Math 194, problem set #7

For discussion Tuesday November 27

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Prove for $n \geq 1$ that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}. \qquad \text{(Engel)}$$

2. Let $a_1/b_1, a_2/b_2, \ldots, a_n/b_n$ be n fractions with $b_i > 0$ for $i = 1, 2, \ldots, n$. Show that the fraction
$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$
is a number between the largest and smallest of these fractions.
(Larson 7.1.10)

3. Prove that $\sqrt[n]{n} < 1 + \sqrt{2/n}$ if n is a positive integer. (Larson 7.1.15)

4. Prove that for every integer $n \geq 2$,
$$\prod_{k=1}^{n} \binom{n}{k} \leq \left(\frac{2^n - 2}{n-1} \right)^{n-1} \qquad \text{(Storey)}$$

5. Prove that if $a_1, \ldots, a_n$ are real numbers and $a_1 + \cdots + a_n = 1$, then
$$a_1^2 + \cdots + a_n^2 \geq 1/n.$$

6. Prove that the sequence $\{a_n\}$ defined by
$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n)$$
converges.

7. Show that for all x,

$$1 + x + \frac{x^2}{2!} + \cdots \frac{x^{2n}}{(2n)!} > 0.$$

8. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

(Putnam, 1998)

9. Prove that for every positive n, $n! > (n/e)^n$. (Larson 7.1.12)

10. Prove that

$$\left(\frac{a+1}{b+1}\right)^{b+1} > \left(\frac{a}{b}\right)^{b}$$

for every $a, b > 0$, $a \neq b$. (Larson 7.4.17)

11. Find all positive integers $n, k_1, k_2, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

(Putnam 2005)