Math 194, problem set #9

For discussion Tuesday December 1

(1) Let
$$T_0 = 2, T_1 = 3, T_2 = 6,$$
and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are
$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$
Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences. (Putnam 1990)

(2) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that
$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$
holds identically? (Putnam 2003)

(3) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$. (Putnam 2000)

(4) Find the minimum value of
$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$
for $x > 0$. (Putnam 1998)

(5) Evaluate
$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$
(Putnam 1987)

(6) Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area? (Putnam 1986)

(7) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that
$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$
(Putnam 2003)