

Math 194, problem set #5

For discussion Thursday October 31

(1) Compute the following infinite sums:

(a) $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$

(b) $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots$

(c) $1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \frac{1}{16} - \frac{1}{19} + \frac{1}{21} - \cdots$

(2) Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$.

(3) Suppose k is an integer, $k \geq 2$. Evaluate $\sum_{n=1}^{\infty} \frac{a_n}{n}$, where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is not a multiple of } k, \\ 1 - k & \text{if } n \text{ is a multiple of } k. \end{cases}$$

(4) Express $\sum_{n=0}^{\infty} \frac{x^{2n}}{1 - x^{2n+1}}$ as a rational function of x (a quotient of polynomials).

(5) Evaluate $\sum_{n=1}^{\infty} \frac{n^3}{n!}$.

(6) Given a sequence (x_n) such that $\lim_{n \rightarrow \infty} (x_n - x_{n-2}) = 0$, prove that

$$\lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{n} = 0.$$

(Putnam 1970)

(7) Show that

$$\frac{x}{1 - x - x^2} = \sum_{n=1}^{\infty} F_n x^n,$$

where F_n is the n -th Fibonacci number.