Math 194, problem set #7

For discussion Tuesday November 17

(1) Let $0 < x_i < \pi$, $i = 1, ..., n$ and set $x = (x_1 + \cdots + x_n)/n$. Prove that

$$\prod_{i=1}^{n} \left(\frac{\sin x_i}{x_i} \right) \leq \left(\frac{\sin x}{x} \right)^n. \qquad \text{(Putnam, 1978)}$$

(2) If a, b, c are positive real numbers, show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

(3) For every positive integer n, show that

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}} < 2.$$

(4) If $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ is an injective function, then for every n

$$\sum_{k=1}^{n} \frac{f(k)}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

(5) Prove that for every positive integer n

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}.$$

(6) Let $f(x)$ be a function such that $f(1) = 1$ and for $x \geq 1$

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is less than $1 + \frac{\pi}{4}$. (Putnam, 1947)

(7) Show that if $\epsilon(n) = 1/n$, then for every $n \geq 1$,

$$2\sqrt{n^2+n} - 2n - \epsilon(n) < \sum_{i=1}^{n} \frac{1}{\sqrt{n^2+i}} < 2\sqrt{n^2+n} - 2n.$$

Can you improve the "error term" $\epsilon(n)$?