Math 194, problem set #5

For discussion October 29

(1) Compute the following infinite sums:

(a) $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots$

(b) $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{2\cdot 3\cdot 4}+\dfrac{1}{3\cdot 4\cdot 5}+\cdots$

(c) $1-\dfrac{1}{4}+\dfrac{1}{6}-\dfrac{1}{9}+\dfrac{1}{11}-\dfrac{1}{14}+\dfrac{1}{16}-\dfrac{1}{19}+\dfrac{1}{21}-\cdots$

(2) Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$.

(3) Suppose k is an integer, $k\geq 2$. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n}$, where

$$a_n=\begin{cases}1 & \text{if } n \text{ is not a multiple of } k,\\ 1-k & \text{if } n \text{ is a multiple of } k.\end{cases}$$

(4) Express $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}$ as a rational function of x (a quotient of polynomials).

(5) Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{n!}$.

(6) Given a sequence (x_n) such that $\lim_{n\to\infty}(x_n-x_{n-2})=0$, prove that

$$\lim_{n\to\infty}\frac{x_n-x_{n-1}}{n}=0.$$

(Putnam 1970)

(7) Show that

$$\frac{x}{1-x-x^2}=\sum_{n=1}^{\infty}F_n x^n,$$

where F_n is the n-th Fibonacci number.