Math 194, problem set #1

For discussion Thursday October 3

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems. These can be either submitted by email, or as a hard copy.

1. Show that some multiple of 1232123432123454321 contains all 10 digits (at least once) when written in base 10.

2. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant? (Putnam, 1991)

3. Determine $F(x)$ if, for all real x and y, $F(x)F(y) - F(xy) = x + y$.

4. A two-person game is played as follows. The players alternate placing a penny on a circular table. Each penny must lie completely on the table, and not overlap any previously-placed pennies. The first player unable to fit a penny on the table loses. (You can assume they have all the pennies they need.) Is it better to go first or second? Is there a winning strategy? Is the answer any different if the table is square? triangular?

5. Compute the determinant of the $n \times n$ matrix all of whose diagonal entries are 0, and all of whose off-diagonal entries are 1.

6. Show that every infinite sequence of distinct real numbers contains either a strictly increasing infinite subsequence or a strictly decreasing infinite subsequence.

7. (a) Give a sensible definition of the infinite tower of exponentials $t(x) := x^{x^{x^{x^{x^{\cdots}}}}}$ for real numbers $x \geq 1$, when it makes sense.
   (b) Show that $t(\sqrt{2}) = 2$.
   (c) Show that there is no real number a such that $t(a) = 4$.
   (d) What can you say about the domain and range of t?

8. Suppose x_0 and x_1 are given real numbers, and for $n \geq 2$ define

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}.$$

Find $\lim_{n \to \infty} x_n$.