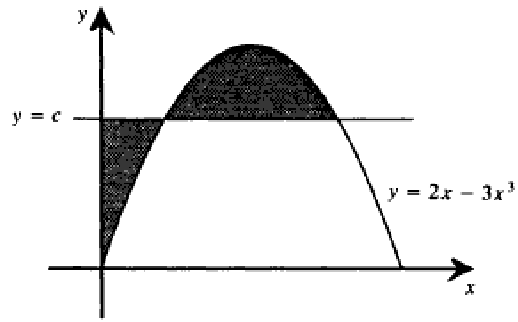
Math 194, problem set #6

For discussion November 12

(1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. (Putnam, 1993)



(2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).
(Putnam 1988)

(3) If n is a positive integer, prove for $x > 0$ that $\dfrac{x^n}{(x+1)^{n+1}} \leq \dfrac{n^n}{(n+1)^{n+1}}$.

(4) (a) Assuming that temperature is a continuous function, show that at any given time on the earth's equator there are two points directly opposite points that have the same temperature.
(b) A rock climber starts to climb a mountain at 7:00 AM on Saturday and gets to the top at 5:00 PM. She camps on top and climbs back down on Sunday, starting at 7:00 AM. Show that at some time of day on Sunday she was at the same elevation as she was at that time on Saturday.

(5) Suppose f and g are differentiable functions and for every x, $f'(x)g(x) \neq f(x)g'(x)$. Show that between every two zeros of f there is a zero of g.

(6) (a) Suppose that $f(x)$ is continuous and $f(x) \geq 0$ on $[0, 1]$. Show that if $\int_0^1 (x-1)^2 f(x)dx = 0$, then $f(x) = 0$ on $[0, 1]$.
(b) Find all continuous functions $f(x)$ on $[0, 1]$ such that $f(x) \geq 0$ and

$$\int_0^1 f(x)dx = 1, \quad \int_0^1 xf(x)dx = \alpha, \quad \int_0^1 x^2 f(x)dx = \alpha^2$$

where α is a given real number. (Putnam, 1964)

(7) Suppose f is a differentiable function on $[0, 1]$, $f(0) = 0$, and $f'(x)$ is strictly increasing. Show that $f(x)/x$ is strictly increasing.

(8) Suppose f is a continuous function on $[0, 1]$, $n \in \mathbf{Z}^+$, $\int_0^1 x^k f(x)dx = 0$ for $k = 0, 1, \ldots, n-1$, and $\int_0^1 x^n f(x)dx = 1$. Show that there is a $c \in [0, 1]$ such that $|f(c)| > 2^n(n+1)$.