-\partial_+\,.$$

Hence, (∂_+, ∂_-) share similar properties with those of the Dolbeault operators $(\partial, \overline{\partial})$.

In fact, (∂_+, ∂_-) can be used to build an elliptic differential complex:

$$\begin{array}{ccccccccccccc}
0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \mathcal{P}^1 & \xrightarrow{\partial_+} & \dots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\
 & & & & & & & & & & \downarrow{\scriptstyle \partial_+\partial_-} \\
0 & \xleftarrow{\partial_-} & \mathcal{P}^0 & \xleftarrow{\partial_-} & \mathcal{P}^1 & \xleftarrow{\partial_-} & \dots & \xleftarrow{\partial_-} & \mathcal{P}^{n-1} & \xleftarrow{\partial_-} & \mathcal{P}^n
\end{array} \tag{2.3}$$

The four-dimensional ($n = 2$) version of this primitive complex was written down by R.T. Smith [**15**] back in 1976. Tseng-Yau [**19**] and also M. Eastwood and his collaborators [**2, 4**] independently wrote down the above complex more recently.

It turns out that this elliptic complex for primitive form is only the special $p = 0$ case of elliptic complexes for more general p-filtered forms $F^p\Omega$ for $p = 0, 1, \dots, n$ as found by Tsai-Tseng-Yau [**18**]:

$$\begin{array}{ccccccccccccc}
0 & \xrightarrow{d_+} & F^p\Omega^0 & \xrightarrow{d_+} & F^p\Omega^1 & \xrightarrow{d_+} & \dots & \xrightarrow{d_+} & F^p\Omega^{n+p-1} & \xrightarrow{d_+} & F^p\Omega^{n+p} \\
 & & & & & & & & & & \downarrow{\scriptstyle \partial_+\partial_-} \\
0 & \xleftarrow{d_-} & F^p\Omega^0 & \xleftarrow{d_-} & F^p\Omega^1 & \xleftarrow{d_-} & \dots & \xleftarrow{d_-} & F^p\Omega^{n+p-1} & \xleftarrow{d_-} & F^p\Omega^{n+p}
\end{array}$$

where (d_+, d_-) are first-order linear operators defined such that $d_\pm : F^p\Omega^k \longrightarrow F^p\Omega^{k\pm1}$ and also $(d_+)^2 = (d_-)^2 = 0$. They are in some sense generalizations of (∂_+, ∂_-) since $(d_+, d_-) = (\partial_+, \partial_-)$ when acting on primitive forms. (See [**18**] for more details.)

For simplicity, we will in the following, focus mostly on the $p = 0$ primitive case.

2.3. Symplectic Laplacians and Cohomologies. Associated with the p-filtered elliptic complex above are four distinct finite-dimensional cohomologies and Laplacians for p-filtered forms. Specifically, for the elliptic complex of (2.3), we have the following $p = 0$ primitive cohomologies

$$PH(M) = \{PH^0_+, PH^1_+, \ldots, PH^n_+, PH^n_-, \ldots, PH^1_-, PH^0_-\}$$

where

$$PH^k_+(M) = \frac{\ker \partial_+ \cap \mathcal{P}^k}{\text{im } \partial_+ \cap \mathcal{P}^k}\,, \qquad PH^k_-(M) = \frac{\ker \partial_- \cap \mathcal{P}^k}{\text{im } \partial_- \cap \mathcal{P}^k}\,, \tag{2.4}$$

for $k = 0, 1, \ldots, n-1$, and,

$$PH^n_+(M) = \frac{\ker \partial_+\partial_- \cap \mathcal{P}^n}{\text{im } \partial_+ \cap \mathcal{P}^n}\,, \qquad PH^n_-(M) = \frac{\ker \partial_- \cap \mathcal{P}^n}{\text{im } \partial_+\partial_- \cap \mathcal{P}^n}\,. \tag{2.5}$$

Certainly, their dimensions represent new symplectic invariants especially for non-Kähler symplectic manifolds. Furthermore, they provide us with the following symplectic *elliptic* Laplacians associated with the cohomologies:

$$\Delta_+ = \partial_+^*\partial_+ + \partial_+\partial_+^*\,, \qquad \Delta_- = \partial_-^*\partial_- + \partial_-\partial_-^*\,, \tag{2.6}$$

$$\Delta_{++} = (\partial_+\partial_-)^*\partial_+\partial_- + (\partial_+\partial_+^*)^2\,, \qquad \Delta_{--} = \partial_+\partial_-(\partial_+\partial_-)^* + (\partial_-^*\partial_-)^2\,. \tag{2.7}$$

The two second-order Laplacians, Δ_+ and Δ_-, look very similar to the standard De Rham Laplacian. In fact, one may quickly obtain them by replacing (d, d^*) in the standard Laplacian with $(\partial_+, \partial_+^*)$ and $(\partial_-, \partial_-^*)$, respectively. The fourth-order ones Δ_{++} and Δ_{--} are non-standard. They are associated with the middle part of the elliptic complex in (2.3). For these fourth-order Laplacians, the squaring of the second term, $(\partial_+\partial_+^*)$ and $(\partial_-^*\partial_-)$, ensures that the principal symbols are positive, and not just the symbol as in the case of the Laplacian in (1.10).

3. Analysis on Symplectic Manifolds with Boundary

A basic setting in considering analysis is to study the local properties of the operators and the differential forms which they act upon within some open or closed region. If the region has a boundary, it would be most advantageous to impose some boundary conditions on the differential forms. A good boundary condition can be very useful as it can for instance ensures that no boundary integral contributions arise when doing integration by parts. Thus, a fundamental question for these novel symplectic operators $(\partial_+, \partial_-, \partial_+\partial_-)$ is whether there are natural boundary conditions that we can impose on differential forms when working in regions with boundary?

In [**22**], we studied this issue of boundary conditions within the general setting of symplectic manifolds with a smooth boundary, i.e. $\partial M \neq 0$. Symplectic manifolds with boundary appear in many different contexts in symplectic geometry, for instance, in the study of symplectic filling and symplectic cobordisms of contact manifolds. Indeed, we found symplectic analogues of the standard Dirichlet and Neumann boundary conditions commonly used for the d and $\overline{\partial}$ operators. Moreover, natural boundary conditions for the second-order operators $\partial_+\partial_-$ were also found. These symplectic boundary conditions are very useful; for instance, they are needed to establish Hodge decomposition of forms and also define "relative" cohomology of forms with respect to the boundary ∂M. We will describe these boundary conditions below.

3.1. Standard Dirichlet and Neumann Boundary Condition. To begin, let us first review the usual Dirichlet and Neumann boundary conditions on differential forms. In our discussion of boundary conditions, we will make extensive use of the local boundary defining function which we denote by ρ. Specifically, $\rho = 0$ on ∂M and $d\rho \neq 0$ is dual to the inward normal vector $\vec{n}$.

The well-known Dirichlet (D) boundary condition on a function $f \in \Omega^0$ is the requirement that $f\,|_{\partial M} = 0$. But more generally, the Dirichlet boundary condition for a k-form $\eta \in \Omega^k$ can be expressed simply in terms of the symbol of the exterior derivative d, i.e. $\sigma_d(d\rho)\,\eta\,|_{\partial M} = 0$. Alternatively, this implies

$$(D) \quad \sigma_d(d\rho)\,\eta\,|_{\partial M} = d\rho \wedge \eta\,|_{\partial M} = d(\rho\,\eta)\,|_{\partial M} = 0\,. \tag{3.1}$$

Notice that the Dirichlet condition on forms requires that a form with only components in the tangential directions of ∂M vanishes on the boundary. The Neumann (N) boundary condition can be defined in a similar way using the adjoint operator d^*:

$$(N) \quad \sigma_{d^*}(d\rho)\,\eta\,|_{\partial M} = \iota_{\vec{n}}\,\eta\,|_{\partial M} = d^*(\rho\,\eta)\,|_{\partial M} = 0\,. \tag{3.2}$$

Let us mention here several desirable properties of the Dirichlet and Neumann boundary conditions:

(1) For both Dirichlet and Neumann boundary conditions, the boundary contributions vanishes:

$$(d\eta, \xi) - (\eta, d^*\xi) = \int_{\partial M} \langle \sigma_d(d\rho)\,\eta, \xi\rangle\, dS = -\int_{\partial M} \langle \eta, \sigma_{d^*}(d\rho)\,\xi\rangle\, dS\,.$$

(2) The Dirichlet and Neumann boundary conditions are preserved under the operation of d and d^*, respectively:

$$\eta \in D \Longrightarrow d\eta \in D\,,$$
$$\eta \in N \Longrightarrow d^*\eta \in N\,.$$

(3) The boundary conditions play an essential role in establishing the Hodge decompositions of forms on manifolds with boundary [**6, 14**]. The Hodge decompositions in turn implies the finite-dimensionality of the absolute and relative cohomologies on M:

$$H^k(M) = \frac{\ker d \cap \Omega^k}{\mathrm{im}\ d \cap \Omega^k} \cong \mathcal{H}^k_N(M) \qquad (absolute)$$
$$H^k(M, \partial M) = \frac{\ker d \cap \Omega^k_D}{\mathrm{im}\ d \cap \Omega^k_D} \cong \mathcal{H}^k_D(M) \qquad (relative)$$

where Ω^k_D denotes the space of k-forms with Dirichlet boundary condition and $\mathcal{H}^k_N$ and $\mathcal{H}^k_D(M)$ are the space of harmonic fields of degree k that satisfy Neumann and Dirichlet boundary conditions, respectively. Recall also that there is a natural pairing duality between the absolute and relative cohomologies on manifolds with boundary:

$$H^k(M) \cong H^{2n-k}(M, \partial M)\,.$$

3.2. Symplectic Boundary Conditions on Forms. Are there boundary conditions for the symplectic operators $(\partial_+, \partial_-, \partial_+\partial_-)$ with similar properties as the Dirichlet and Neumann boundary conditions for the exterior derivative d?

It turns out there are. As we saw in (3.1) and (3.2), Dirichlet and Neumann boundary conditions on forms are defined by d and d^*, respectively. Naively, if we replace (d, d^*) with $(\partial_+, \partial_+^*)$ in the definitions of Dirichlet and Neumann boundary conditions, we obtain the following Dirichlet- and Neumann-type boundary conditions:

$$\begin{aligned} &(D_+) \quad \sigma_{\partial_+}(d\rho)\,\eta\,|_{\partial M} = \partial_+(\rho\,\eta)\,|_{\partial M} = 0\,, \\ &(N_+) \quad \sigma_{\partial_+^*}(d\rho)\,\eta\,|_{\partial M} = \partial_+^*(\rho\,\eta)\,|_{\partial M} = 0\,. \end{aligned}$$

Similarly for ∂_-, we can define

$$\begin{aligned} &(D_-) \quad \sigma_{\partial_-}(d\rho)\,\eta\,|_{\partial M} = \partial_-(\rho\,\eta)\,|_{\partial M} = 0\,, \\ &(N_-) \quad \sigma_{\partial_-^*}(d\rho)\,\eta\,|_{\partial M} = \partial_-^*(\rho\,\eta)\,|_{\partial M} = 0\,. \end{aligned}$$

If these symplectic boundary conditions are defined on primitive forms, $\beta \in P^k$ for $k = 0, 1, \ldots, n-1$, then they also have the good properties stated above for standard Dirichlet and Neumann boundary conditions. Specifically, for $\beta, \lambda \in P^k(M)$ and $k = 0, 1, \ldots, n-1$,

(1) No boundary contribution:

$$\begin{aligned} (\partial_+\beta, \lambda) - (\beta, \partial_+^*\lambda) &= \int_{\partial M} \langle \sigma_{\partial_+}(d\rho)\,\beta, \lambda\rangle\, dS = -\int_{\partial M} \langle \beta, \sigma_{\partial_+^*}(d\rho)\,\lambda\rangle\, dS\,, \\ (\partial_-\beta, \lambda) - (\beta, \partial_-^*\lambda) &= \int_{\partial M} \langle \sigma_{\partial_-}(d\rho)\,\beta, \lambda\rangle\, dS = -\int_{\partial M} \langle \beta, \sigma_{\partial_-^*}(d\rho)\,\lambda\rangle\, dS\,. \end{aligned}$$

(2) Boundary condition is preserved:

$$\begin{aligned} &\beta \in D_+ \Longrightarrow \partial_+\beta \in D_+\,, \qquad &\beta \in D_- \Longrightarrow \partial_-\beta \in D_-\,, \\ &\beta \in N_+ \Longrightarrow \partial_+^*\beta \in N_+\,, \qquad &\beta \in N_- \Longrightarrow \partial_-^*\beta \in N_-\,. \end{aligned}$$

(3) There are also Hodge decompositions of primitive forms. They imply the finite dimensionality of the absolute and relative primitive cohomologies:

$$PH_+^k(M) = \frac{\ker \partial_+ \cap P^k}{\operatorname{im}\, \partial_+ \cap P^k} \cong P\mathcal{H}^k_{+,N_+}(M) \qquad (absolute)$$

$$PH_-^k(M) = \frac{\ker \partial_+ \cap P^k}{\operatorname{im}\, \partial_+ \cap P^k} \cong P\mathcal{H}^k_{-,N_-}(M) \qquad (absolute)$$

$$PH_+^k(M, \partial M) = \frac{\ker \partial_+ \cap P^k_{D_+}}{\operatorname{im}\, \partial_+ \cap P^k_{D_+}} \cong P\mathcal{H}^k_{+,D_+}(M) \qquad (relative)$$

$$PH_-^k(M, \partial M) = \frac{\ker \partial_+ \cap P^k_{D_-}}{\operatorname{im}\, \partial_+ \cap P^k_{D_-}} \cong P\mathcal{H}^k_{-,D_-}(M) \qquad (relative)$$

where the subscript $\{D_+, N_+, D_-, N_-\}$ denotes imposing the boundary condition on the space of primitive forms $P^k(M)$. Also, for instance, $P\mathcal{H}^k_{+,N_+}(M)$ denotes the space of primitive harmonic fields with (N_+) boundary condition, i.e. $\beta \in P^k_{N_+}$ such that $\partial_+\beta = \partial_+^*\beta = 0$. The other three harmonic spaces – $P\mathcal{H}^k_{-,N_-}(M)$, $P\mathcal{H}^k_{+,D_+}(M)$, $P\mathcal{H}^k_{-,D_-}(M)$ – are similarly defined. Interestingly, there is also pairing dualities:

$$PH^k_+(M) \cong PH^k_-(M, \partial M)\,,$$
$$PH^k_-(M) \cong PH^k_+(M, \partial M)\,.$$

Now concerning primitive forms of middle degree, $k = n$, the symplectic cohomology as in (2.5) involves the $\partial_+\partial_-$ operator. This is no doubt unusual. Indeed, just replacing $d \to \partial_+\partial_-$ and $d^* \to (\partial_+\partial_-)^*$ in the definition of the (D) and (N) boundary conditions is not sufficient to obtain the above three properties when the degree of the primitive form $k = n$. In [**22**], we were able to define boundary conditions D_{++} and N_{--} that impose an additionally first-order differential equation on $\beta \in P^n(M)$. These novel boundary conditions lead to properties analogous to the three listed above for the $\{D_+, N_+, D_-, N_-\}$ boundary conditions. (As the definition for $\{D_{++}, N_{--}\}$ requires the introduction of additional terminology and notations, we will refer the reader to [**22**] for further details.) It is nevertheless worthwhile to point out that the symplectic Hodge decompositions of Property (3) above require showing that certain boundary value problems are elliptic. In the $k = n$ case, they lead to interesting elliptic boundary value problems such as the following system:

$$\Delta_{++}\,\beta = \left[(\partial_+\partial_-)^*(\partial_+\partial_-) + (\partial_+\partial_+^*)^2\right]\beta = \lambda\,, \qquad \text{on } M$$
$$\begin{cases} \beta \in D_{++}\,, \\ \partial_+(\rho\,\partial_+^*\beta) = 0\,, & \qquad \text{on } \partial M \\ \partial_+(\rho\,\partial_+^*\partial_+\partial_+^*\beta) = 0\,, \end{cases}$$

for $\beta, \lambda \in P^n(M)$.

3.3. Example: $M = B^3 \times T^3$. To conclude this section, let us mention that the symplectic cohomologies of primitive forms are computable in many instances. Especially, if the de Rham cohomology ring is understood, then the symplectic cohomologies can be easily computed by means of Lefschetz maps as described in [**18, 22**]. One can also compute the cohomologies with respect to different symplectic structures on a manifold. For instance, let us very briefly describe the cohomology on a six-dimensional manifold, $M = B^3 \times T^3$, that is the product of a unit ball in $\mathbb{R}^3$ with a three torus:

$$(x_1, x_2, x_3, y_1, y_2, y_3) \sim (x_1, x_2, x_3, y_1 + a, y_2 + b, y_3 + c)\,,$$

with $a, b, c \in \mathbb{Z}$ and $x_1^2 + x_2^2 + x_3^2 \leq 1$. The boundary is

$$\partial M = S^2 \times T^3 : \{x_1^2 + x_2^2 + x_3^2 = 1\}\,.$$

We can consider two distinct different symplectic (Kähler) forms:

(1) A symplectic structure that is non-trivial in $H^2(M)$.

$$\omega_1 = dx_1 \wedge dx_2 + dy_1 \wedge dy_2 + dy_3 \wedge dx_3\,;$$

(2) A symplectic structure that is trivial in $H^2(M)$

$$\omega_2 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3\,,$$

since $\omega_2 = d\alpha$ where $\alpha = \sum x_i dy_i$.

Interestingly, $PH^k_+(M, \omega_1) \ncong PH^k_+(M, \omega_2)$ for $k = 1, 2, 3$. (For details see [**22**].)

4. Developing Further Symplectic Analysis

Clearly, symplectic manifolds have their own distinctive building blocks of linear differential operators, $(\partial_+, \partial_-, \partial_+\partial_-)$, to write down systems of partial differential equations. As we have seen, these linear symplectic operators have many properties that are analogous to those of the standard exterior derivative operator, d. This is suggestive of developing a symplectic analysis based on $(\partial_+, \partial_-, \partial_+\partial_-)$.

There are certainly some significant challenges in pursuing a symplectic analysis of these operators. The first relates to the fact that interesting analysis for them takes place on forms of degree one or greater. The analysis on the function case, which is the the most approachable and widely studied case in geometric analysis, is not so interesting. We note that acting on any function, f,

$$d\,f = \partial_+ f + \omega \wedge \partial_- f\,,$$

and hence, $\partial_+ f = d\,f$ and $\Delta_+ f = \Delta f$. This implies that symplectic invariants lie within forms of degree one or higher.

Furthermore, there do not seem to be any simplifying coordinate system analogous to the holomorphic and anti-holomorphic coordinates of complex analysis to simplify the calculations on symplectic manifolds. In complex analysis, with local $\{z_i, \bar{z}_i\}$ coordinates, we can for instance write the Dolbeault operator rather simply as $\overline{\partial} = d\bar{z}_i \otimes \dfrac{\partial}{\partial \bar{z}_i}$. On symplectic manifolds, we have at best the Darboux coordinates $\{dx_i, dy_i\}$ with $\omega = \sum dx_i \wedge dy_i$. However, the operators (∂_+, ∂_-) take on the following form:

$$\begin{aligned}
\partial_+ &= \Pi^0\, d = \Pi^0 \left[dx_i \frac{\partial}{\partial x_i} + dy_i \frac{\partial}{\partial y_i} \right]\,, \\
\partial_- &= \left[\frac{1}{2} (\omega^{-1})^{ij}\, i_{\partial_{x_i}} i_{\partial_{y_j}} \right] d\,, \\
\partial_+\partial_- &= d^\Lambda d\,,
\end{aligned}$$

which are not as easy to work with.

Despite these challenges, Tanaka-Tseng [**16**] have recently provided a noteworthy insight that strongly suggests that developing this new symplectic analysis will be fruitful. Interestingly, their work shows that symplectic analysis involving $(\partial_+, \partial_-, \partial_+\partial_-)$ on (M^{2n}, ω) are directly related to the Riemannian analysis on a higher odd-dimensional sphere bundle over M. In particular, for the primitive $p = 0$ case, we can consider

$$\begin{array}{ccc}
S^1 & \longrightarrow & E \\
 & & \downarrow \\
 & & (M^{2n}, \omega)
\end{array}$$

where the Euler class $e(E) = \omega$ and we have assumed $[\omega] \in H^2(M, \mathbb{Z})$. Tanaka-Tseng observed that the data of E comes from two sources only: (i) M^{2n}; (ii) twist of the fiber given by ω. This implies in particular that the *topology* of the $(2n+1)$-dimensional E should reflect the *symplectic geometry* of (M^{2n}, ω) [**16**]. We have shown some of the relations in Table 2. For instance, the de Rham differential graded algebra on E is now known to be "equivalent" to the symplectic algebra of primitive forms of Tsai-Tseng-Yau [**18**] on M. Such relations between E and M

Analysis of (E, g_E, d)	$\longleftrightarrow$	Analysis of (M^{2n}, ω, g, d')
$H(E)$	$\cong$	$PH(M)$
$(\Omega, d, \wedge)$ DGA	$\sim$	$(\mathcal{F}, d', \times, m_3)$ A_∞-algebra
contact homology	$\sim$	genus zero Gromov-Witten
$\vdots$	$\sim$	$\vdots$
Geometric Analysis on E	$\longleftrightarrow$	Symplectic Analysis on (M^{2n}, ω)

TABLE 2. Comparing the analysis on a symplectic manifold (M, ω) with the associated circle bundle E over M where the Euler class of E is ω The linear symplectic operators on M are grouped together and labelled by $d' = (\partial_+, \partial_-, \partial_+\partial_-)$ in the table.

also extends to the quantum level. From the work of Bourgeois [**1**], the data of the contact homology of E corresponds to the genus zero Gromov-Witten invariants on M. In short, we should expect that geometrical analytical relations on the odd-dimensional sphere bundle E based on the exterior derivative d will have analogous symplectic analytical relations on M involving $(\partial_+, \partial_-, \partial_+\partial_-)$.

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