Math 194

Problem set #6

1. Evaluate
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k^2+n^2}}.$$
(Larson 6.8.5)

2. Let $f(x)=\sum_{k=1}^{n} a_k \sin(kx)$ with $a_i \in \mathbf{R}$, $n \geq 1$. Prove that if $f(x) \leq |\sin(x)|$ for every x, then
$$\left|\sum_{k=1}^{n} k a_k\right| \leq 1.$$
(Putnam 1967)

3. Let $f(x)$ be a continuous function on $[0,1]$ such that $f(0)=f(1)=0$ and $2f(x)+f(y)=3f(\frac{2x+y}{3})$ for all $x,y\in[0,1]$. Prove that $f(x)=0$ for all $x\in[0,1]$. (Vietnamese Mathematical Olympiad, 1999)

4. Suppose $f:\mathbf{R}\to\mathbf{R}$ is a continuous function such that $|f(x)-f(y)|\geq|x-y|$ for every $x,y\in\mathbf{R}$. Show that the range of f is $\mathbf{R}$, i.e., for every $c\in\mathbf{R}$ there is an x such that $f(x)=c$.
(De Souza & Silva, *Berkeley Problems in Mathematics*)

5. Suppose that $f:\mathbf{R}\to\mathbf{R}$ is a continuous function, and define
$$g(x)=f(x)\int_0^x f(t)dt.$$
Prove that if g is a nonincreasing function, then $f(x)=0$ for every x.
(Romanian Olympiad 1978)

6. Let $f:[0,1]\to\mathbf{R}$ be a function with a continuous derivative, such that $f(0)=0$ and $0<f'(x)\leq 1$ for every x. Show that
$$\left(\int_0^1 f(x)dx\right)^2 \geq \int_0^1 (f(x))^3 dx.$$
(Putnam 1973)

7. Suppose f and g are n-times continuously differentiable functions in a neighborhood of a point a, such that $f(a)=g(a)$, $f'(a)=g'(a)$, $\ldots$, $f^{(n-1)}(a)=g^{(n-1)}(a)$, and $f^{(n)}(a)\neq g^{(n)}(a)$. Evaluate
$$\lim_{x\to a}\frac{e^{f(x)}-e^{g(x)}}{f(x)-g(x)}.$$

(N. Georgescu-Roegen)

8. Let $n > 1$ be an integer, and $f : [a, b] \to \mathbf{R}$ a continuous function, n-times differentiable on (a, b). Prove that if the graph of f has $n + 1$ collinear points, then there is a point $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

(G. Sireţchi, *Mathematics Gazette, Bucharest*)

9. Suppose $x_1, x_2, \ldots, x_n \in \mathbf{R}$. Find the real number(s) a that minimize the expression

$$|a - x_1| + |a - x_2| + \cdots + |a - x_n|.$$

(Andreescu & Gelca)

10. Prove that for every natural number $n \geq 2$ and every $x \in [-1, 1]$,

$$(1 + x)^n + (1 - x)^n \leq 2^n.$$

(*Exercises and Problems in Algebra*, Bucharest 1983)