Math 194

October 29, 2015

1. Prove that the polynomial $x^3 - 3x^2 + 1$ has exactly one zero in the interval $[0, 1]$.

2. (a) Suppose f is a continuous function from the closed interval $[0, 1]$ to $[0, 1]$. Prove that there is some c such that $f(c) = c$.

   (b) Find a continuous function f from the *open* interval $(0, 1)$ to $(0, 1)$ such that there is *no* c satisfying $f(c) = c$.

3. For any pair of triangles, prove that there exists a line that bisects them (divides each one into two equal areas) simultaneously.

4. Suppose $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$ with real numbers $a_1, \ldots, a_n$, and $|f(x)| \leq |\sin(x)|$ for every x. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

5. Suppose f is a differentiable function and $a \in \mathbf{R}$. Show that between any two zeros of f there is an x such that $f'(x) = af(x)$.