

## Math 194, problem set #1

For discussion October 1

1. Show that some multiple of 1232123432123454321 contains all 10 digits (at least once) when written in base 10.
2. For each integer  $n \geq 0$ , let  $S(n) = n - m^2$ , where  $m$  is the greatest integer with  $m^2 \leq n$ . Define a sequence  $(a_k)_{k=0}^{\infty}$  by  $a_0 = A$  and  $a_{k+1} = a_k + S(a_k)$  for  $k \geq 0$ . For what positive integers  $A$  is this sequence eventually constant? (Putnam, 1991)
3. Determine  $F(x)$  if, for all real  $x$  and  $y$ ,  $F(x)F(y) - F(xy) = x + y$ .
4. A two-person game is played as follows. The players alternate placing a penny on a circular table. Each penny must lie completely on the table, and not overlap any previously-placed pennies. The first player unable to fit a penny on the table loses. (You can assume they have all the pennies they need.) Is it better to go first or second? Is there a winning strategy? Is the answer any different if the table is square? triangular?
5. Compute the determinant of the  $n \times n$  matrix all of whose diagonal entries are 0, and all of whose off-diagonal entries are 1.
6. Show that every infinite sequence of distinct real numbers contains either a strictly increasing infinite subsequence or a strictly decreasing infinite subsequence.
7. (a) Give a sensible definition of the infinite tower of exponentials  $t(x) := x^{x^{x^{x^{\dots}}}}$  for real numbers  $x \geq 1$ , when it makes sense.  
(b) Show that  $t(\sqrt{2}) = 2$ .  
(c) Show that there is no real number  $a$  such that  $t(a) = 4$ .  
(d) What can you say about the domain and range of  $t$ ?
8. Suppose  $x_0$  and  $x_1$  are given real numbers, and for  $n \geq 2$  define

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}.$$

Find  $\lim_{n \rightarrow \infty} x_n$ .