Unitary PSL_2 CM Fields and the Colmez Conjecture

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Definitions

- A **CM field** is a totally imaginary number field E that is a quadratic extension of a totally real field F. E is a **unitary CM field** if E contains an imaginary quadratic field k.
- A **CM type** of E is a subset $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ such that

$$\Phi \cup \overline{\Phi} = \mathrm{Hom}(E, \mathbb{C}), \qquad \Phi \cap \overline{\Phi} = \varnothing.$$

- If $E = kF$ is a unitary CM field of degree $2n$ and Φ is a CM type of E, then the **signature** of Φ is $(n-\epsilon, \epsilon)$ if $n-\epsilon$ of the embeddings of Φ restrict to the identity $k \hookrightarrow \mathbb{C}$.
- Let X be an abelian variety with everywhere good reduction over a number field L with Neron model $\mathcal{X}$ and zero section ϵ. Take a nonzero $\alpha \in \omega := \epsilon^*(\Omega^n_{\mathcal{X}/\mathcal{O}_L})$ and define the **Faltings height** by

$$\mathrm{h_{Fal}}(X) := \frac{-1}{2[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} \log \left| \int_{X^\sigma(\mathbb{C})} \alpha^\sigma \wedge \overline{\alpha^\sigma} \right| + \log \left| \omega / \alpha \mathcal{O}_L \right|.$$

Colmez's Constructions

- Let (E, Φ) be a CM pair and view Φ as a map $\Phi : \mathrm{Hom}(E, \mathbb{C}) \to \{0,1\}$ and extend it to $\Phi^c : \mathrm{Gal}(E^c/\mathbb{Q}) \to \{0,1\}$.
- Let $\widetilde{\Phi}^c$ be the reflex CM type of Φ^c and let $A_\Phi := \Phi^c * \widetilde{\Phi}^c$ be the convolution of Φ^c with $\widetilde{\Phi}^c$.
- Let $A^0_\Phi = \sum_\chi a_\chi \chi$ be the projection of A_Φ onto the space of class functions on $\mathrm{Gal}(E^c/\mathbb{Q})$ and define $Z(s, A^0_\Phi)$ by

$$Z(s, A^0_\Phi) = \sum_\chi a_\chi \left(\frac{L'(s,\chi)}{L(s,\chi)} + \frac{1}{2} \log f_\chi \right).$$

The Colmez Conjecture

Conjecture 1 (Colmez). *Let E be a CM field, Φ a CM type of E, and X_Φ an abelian variety with CM by $(\mathcal{O}_E, \Phi)$. Then,*

$$\mathrm{h_{Fal}}(\Phi) := \frac{1}{[E:\mathbb{Q}]} \mathrm{h_{Fal}}(X_\Phi) = -Z(0, A^0_\Phi).$$

Reduction in the Unitary Case

Theorem 1 (P.). *Let $E = kF$ be a unitary CM field. Then the Colmez conjecture holds for all CM types of E if and only if it holds for all CM types of signature $(n-2, 2)$.*

CM Types and Equivalence

- Let $E = kF$ be a unitary CM field of degree $2n$ and let $\{\sigma_1, \dots, \sigma_n\} = \mathrm{Hom}(F, \mathbb{C})$. Subsets of $\{1, \dots, n\}$ of size ϵ parametrize CM types of E of signature $(n-\epsilon, \epsilon)$ in the following manner. For example, take $\{1,2\} \subseteq \{1, \dots, n\}$.

$$\{1,2\} \leftrightarrow \Phi_{\{1,2\}} := \{\overline{\sigma_1}, \overline{\sigma_2}, \sigma_3, \dots, \sigma_n\}$$

- Let E^c denote the Galois closure of E. There is a natural $\mathrm{Gal}(E^c/\mathbb{Q})$ action on the CM types of E.
- If Φ_1 and Φ_2 are CM types equivalent under the Galois action, then $\mathrm{h_{Fal}}(\Phi_1) = \mathrm{h_{Fal}}(\Phi_2)$ and $A^0_{\Phi_1} = A^0_{\Phi_2}$.
- The Galois action on CM types of signature $(n-\epsilon, \epsilon)$ is equivalent to the action of G on ϵ many elements of G/H, where $G = \mathrm{Gal}(F^c/\mathbb{Q})$ and $H = \mathrm{Gal}(F^c/F)$.

Average amongst Signatures

- Recent work of Yang and Yin has computed the average Faltings height amongst CM abelian varieties with the same signature.

Theorem 2 (Yang-Yin). *Let $E = kF$ be a unitary CM field of degree $2n$ and denote by $\Phi(E)_\epsilon$ the set of all CM types of E of signature $(n-\epsilon, \epsilon)$. Then,*

$$\begin{aligned} \sum_{\Phi \in \Phi(E)_\epsilon} \mathrm{h_{Fal}}(\Phi) &= \sum_{\Phi \in \Phi(E)_\epsilon} -Z(0, A^0_\Phi) \\ &= \frac{-1}{4}\binom{n}{\epsilon} Z(0, \zeta_k) + \binom{n-2}{\epsilon-1} Z(0, \chi_{k/\mathbb{Q}}) \\ &\quad - \frac{1}{n}\binom{n-2}{\epsilon-1} Z(0, \chi_{E/F}). \end{aligned}$$

Dealing with $\mathbf{A^0_\Phi}$

Proposition 1 (P.). *Let $E = kF$ be a unitary CM field, let $S \subseteq \{1, \dots, n\}$, and let Φ_S be the CM type of E corresponding to S. Then,*

$$A^0_{\Phi_S} = \sum_{\{i,j\} \subseteq S} A^0_{\Phi_{\{i,j\}}} - (\epsilon - 2) \sum_{i \in S} A^0_{\Phi_{\{i\}}} + \frac{(\epsilon-1)(\epsilon-2)}{2} A^0_{\Phi_{\{\varnothing\}}}.$$

- A reformulation of the Chowla-Selberg formula implies that the Colmez conjecture holds for $\Phi_{\{\varnothing\}}$.
- Any group G acts transitively on G/H for any subgroup $H \leq G$. Therefore, the result of Yang-Yin implies that the Colmez conjecture holds for $\Phi_{\{i\}}$ for any $i \in S$.
- Yang and Yin's theorem, Proposition 1, and the above known cases of the Colmez conjecture imply the following.

Proposition 2 (P.). *Let $E = kF$ be a unitary CM field, let $G = \mathrm{Gal}(F^c/\mathbb{Q})$ and $H = \mathrm{Gal}(F^c/F)$. If G acts doubly transitively on G/H, then the Colmez conjecture holds for any CM type of E.*

Colmez for Unitary $\mathbf{PSL_2}$ CM fields

- Let B be the Borel subgroup of $\mathrm{PSL}_2(\mathbb{F}_q)$ where q is an odd prime power. The action of $\mathrm{PSL}_2(\mathbb{F}_q)$ on $\mathrm{PSL}_2(\mathbb{F}_q)/B$ is isomorphic to the action of $\mathrm{PSL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$ and is doubly transitive.

Theorem 3 (P.). *Let F be a totally real number field such that $\mathrm{Gal}(F^c/\mathbb{Q}) \cong \mathrm{PSL}_2(\mathbb{F}_q)$ and $\mathrm{Gal}(F^c/F) \cong B$. Then for any imaginary quadratic field k and any CM type Φ of $E = kF$, the Colmez conjecture holds for Φ.*

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