

Math 194
Problem set #5

1. For what positive x does the series

$$(x - 1) + (\sqrt{x} - 1) + (\sqrt[3]{x} - 1) + \cdots + (\sqrt[n]{x} - 1) + \cdots$$

converge?

(The Wohascum County Problem Book, MAA, 1996)

2. Does the series $\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2 + 1})$ converge?

(Gh. Siretchi)

3. Show that

$$\sum_{n=1}^{9999} \frac{1}{(\sqrt{n} + \sqrt{n+1})(\sqrt[4]{n} + \sqrt[4]{n+1})} = 9.$$

(*Mathematical Reflections*)

4. Compute the product

$$\left(1 - \frac{4}{1}\right) \left(1 - \frac{4}{9}\right) \left(1 - \frac{4}{25}\right) \left(1 - \frac{4}{49}\right) \cdots$$

5. Compute the product

$$\prod_{n=1}^{\infty} (1 + x^{2^n}).$$

6. Find a formula for the general term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \dots$$

(Graham, Knuth & Patashnik, *Concrete Mathematics*)

7. Define the sequence a_0, a_1, a_2, \dots by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \quad \text{for } n \geq 0.$$

Show that n divides a_n for every $n \geq 1$.

(D. Andrica, *Timișoara Mathematics Gazette*)

8. Suppose x_1, x_2, \dots is a sequence of positive real numbers satisfying $x_{n+1} \leq x_n + 1/n^2$ for all $n \geq 1$. Prove that $\lim_{n \rightarrow \infty} x_n$ exists.

(De Souza & Silva, *Berkeley Problems in Mathematics*)

9. Prove that if a and b are relatively prime odd positive integers, then

$$\sum_{x=1}^{b-1} \left\lfloor \frac{ax}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

10. Evaluate

$$\sum_{n=1}^{\infty} \frac{n}{3^n}.$$