Math 194

November 12, 2015

Arithmetic-geometric mean inequality: If $N > 0$ and $A_1, A_2 \cdots A_N > 0$, then

$$\frac{A_1 + A_2 \cdots + A_N}{N} \geq (A_1 A_2 A_3 \cdots A_N)^{1/N}$$

Convexity inequality: If $f''(x) \geq 0$ on an interval $[s, t]$, $a_1 + \cdots + a_n = 1$ with $a_i \geq 0$, and $x_1, \ldots, x_n \in [s, t]$, then

$$f(a_1 x_1 + \cdots + a_n x_n) \leq a_1 f(x_1) + \cdots + a_n f(x_n).$$

(1) Prove that for any positive integer n and positive numbers a_i, b_i that either

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n \qquad \text{or} \qquad \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq n.$$

(2) Show that if $x, y \geq 0$, then

$$\frac{x+y}{2} \leq \sqrt{\frac{x^2 + y^2}{2}}.$$

(3) Suppose $x_1, x_2, \ldots, x_n \in \mathbf{R}$ and $\sum_{i=1}^n x_i = 1$. If k is a positive integer show that

$$\sum_{i=1}^{n} x_i^k \geq \frac{1}{n^{k-1}}.$$

(4) Show that $1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1) < n^n$.

(5) If a, b, c are positive real numbers and $1/a + 1/b + 1/c = 1$, show that

$$(a-1)(b-1)(c-1) \geq 8.$$