Putnam problems and solutions A1

(Many solutions are taken directly from http://www.unl.edu/amc/a-activities/a7-problems/putnamindex.shtml where the authors are properly attributed. Others are from "The William Lowell Putnam Mathematical Competition, 1985-2000, Problems, Solutions, and Commentary," by Kedlaya, Poonen, and Vakil. The remaining solutions are modified versions of these or originals by Dennis.)

2006. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Solution:

We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2 + y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r - 3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus) with volume

$$\pi \int_{-1}^{1} (3 + \sqrt{1 - z^2})^2 - (3 - \sqrt{1 - z^2})^2 dz = \pi \int_{-1}^{1} 12\sqrt{1 - z^2} dz = 6\pi^2,$$

where we have computed the final integral by realizing that it is equal to 12π times the area of a half-disc with radius 1.

2005. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Solution:

We will show by induction that we can construct a set S_n whose elements are each of the form $2^r 3^s$ with r and s nonnegative integers, no element of S_n divides another, and the elements of S_n sum to n. Let $S_0 = \emptyset$ and $S_1 = \{1\}$, and notice that S_0 and S_1 both have the desired properties. Assume that we have sets S_k with the desired properties for all $k < n$. If n is even, let $S_n = \{2a : a \in S_{n/2}\}$, and we see that S_n will inherit all of the desired properties from $S_{n/2}$. If n is odd, let $3^t \leq n$ be as large as possible with t an integer, and let $S_n = \{3^t\} \cup \{2a : a \in S_{(n-3^t)/2}\}$. Since $(n - 3^t)/2 < 3^t$, 3^t cannot divide any other element of S_n, and since $S_n \setminus \{3^t\}$ contains only even elements, no other element of S_n can divide 3^t. Thus S_n has the desired properties, and so sets S_n exist for all n by induction.

2004. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

How to solve it:

After looking at the possibilities for small N, it appears that $S(N)$ will be exactly 80% of N at some point. It makes sense to consider some free throw where, on the next throw, Shanille reaches or exceeds 80%.

Solution:

Yes. Since Shanille's free throw percentage is under 80% at some point, and above 80% at the end of the season, there must be some N_0 such that $S(N_0) <$ 80% of N_0 and $S(N_0+1) \geq$ 80% of $N_0 + 1$. She must make the $(N_0 + 1)$st free throw for this to happen, so $\frac{S(N_0)}{N_0} < \frac{4}{5}$ and $\frac{S(N_0+1)}{N_0+1} = \frac{S(N_0)+1}{N_0+1} \geq \frac{4}{5}$. Thus $5S(N_0) < 4N_0$ and $5S(N_0) + 5 \geq 4N_0 + 4$ so that $5S(N_0) < 4N_0 \leq 5S(N_0) + 1$. Since this is an inequality among integers, it must be that, in fact, $4N_0 = 5S(N_0) + 1$. Thus, $S(N_0 + 1) = S(N_0) + 1 = \frac{4N_0-1}{5} + 1 = \frac{4}{5}(N_0 + 1)$, and so $S(N_0 + 1)$ is exactly 80% of $N_0 + 1$.

Remark: This same argument works for any fraction of the form $(n-1)/n$ for some integer $n > 1$, but not for any other real number between 0 and 1.

2003. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

How to solve it:

A. If we look at a few particular values of n, we are soon led to conjecture that the number of ways to write n in this manner is n.

B. Introduce suitable notation.

C. The method of our solution will now depend on *why* we think our conjecture is true. If we see it inductively, we should use induction. If we see it constructively, we should think about how the representations can be constructed. If we notice a refinement (that there is exactly one representation with k parts for each $0 < k \leq n$, for example), we should try to prove our refinement.

Solution 1:

Let $f(n)$ be the number of ways to write n as described above. We conjecture that $f(n) = n$ for all n, which we now prove by induction. We see that $f(1) = 1$. Assume as our induction hypothesis that $f(j) = j$. If we add one to the last (rightmost) occurrence of the smallest part of each of our representations of j, we get a representation of $j+1$. We see that each of these representations is counted by $f(j+1)$, since adding one to the last occurrence of the smallest part of a representation preserves the inequalities $a_1 \le a_2 \le \cdots \le a_k \le a_1+1$ (before adding one, exactly one of the above inequalities is strict; adding one to the last occurrence of the smallest part effectively "moves the strict inequality one to the left"). Similarly, if we have a representation counted by $f(j+1)$ that is not all ones, subtracting one from the first occurrence of the largest part will give us a representation counted by $f(j)$. We see that each of these operations is one-to-one, and so the representations counted by $f(j+1)$ are exactly the representation that is $j+1$ ones plus the representations counted by $f(j)$ with an increased smallest part. Thus $f(j+1) = 1 + f(j) = j+1$.

Solution 2:

If we perform Euclidean Division on n with any positive integer $k \le n$, we have $n = k \cdot q + r$, for a unique q and r with $q > 0$ and $0 \le r < k$. Letting $q = a_1 = a_2 = \cdots = a_{k-r}$ and any remaining $a_i = q+1$, we have a representation of n with k parts satisfying $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$. It is clear that there can be no other representation of n into k parts of this type (e.g., any representation with the same smallest part must be the same, and any representation with smaller or larger smallest part would be sum to either too little or too much, respectively). Since there is exactly one representation of n into k parts for each $0 < k \le n$, we have that the number of representations of n of the desired type is n.

2002. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Solution:

Notice that the $(n+1)$st derivative of $\frac{1}{x^k-1}$ is just

$$\left[\frac{P_n(x)}{(x^k-1)^{n+1}}\right]' = \frac{P_n'(x)}{(x^k-1)^{n+1}} + \frac{-(n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}} = \frac{P_n'(x)(x^k-1)-(n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}},$$

and so $P_{n+1}(x) = P_n'(x)(x^k-1) - (n+1)kx^{k-1}P_n(x)$.
Thus

$$P_{n+1}(1) = P_n'(1)(1^k-1) - (n+1)k1^{k-1}P_n(1) = -(n+1)kP_n(1).$$

Since $P_0(1) = 1$, we see that $P_n(1) = (-1)^n k^n n!$.

2001. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Solution:

The hypothesis implies $((b*a)*b)*(b*a) = b$ for all $a, b \in S$ (by replacing a by $b*a$), and hence $a*(b*a) = b$ for all $a, b \in S$ (using $(b*a)*b = a$).

2000. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Solution:

The possible values comprise the interval $(0, A^2)$.

To see that the values must lie in this interval, note that

$$\left(\sum_{j=0}^{m} x_j\right)^2 = \sum_{j=0}^{m} x_j^2 + \sum_{0\le j<k\le m} 2x_j x_k,$$

so $\sum_{j=0}^{m} x_j^2 \le A^2 - 2x_0x_1$. Letting $m \to \infty$, we have $\sum_{j=0}^{\infty} x_j^2 \le A^2 - 2x_0x_1 < A^2$.

To show that all values in $(0, A^2)$ can be obtained, we use geometric progressions with $x_1/x_0 = x_2/x_1 = \cdots = d$ for variable d. Then $\sum_{j=0}^{\infty} x_j = x_0/(1-d)$ and

$$\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d}\left(\sum_{j=0}^{\infty} x_j\right)^2.$$

As d increases from 0 to 1, $(1-d)/(1+d)$ decreases from 1 to 0. Thus if we take geometric progressions with $\sum_{j=0}^{\infty} x_j = A$, $\sum_{j=0}^{\infty} x_j^2$ ranges from 0 to A^2. Thus the possible values are indeed those in the interval $(0, A^2)$, as claimed.

1999. Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \le x \le 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

How to solve it:

One may use the piecewise definition of the absolute value function to select possible forms for f, g, and h, and then solve for the free variables to find them explicitly.

Solution:

Let $f(x) = (3x+3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

For $x < -1$,

$$\left|\frac{3x+3}{2}\right| - \left|\frac{5x}{2}\right| - x + \frac{1}{2} = -\frac{3x+3}{2} + \frac{5x}{2} - x + \frac{1}{2} = -1.$$

For $-1 \le x \le 0$,

$$\left|\frac{3x+3}{2}\right| - \left|\frac{5x}{2}\right| - x + \frac{1}{2} = \frac{3x+3}{2} + \frac{5x}{2} - x + \frac{1}{2} = 3x+2.$$

For $x > 0$,

$$\left|\frac{3x+3}{2}\right| - \left|\frac{5x}{2}\right| - x + \frac{1}{2} = \frac{3x+3}{2} - \frac{5x}{2} - x + \frac{1}{2} = -2x+2.$$

1998. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Solution:

After drawing a picture, let the side length of the cube be s. Consider the rectangle created by two parallel non-adjacent edges of the cube perpendicular to the base of the cone and the connecting diagonals of the two cube faces parallel to the base of the cone. Since the slope of the edge of the cone is -3, we see that

$$3 - 3\frac{\sqrt{2}}{2}s = s, \quad \text{so that} \quad s = \frac{6}{2+3\sqrt{2}} = \frac{-6+9\sqrt{2}}{7}.$$

1997. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

How to solve it:

First, draw a picture (the solution may seem like gibberish without it). Label all of the points described and all of the known lengths. Establish some notation. For example, for two points X and Y, let $|XY|$ denote the length of XY. Think about BC, and consider what other lengths would allow us to calculate $|BC|$. Use the information given to establish relationships that will allow us to find one of these lengths.

There is more than one way to solve this problem. We could introduce coordinate axes, or we could do without. In thinking about the problem, we may realize that $|AH|$ and $|BF|$ are two critical unknown lengths. If we can use two different facts about the figure to establish two different relationships between $|AH|$ and $|BF|$, we can solve for $|BF|$, and then find $|BC|$. In the solution below, we avoid introducing axes, although doing so would be valid (if we introduced axes, we would find the coordinates of B instead of finding $|BF|$, and we would replace the use of similar triangles below by using the fact that the slope of AC must be the negative reciprocal of the slope of BH, since they are perpendicular).

Solution:

Let P be the foot of the altitude from B to AC. Notice that $\triangle BPC$ is similar to $\triangle AFC$ since they both have one right angle, and they share $\angle C$. Also notice that $\triangle BPC$ is similar to $\triangle BFH$ since they both have one right angle, and they share $\angle PBF$ (recall that H is on BP by our hypotheses). Thus $\triangle AFC$ is similar to $\triangle BFH$. Thus $\frac{|AF|}{|FC|} = \frac{|BF|}{|FH|}$. Notice that $|FC| = |BF| + 22$ since M is the midpoint of BC. Then, substituting $|AF| = |AH| + 5, |FC| = |BF| + 22$, and $|FH| = 5$, we have $\frac{|AH|+5}{|BF|+22} = \frac{|BF|}{5}$, or

$$5|AH| + 25 = |BF|^2 + 22|BF|, \tag{1}$$

our first relationship between $|AH|$ and $|BF|$.

Now notice that $|OA| = |OB|$ since they are both radii of the same circle. Thus, by the Pythagorean Theorem, $|HO|^2 + |AH|^2 = |OM|^2 + |BM|^2$, or equivalently, $11^2 + |AH|^2 = 5^2 + (|BF| + 11)^2$, which we can rewrite as

$$|AH|^2 - 25 = |BF|^2 + 22|BF|, \tag{2}$$

a second relationship between $|AH|$ and $|BF|$.

Combining (1) and (2), we have $|AH|^2 - 25 = 5|AH| + 25$ which, after solving this quadratic, implies $|AH| = 10$. Then, solving for $|BF|$, we find $|BF| = 3$, so that $|BC| = 28$.

1996. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

Solution:

If x and y are the sides of two squares with combined area 1, then $x^2 + y^2 = 1$. Suppose without loss of generality that $x \geq y$. Then the shorter side of a rectangle containing both squares without overlap must be at least x, and the longer side must be at least $x+y$. Hence the desired value of A is the maximum of $x(x + y)$.

To find this maximum, we let $x = \cos\theta, y = \sin\theta$ with $\theta \in [0, \pi/4]$. Then we are to maximize $\cos^2\theta + \sin\theta\cos\theta$. Setting

$$\left[\cos^2\theta + \sin\theta\cos\theta\right]' = -2\cos\theta\sin\theta + \cos^2\theta - \sin^2\theta = -\sin 2\theta + \cos 2\theta$$

equal to 0, we have $\theta = \pi/8$. After verifying that a maximum does occur here, we have that the desired value of A is

$$x(x+y) = \cos^2\frac{\pi}{8} + \sin\frac{\pi}{8}\cos\frac{\pi}{8} = \frac{1 + \cos 2\frac{\pi}{8}}{2} + \frac{1}{2}\sin 2\frac{\pi}{8} = \frac{1+\sqrt{2}}{2}.$$

1995. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Solution:

Suppose on the contrary that there exist $t_1, t_2 \in T$ with $t_1t_2 \in U$ and $u_1, u_2 \in U$ with $u_1u_2 \in T$. Then $(t_1t_2)u_1u_2 \in U$ while $t_1t_2(u_1u_2) \in T$, contradiction.

1994. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Solution:

For, $n \ge 0$, let $b_n = \displaystyle\sum_{j=2^n}^{2^{n+1}-1} a_j$. Since $a_n \le a_{2n} + a_{2n+1}$, we have

$$b_n = \sum_{j=2^n}^{2^{n+1}-1} a_j = \sum_{j=2^{n-1}}^{2^n-1} a_{2j} + a_{2j+1} \ge \sum_{j=2^{n-1}}^{2^n-1} a_j = b_{n-1}.$$

Thus, $b_n \not\to 0$, and we have that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} \sum_{j=2^n}^{2^{n+1}-1} a_j = \sum_{n=0}^{\infty} b_n$$

diverges.

1993. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

Solution:

Let the final intersection of $y = c$ and $y = 2x - 3x^3$ occur at $x = d$. We then have $\displaystyle\int_0^d 2x - 3x^3 - c = 0$ so that $d^2 - \dfrac{3}{4}d^4 - cd = 0$. Solving for c, we have $c = d - \frac{3}{4}d^3$, and by the definition of d, we have $c = 2d - 3d^3$. Thus $0 = d - \frac{9}{4}d^3 = d(1 - \frac{3}{2}d)(1 + \frac{3}{2}d)$, so our d must be $\frac{2}{3}$. Thus $c = 2\frac{2}{3} - 3(\frac{2}{3})^3 = \frac{4}{9}$.

1992. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

Solution:

By (i), $f(f(f(n+2)+2)) = f(n+2)+2$, and by (ii), $f(f(f(n+2)+2)) = f(n)$, each holding for all integers n. Thus, $f(n) = f(n+2)+2$. Also notice that $f(f(0)) = f(1) = 0$ by (iii) and (i). Since $f(2m) = f(2m+2)+2 = f(2(m+1))+2$, $f(2m) = f(2m-2)-2 = f(2(m-1))-2$, $f(2m+1) = f(2m+3) + 2 = f(2(m+1)+1) + 2$, and $f(2m+1) = f(2m-1) - 2 = f(2(m-1)+1) - 2$, we have that $f(n) = 1 - n$ for all integers n by induction.

1991. A 2×3 rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

Solution:

If we sketch the path of the point whose initial position is (1,1), we see that the area of the region in question can be broken up into five solid triangles and four quarter discs. Adding the area of each of these, we see that the region has area $7\pi/2 + 6$.

1990. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Solution:

From either the recurrence itself or from looking at the ratios of the first several pairs of consecutive values, we might guess that T_n grows in a way similar to $n!$. If we consider the first several values of $T_n - n!$, it looks like $T_n = n! + 2^n$, which we now verify by induction.

For $n = 0, 1$, and 2, our formula holds. Assume our formula holds for all $n \leq k-1$. Then for $k \geq 3$,

$$\begin{aligned}
T_k &= (k+4)T_{k-1} - 4kT_{k-2} + (4k-8)T_{k-3} \\
&= (k+4)[(k-1)! + 2^{k-1}] - 4k[(k-2)! + 2^{k-2}] + (4k-8)[(k-3)! + 2^{k-3}] \\
&= (k-3)! \cdot [(k+4)(k-1)(k-2) - 4k(k-2) + (4k-8)] + \\
&\quad 2^{k-3} \cdot [4(k+4) - 2(4k) + (4k-8)] \\
&= (k-2)! \cdot [(k+4)(k-1) - 4k + 4] + 2^k \\
&= (k-1)! \cdot [(k+4) - 4] + 2^k = k! + 2^k.
\end{aligned}$$

Thus $T_n = n! + 2^n$ for all $n \geq 0$ by induction.

1989. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

Solution:
Suppose $N = 101\cdots 0101$ with k ones for some $k \geq 2$. Then $99N = 10^{2k} - 1 = (10^k + 1)(10^k - 1)$. If N is prime, then N divides either $(10^k + 1)$ or $(10^k - 1)$, and thus one of $\frac{99}{10^k-1} = \frac{10^k+1}{N}$ and $\frac{99}{10^k+1} = \frac{10^k-1}{N}$ is an integer. For $k > 2$, $(10^k + 1)$ and $(10^k - 1)$ are both greater than 99, so we get a contradiction. Therefore $k = 2$, and $N = 101$ must be the only prime of the desired form.

1988. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

Solution: (sketch omitted)
Upon sketching the region, we see that it is the area inside the polygon with vertices $(1, 0), (2, 1), (-2, 1), (-1, 0), (-2, -1)$, and $(2, -1)$. This region can be divided into two congruent trapezoids with height 1 and base lengths 2 and 4, so their combined area is 6.

1987. Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\}, \\ B &= \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\}, \\ C &= \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x, y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

Solution:
Let $z = x + iy$. The equations defining A and B are the real and imaginary parts of the equation $z^2 = z^{-1} + 3i$, and similarly the equations defining C and D are the real and imaginary parts of the equation $z^3 - 3iz = 1$. Hence for all real x and y, we have

$$(x, y) \in A \cap B \Longleftrightarrow z^2 = z^{-1} + 3i \Longleftrightarrow z^3 - 3iz = 1 \Longleftrightarrow (x, y) \in C \cap D.$$

Thus $A \cap B = C \cap D$.

1986. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

Solution:

Notice that $x^4 + 36 \leq 13x^2$ is equivalent to $x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9) \leq 0$, which holds exactly when $x \in [-3, -2] \cup [2, 3]$. Also, $f'(x) = 3x^2 - 3$ is positive on both of these intervals, so that $f(x)$ is increasing there. Thus, the maximum of $f(x)$ in this region is $\max\{f(-2), f(3)\} = \max\{-2, 18\} = 18$.

1985. Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

Solution:

Each element of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ must belong to exactly one of $A_1 \cap A_2, A_1 \cap A_3, A_2 \cap A_3, A_3 \setminus (A_1 \cup A_2), A_2 \setminus (A_1 \cup A_3)$, and $A_1 \setminus (A_2 \cup A_3)$. Furthermore, every such distribution is possible, and each uniquely determines the triple (A_1, A_2, A_3). Thus, there are $6^{10} = 2^{10} 3^{10} 5^0 7^0$ such triples.

Putnam problems and solutions B1

2006. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

Solution:
The "curve" $x^3 + 3xy + y^3 - 1 = 0$ is actually reducible, because the left side factors as

$$(x + y - 1)(x^2 - xy + y^2 + x + y + 1).$$

Moreover, the second factor is

$$\frac{1}{2}((x + 1)^2 + (y + 1)^2 + (x - y)^2),$$

so it only vanishes at $(-1, -1)$. Thus the curve in question consists of the single point $(-1, -1)$ together with the line $x+y = 1$. To form a triangle with three points on this curve, one of its vertices must be $(-1, -1)$. The other two vertices lie on the line $x + y = 1$, so the length of the altitude from $(-1, -1)$ is the distance from $(-1, -1)$ to $(1/2, 1/2)$, or $3\sqrt{2}/2$. The area of an equilateral triangle of height h is $h^2\sqrt{3}/3$, so the desired area is $3\sqrt{3}/2$.

2005. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a\rfloor, \lfloor 2a\rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu\rfloor$ is the greatest integer less than or equal to ν.)

Solution:
If $a - \lfloor a\rfloor < 0.5$, then $\lfloor 2a\rfloor = 2\lfloor a\rfloor$. Otherwise, $\lfloor 2a\rfloor = 2\lfloor a\rfloor + 1$. Thus $P(\lfloor a\rfloor, \lfloor 2a\rfloor) = P(\lfloor a\rfloor, 2\lfloor a\rfloor)$ or $P(\lfloor a\rfloor, 2\lfloor a\rfloor + 1)$, so $P(x, y) = (2x - y)(2x + 1 - y)$ is a nonzero polynomial such that $P(\lfloor a\rfloor, \lfloor 2a\rfloor) = 0$ for all real numbers a.

2004. Let $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_nr,\ c_nr^2 + c_{n-1}r,\ c_nr^3 + c_{n-1}r^2 + c_{n-2}r,$$
$$\ldots,\ c_nr^n + c_{n-1}r^{n-1} + \cdots + c_1r$$

are integers.

Solution:
Let k be an integer, $0 \le k \le n - 1$. Since $P(r)/r^k = 0$, we have

$$\begin{aligned} &c_nr^{n-k} + c_{n-1}r^{n-k+1} + \cdots + c_{k+1}r \\ &\quad = -(c_k + c_{k-1}r^{-1} + \cdots + c_0r^{-k}). \end{aligned}$$

Write $r = p/q$ where p and q are relatively prime. Then the left hand side of the above equation can be written as a fraction with denominator q^{n-k}, while the right hand side is a fraction with denominator p^k. Since p and q are relatively prime, both sides of the equation must be an integer, and the result follows.

2003. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Solution:

No, there do not. Suppose such polynomials do exist. By setting $y = -1, 0$, and 1, we see that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are linear combinations of $a(x)$ and $b(x)$. However, these three polynomials are linearly independent, so cannot all be written as linear combinations of two other polynomials, a contradiction.

2002. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

How to solve it: Can we solve a more accessible related problem? A more general problem? When asked to find something involving a fixed integer (like 100, for example), we should often consider trying to solve the problem for general n. When given two fixed integers (such as 50 and 100), we should consider trying to solve for general k and n.

Solution:

Let $P(n, k)$ be the probability that after n shots, Shanille will have made k of them. After computing a few values, we are soon led to conjecture that $P(n, k) = \frac{1}{n-1}$ for all $1 \leq k \leq n-1$ and all $n \geq 2$, which we now prove by induction. Notice $P(2, 1) = 1$ by the conditions of the problem. Assume as our induction hypothesis that $P(m, k) = \frac{1}{m-1}$ for all $1 \leq k \leq m - 1$. The probability of Shanille making k shots after $m + 1$ attempts is the probability of her making k out of m and then missing plus the probability of her making $k - 1$ out of m and then hitting. Thus

$$\begin{aligned} P(m+1, k) &= P(m, k)\frac{m-k}{m} + P(m, k-1)\frac{k-1}{m} = \frac{1}{m-1}\left(\frac{m-k}{m}\right) + \frac{1}{m-1}\left(\frac{k-1}{m}\right) \\ &= \frac{1}{m-1}\left(\frac{m-k+k-1}{m}\right) = \frac{1}{m} \end{aligned}$$

as desired. Thus, our conjecture holds, and $P(100, 50) = \frac{1}{99}$.

2001. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution:

Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n+g(s)+1$ denote the number written in square s, where $0 \leq f(s), g(s) \leq n-1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s \in R} f(s)n + g(s) + 1 = \sum_{s \in B} f(s)n + g(s) + 1,$$

as desired.

2000. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Solution:

Since at least one of a_j, b_j, c_j is odd, the parity of $ra_j + sb_j + tc_j$ changes if we change the parity of any r, s, or t that is coupled with an odd value. Thus, for each j, if we allow r, s, and t to range over the eight triples (r, s, t) with $r, s, t \in \{0, 1\}$, half of the sums $ra_j + sb_j + tc_j$ are even and half are odd. The sum with $r = s = t = 0$ is even, so at least four of the seven triples with r, s, t not all zero yield an odd sum. In other words, at least $4N$ of the $7N$ 4-tuples (r, s, t, j) with r, s, t not all zero yield odd sums. By the pigeonhole principle, there is a triple (r, s, t) for which at least $4N/7$ of the sums are odd.

1999. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

Solution:
The answer is $1/3$. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta\to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3.$$

1998. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)}$$

for $x > 0$.

Solution:
Let $y = x + 1/x$ and $z = x^3 + 1/x^3$. Thus

$$\begin{aligned}\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)} &= \frac{y^6 - z^2}{y^3+z} = \frac{y^6 - z^2}{y^3+z}\frac{y^3-z}{y^3-z} \\ &= \frac{(y^6-z^2)(y^3-z)}{y^6-z^2} = y^3 - z \\ &= (x+1/x)^3 - (x^3+1/x^3) = 3(x+1/x).\end{aligned}$$

The latter is easily seen (using the AM-GM inequality or calculus) to have minimum value 6 (achieved at $x = 1$).

1997. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

Solution:

It is trivial to check that $\frac{m}{6n} = \{\frac{m}{6n}\} \le \{\frac{m}{3n}\}$ for $1 \le m \le 2n$, that $1 - \frac{m}{3n} = \{\frac{m}{3n}\} \le \{\frac{m}{6n}\}$ for $2n \le m \le 3n$, that $\frac{m}{3n} - 1 = \{\frac{m}{3n}\} \le \{\frac{m}{6n}\}$ for $3n \le m \le 4n$, and that $1 - \frac{m}{6n} = \{\frac{m}{6n}\} \le \{\frac{m}{3n}\}$ for $4n \le m \le 6n$. Therefore the desired sum is

$$\sum_{m=1}^{2n-1} \frac{m}{6n} + \sum_{m=2n}^{3n-1} \left(1 - \frac{m}{3n}\right) + \sum_{m=3n}^{4n-1} \left(\frac{m}{3n} - 1\right) + \sum_{m=4n}^{6n-1} \left(1 - \frac{m}{6n}\right).$$

Rewriting each sum as the number of terms times its average term, we have

$$\frac{2n-1}{6} + \frac{n+1}{6} + \frac{n-1}{6} + \frac{2n+1}{6} = n.$$

1996. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Solution:

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let f_n denote the number of minimal selfish subsets of $[n]$. Then the number of minimal selfish subsets of $[n]$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish subset of $[n]$ containing n, by subtracting 1 from each element, and then taking away the element $n-1$ from the set, we obtain a minimal selfish subset of $[n-2]$ (since 1 and n cannot both occur in a selfish set). Conversely, any minimal selfish subset of $[n-2]$ gives rise to a minimal selfish subset of $[n]$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $[n]$ containing n is f_{n-2}. Thus we obtain $f_n = f_{n-1} + f_{n-2}$. Since $f_1 = f_2 = 1$, we have $f_n = F_n$, where F_n denotes the nth term of the Fibonacci sequence.

1995. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Solution:

For a given π, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct partitions of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.

1994. Find all positive integers n that are within 250 of exactly 15 perfect squares.

Solution:

This happens if and only if, for some m, $(m-1)^2+1 \leq n-250 \leq m^2$ and $(m+14)^2 \leq n+250 \leq (m+15)^2-1$. Flipping the first string of inequalities and subtracting it from the second, we have $28m+196 \leq 500 \leq 32m+222$, which implies $m=9$ or 10. For $m=9, 8^2+1 \leq n-250$ and $n+250 \leq (24)^2-1$ imply $315 \leq n \leq 325$. For $m=10, 9^2+1 \leq n-250$ and $n-250 \leq (10)^2$ imply $332 \leq n \leq 350$. Thus, n is within 250 of exactly 15 perfect squares if and only if $315 \leq n \leq 325$ or $332 \leq n \leq 350$.

1993. Find the smallest positive integer n such that for every integer m with $0<m<1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Solution:

The smallest such $n=3987$.

Lemma: If $a,b,c,d \in \mathbb{R}^+$ and $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Proof of Lemma: Notice $\frac{a}{b} < \frac{c}{d}$ implies $ad<bc$. Thus, $ab+ad<ab+bc$, so that $\frac{a}{b} < \frac{a+c}{b+d}$. Also, $ad+cd<bc+cd$, so that $\frac{a+c}{b+d} < \frac{c}{d}$.

By our Lemma,

$$\frac{m}{1993} < \frac{2m+1}{3987} < \frac{m+1}{1994},$$

so that the smallest such $n \leq 3987$. Also, if $\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994}$, then $\frac{1}{1993} > \frac{n-k}{n} > \frac{1}{1994}$, and so $1993 < \frac{n}{n-k} < 1994$. From this we see that $n-k \neq 1$, so $n-k \geq 2$, which implies that $3986 < n$.

1992. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

Solution:

The smallest possible number of elements in A_S is $2n-3$.

Proof: To see that the smallest possible number of elements in A_S is $\leq 2n-3$, observe that for any n, if $S=\{0,2,\ldots,2n-2\}$, then $A_S=\{1,2,\ldots,2n-3\}$, which has $2n-3$ elements. To prove that the smallest possible number of elements in A_S is $\geq 2n-3$, we proceed by induction on n. For $n=2$, $|A_S|=1 \geq 2(2)-3$. Assume that the smallest possible number of elements in A_S is at least $2k-3$ for any set S of k distinct real numbers. Let S' be a set of $k+1$ distinct real numbers. Let $s_0 < s_1 < s_2$ be the three smallest elements of S'. By our induction hypothesis, $A_{S' \setminus \{s_0\}}$ has at least $2k-3$ elements. Also, $(s_0+s_1)/2$ and $(s_0+s_2)/2$ are not equal, and they are both smaller than every element of $S' \setminus \{s_0\}$. Thus, $A_{S'}$ must contain at least $2k-3+2 = 2(k+1)-3$ elements. Hence, by induction, we have shown that if S is a set of n distinct real numbers, the smallest possible number of elements in A_S is $\geq 2n-3$. Combining this with our first observation, our result follows.

1991. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

Solution:

This sequence eventually constant if and only if A is an integer square.

Proof: If $a_0 = A$ is a square, then $a_n = A$ for all n. Suppose $a_0 = A$ is not a square. If $a_k = B$ is not a square, then for some m, $(m+1)^2 > B > m^2$, so that $a_{k+1} = B + (B - m^2) \leq B + 2m < (m+2)^2$. Also, $(m+1)^2 - a_{k+1} = (m+1)^2 - 2B + m^2$ is odd, so in particular nonzero. Thus, since $m^2 < a_{k+1} < (m+2)^2$ and $a_{k+1} \neq (m+1)^2$, a_{k+1} cannot be a square. That being the case, since $a_0 = A$ is not a square, no term of $(a_k)_{k=0}^{\infty}$ is a square, and so $S(a_n) > 0$ for all n, so that $(a_k)_{k=0}^{\infty}$ is eventually not constant.

1990. Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

Solution:

Differentiating both sides of this equation, we have

$$2f(x) \cdot f'(x) = (f(x))^2 + (f'(x))^2,$$

or equivalently,

$$[f(x) - f'(x)]^2 = 0,$$

which implies $f(x) = f'(x)$. Thus, $f(x) = ce^x$ for some constant c. Setting $x = 0$ in our original equation, we have that $f(0) = \pm\sqrt{1980} = c$, so that $f(x) = \sqrt{1980}e^x$ and $f(x) = -\sqrt{1980}e^x$ are the only real-valued continuously differentiable functions that satisfy our condition.

1989. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

Solution:

Without loss of generality, assume the dartboard has corners at $(\pm 1, \pm 1)$. A point (x, y) in the square is closer to the center than to the top edge if and only if $\sqrt{x^2+y^2} \le 1-y$, which is equivalent to $x^2+y^2 \le (1-y)^2$, and to $y \le (1-x^2)/2$. This describes a region below a parabola. The region consisting of points in the board closer to the center than to any edge is the intersection of the four symmetrical parabolic regions inside the board: it is the union of eight symmetric copies of the region A bounded by $x \ge 0, y \ge x$, and $y \le (1-x^2)/2$. A short calculation shows that the bounding curves $y = x$ and $y \le (1-x^2)/2$ intersect at $(x, y) = (\sqrt{2}-1, \sqrt{2}-1)$. Thus the desired probability is

$$\frac{8\ \text{Area}(A)}{\text{Area(board)}} = 2\ \text{Area}(A) = 2\int_0^{\sqrt{2}-1} \left(\frac{1-x^2}{2} - x\right) dx = \frac{4\sqrt{2}-5}{3}.$$

1988. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

Solution:

Let $x = a-1, y = b-1$, and $z = 1$. Then $xy+xz+yz+1 = (a-1)(b-1)+(a-1)+(b-1)+1 = ab$.

1987. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

Solution:

Let

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

Making the change of variables $y = 6 - x$, we have

$$I = -\int_4^2 \frac{\sqrt{\ln(y+3)}\,dy}{\sqrt{\ln(y+3)} + \sqrt{\ln(9-y)}} = \int_2^4 \frac{\sqrt{\ln(y+3)}\,dy}{\sqrt{\ln(y+3)} + \sqrt{\ln(9-y)}}.$$

Thus

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} = \int_2^4 1\,dx = 2,$$

and so $I = 1$.

1986. Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

Solution:

The radius has length equal to $h/2$ plus the altitude of the triangle, so the altitude of the triangle is $1-h/2$. If the rectangle and the triangle have the same area, then $bh = \frac{1}{2}b(1-h/2)$. Solving for h, we have $h = 2/5$.

1985. Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

Solution:

The leading term of $p(x)$ is x^5, but $p(x) \neq x^5$ since the m_i are not all zero, so $k \neq 1$. Also, $p(x)$ must have either a nonzero constant term or a nonzero x^1 term (or both), because otherwise $x^2|p(x)$, which would imply that two of the m_i are zero (and thus not distinct).

Suppose $k = 2$. Then $p(x) = x^5 + c$ or $p(x) = x^5 + cx$. The roots of $x^5 + c$ are the fifth roots of c, but there is only one real fifth root of c, and we need all of the roots to be integers, so $p(x) \neq x^5 + c$. The roots of $x^5 + cx$ are zero and the fourth roots of c, but there are at most two real fourth roots of c, so $p(x) \neq x^5 + cx$. Thus, $k \neq 2$.

Setting $m_1 = -2, m_2 = 2, m_3 = -1, m_4 = 1$, and $m_5 = 0$, we have $p(x) = (x^2 - 4)(x^2 - 1)x = x^5 - 5x^3 + 4x$, for which $k = 3$.