Math 194

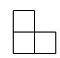
For discussion Oct 16, 2012

You are *strongly urged* to write up and hand in a careful and complete solution to (at least) one of these problems.

1. Show that every positive integer can be written as a sum of distinct Fibonacci numbers.

2. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, ..., \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle. (Putnam, 2000)

3. Show that for every $n \geq 1$, a $2^n \times 2^n$ checkerboard with a single 1×1 corner square removed can be covered by pieces of the form

   (a 2×2 square with a 1×1 corner removed). (Gelca & Andreescu)

4. Prove for every positive integer n that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

5. Prove that for some $n > 0$, the Fibonacci number F_n is divisible by $1,000,000$. (Irish Mathematical Olympiad 1999)

6. Show that every convex polyhedron has 2 faces with the same number of edges. (Moscow Mathematical Olympiad)

7. Seventeen people correspond by email with one another (each one with all 16 others). A total of 3 different topics are discussed, in all the emails, and each pair of correspondents discusses only one of the topics. Prove that there at least 3 people who write to each other about the same topic. (Larson 2.6.11)

8. Let $I_n = \int_0^{2\pi} \sin^n(x)dx$. Find a recurrence relation for I_n. Using this recurrence, what is the value of I_n? (Larson 2.5.15)

9. Suppose there are given a set of 25 points in the plane, such that among any three there exists a pair that are distance less than 1 apart. Prove that there is a circle of radius 1 that contains at least 13 of the given points.

10. Show that if more than half of the subsets of an n-element set are selected, then two of the selected subsets have the property that one is a subset of the other.