Math 194, problem set #1

1. Show that there is no polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree at least one, with integer coefficients a_i, such that $P(k)$ is prime for every nonnegative integer k. (Andreescu & Gelca)

2. Suppose n objects are arranged in a row. A subset of these objects is called *unfriendly* if no two of its elements are consecutive. Show that the number of unfriendly subsets each having k elements is $\binom{n-k+1}{k}$. (Larson, 1.3.12)

3. On an arbitrarily large chessboard, a generalized knight moves by jumping p squares in one direction and q squares in a perpendicular direction, $p, q > 0$. Show that such a knight can return to its original position only after an *even* number of moves. (German Mathematical Olympiad)

4. Use the graph of $y = \sin(x)$ to show the following. Given a triangle with angles A, B, C,
   (a) $\frac{\sin(B)+\sin(C)}{2} \leq \sin\left(\frac{B+C}{2}\right)$,
   (b) $z\sin(B) + (1-z)\sin(C) \leq \sin(zB + (1-z)C)$, if $0 \leq z \leq 1$.

   (Larson, 1.3.12)

5. Every point of 3-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that for every positive real number d, two points of that color are exactly distance d apart. (German Mathematical Olympiad)

6. Let $a_1, a_2, \ldots, a_n$ denote an arbitrary ordering of the numbers $1, 2, \ldots, n$. Prove that if n is odd, then the product $(a_1 - 1)(a_2 - 2)\cdots(a_n - n)$ is even. (Larson, 1.10.8)

7. Show that no set of 9 consecutive integers can be partitioned into two subsets such that the product of the elements in the first set is equal to the product of the elements in the second set. (Andreescu & Gelca)

8. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$. (Putnam, 1986)

9. Define a *selfish* set to a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $1, 2, \ldots, n$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish. (Putnam, 1996)