Math 194

Problem set #4

1. Suppose A and B are two $n \times n$ matrices that commute (i.e., $AB = BA$), and for some positive integers r, s we have $A^r = I_n$ (the $n \times n$ identity matrix) and $B^s = 0$ (the $n \times n$ zero matrix). Prove that $A + B$ is invertible, and find its inverse.
(Andreescu & Gelca)

2. Suppose $a, b \geq 2$ are relatively prime integers. For every $n \geq 0$ show that $a^{2n} + b^{2n}$ is *not* divisible by $a + b$. (Gelca)

3. Find all polynomials $P(x)$ with real coefficients such that
$$(x + 1)P(x) = (x - 2)P(x + 1)$$

4. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, (counted without multiplicities). (Russian Mathematical Olympiad, 2002)

5. Given an integer $n \geq 1$, find all polynomials $P(x)$ with real coefficients such that $P(x)^n = P(x^n)$ for all x.

6. Suppose you have a calculator, but the multiplication and division buttons are broken. You can add, subtract, and take the inverse of a number, but you can't multiply or divide. Show how to find the product of (any) two numbers, using at most 20 operations. (Quantum)

7. Let S be the smallest set of rational functions (i.e., ratios of polynomials) in the variables x and y with real coefficients, containing $f(x, y) = x$ and $g(x, y) = y$ and closed under addition, subtraction, and taking reciprocals. Show that S does *not* contain the constant function $h(x, y) = 1$. (American Mathematical Monthly, 1987)

8. Suppose $f(x)$ is a polynomial with integer coefficients, and for some integer k there are k consecutive integers $n, n + 1, \ldots, n + k - 1$ such that none of the values $f(n)$, $f(n + 1)$, $\ldots$, $f(n + k - 1)$ are divisible by k. Prove that $f(x)$ has no integer roots.
(Putnam, 1940)

9. Suppose that a and b are different roots of $x^3 + x - 1$. Prove that ab is a root of $x^3 - x^2 - 1$.

10. Show that there are infinitely many positive integers a such that $n^4 + a$ is not prime for any natural number n.