Omitting types in C^* algebras

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C* algebras

- In this talk, H denotes a complex Hilbert space. A linear operator $T : B(H) \to B(H)$ is *bounded* if $\|T\| < \infty$, where

$$\|T\| := \sup\{\|Tx\| \ : \ \|x\| = 1\}.$$

- A *concrete C* algebra* is a $*$-subalgebra of $B(H)$ that is closed in the operator norm topology.
- An *abstract C* algebra* is a Banach $*$-algebra A satisfying the C* equality: $\|T^*T\| = \|T\|^2$ for all $T \in A$.
- It is not hard to see that every concrete C* algebra is an abstract C* algebra. Conversely, it can be shown that every abstract C* algebra admits a faithful representation as a norm closed subalgebra of some $B(H)$, so these really are the same notion.

Examples of C* algebras

- $B(H)$ (boring!)
- If X is a compact topological space, then $C(X)$ is a *unital, commutative* C^* algebra when equipped with the sup-norm. By Gelfand theory, every unital commutative C* algebra is isomorphic to $C(X)$ for some compact topological space X, whence C* algebra theory is sometimes dubbed *noncommutative topology*.
- If Γ is a discrete group and one considers the unitary representation $l : \Gamma \to U(\ell^2(\Gamma))$ given by $(l(\gamma)(f))(\eta) := f(\gamma^{-1}\eta)$, the *left-regular representation*, then the C* algebra generated by $l(\Gamma)$ inside of $B(\ell^2(\Gamma))$ is called the *reduced group* C^* *algebra of* Γ, denoted $C^*_r(\Gamma)$.
- Many, many, many more interesting and exotic examples.

$\mathcal{O}_2$

- An *isometry* is a bounded operator $T : H \to H$ that is norm preserving (but is not necessarily onto); equivalently, $T^*T = I$.
- (Made-up terminology) Let us call a pair of isometries $T_1, T_2 \in B(H)$ a *Cuntz pair* if $H = T_1(H) \oplus T_2(H)$.
- The *Cuntz algebra* $\mathcal{O}_2$ is the universal C* algebra generated by a Cuntz pair. So $\mathcal{O}_2$ is generated by a Cuntz pair (T_1, T_2) of isometries and if A is any C* algebra that is generated by a Cuntz pair (S_1, S_2), then there is a *-homomorphism $\mathcal{O}_2 \to A$ sending T_i to S_i for $i = 1, 2$. (It is will actually be an embedding.)
- $\mathcal{O}_2$ is a very interesting C* algebra for many reasons. It has played a crucial role in the classification programme for (simple, separable, nuclear) C* algebras.

Continuous logic-The case of C* algebras

- Rather than define the syntax in general, we will just focus on the case of C* algebras.
- Atomic formulae: $\varphi(\vec{x}) := \|p(\vec{x})\|$, $p(\vec{x})$ a *polynomial (over $\mathbb{C}$).
- Quantifier-free formulae: $\varphi(\vec{x}) := f(\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x}))$, each φ_i atomic, $f : \mathbb{R}^n \to \mathbb{R}$ continuous.
- Quantifiers: If φ is a formula, then so are $\sup_x \varphi$ and $\inf_x \varphi$.
- If A is a C* algebra, $\varphi(\vec{x})$ is a formula, and $\vec{a}$ a tuple from A, then $\varphi(\vec{a})^A$ is a real number.
- For example, $\varphi(x) := \sup_y \|xy - yx\|$ is a formula. If A is a C* algebra and $a \in A$, then $\varphi(a)^A = 0$ if and only if a is in the *center of A*.

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The Omitting Types Theorem

Theorem

Suppose that $\mathcal{K}$ is a class of separable C^* algebras for which there exists a family of existential formulae $(\varphi_{m,n}(\vec{x}_n))$ such that a separable C^* algebra A belongs to $\mathcal{K}$ if and only if, for all m, we have

$$\left(\sup_{\vec{x}_n} \inf_n \varphi_{m,n}(\vec{x}_n)\right)^A = 0.$$

Further suppose that: for each satisfiable condition $p(\vec{x})$, each m, and each $\epsilon > 0$, there is a C^* algebra A and a tuple $\vec{a}$ from A such that $p(\vec{a})$ holds and $(\inf_n \varphi_{m,n}(\vec{a}))^A < \epsilon$. (*) Then there is a separable existentially closed C^* algebra A that belongs to $\mathcal{K}$.

Note the existence of such an A implies (*).

The case of nuclearity

- When $\mathcal{K}$ is the class of nuclear C* algebras, we call (*) the existence of *good nuclear witnesses*: for every satisfiable condition $p(\vec{x})$ and every $\epsilon > 0$, there is a C* algebra A and a tuple $\vec{a}$ from A that satisfies $p(\vec{x})$ and for which $\Delta_{\text{nuc}}(\vec{a}) < \epsilon$.

Theorem (G.-Sinclair)

There is an existentially closed nuclear C* algebra if and only if Kirchberg's embedding problem (KEP) has a positive solution, namely every C* algebra embeds into an ultrapower of $\mathcal{O}_2$.

Corollary

KEP is equivalent to the existence of good nuclear witnesses.

Continua

Definition

A *continuum* is a connected, compact space.

- Bankston studied the model-theory of continua in an unusual way that was supposed to be “dual” to ordinary model theory.
- It is more natural to study the class $C(X)$ for X a continuum as an axiomatizable class of C* algebras as they are simply the projectionless abelian C* algebras.
- From now on, we always work relative to this class.

The pseudo-arc

Definition

Let X be a continuum. We say that X is:

1. *indecomposable* if X cannot be written as the union of two proper subcontinua;
2. *hereditarily indecomposable* if every proper subcontinuum of X is indecomposable;
3. *chainable* if, for any finite open cover $U_1, \ldots, U_n$ of X, there is a refinement $V_1, \ldots, V_m$ with the property that $V_i \cap V_j = \emptyset$ if and only if $|i - j| > 1$.

Definition

The *pseudo-arc* $\mathbb{P}$ is the unique chainable, hereditary indecomposable, metrizable continuum.

Bankston's Question

Theorem

If X is a continuum for which $C(X)$ is existentially closed, then X is hereditarily indecomposable.

Question

Is $C(\mathbb{P})$ existentially closed?

Theorem (Eagle-G.-Vignati)

$C(\mathbb{P})$ is existentially closed.

Arveson's Extension Theorem

Theorem

Let A be a C^* algebra, E an operator system contained in A, and $\phi : E \to \mathcal{B}(H)$ a u.c.p. map. Then there is a u.c.p. map $\psi : A \to \mathcal{B}(H)$ extending ϕ.

Question

Is there a *finitary version* of Arveson Extension? More precisely: Is it true that given an operator system $E \subseteq M_n$, $k \in \mathbb{N}$, and $\epsilon > 0$, there exists $l \in \mathbb{N}$ and $\delta > 0$ such that, for any unital map $\phi : E \to \mathcal{B}(H)$ with $\|\phi\|_l < 1 + \delta$, there is a unital map $\psi : M_n \to X$ with $\|\psi\|_k < 1 + \epsilon$ and $\|\psi|E - \psi\| < \epsilon$?

FAE is false

Let FAE denote the statement that the finitary version of Arveson Extension mentioned above is true.

Theorem (G.-Sinclair; Ozawa)

FAE is false.

Using work of Choi-Effros, it turns out that FAE is equivalent to $\mathcal{B}(H)^\omega$ having WEP. So:

Corollary

$\mathcal{B}(H)^\omega$ *does not have WEP.*

Sketch of the proof

- We first show that if FAE held, then the class of 1-exact operator systems is omitting types.
- We then consider two spaces: $\mathfrak{M}_n$ is the space of "codes" for n-dimensional operator systems, equipped with the logic topology; and $\mathcal{OS}_n$, the space of (isomorphism classes of) n-dimensional operator systems, equipped with its weak topology. Both are Polish spaces.
- There is a "forgetful" map $F : \mathfrak{M}_n \to \mathcal{OS}_n$.
- Let $\mathcal{E}_n$ denote the 1-exact elements of $\mathcal{OS}_n$. We show that the class of 1-exact operator systems being omitting types implies that $F^{-1}(\mathcal{E}_n)$ is G_δ, so Polish.
- By the Open Mapping Theorem for Polish spaces, we get that $\mathcal{E}_n$ is weakly Polish. But this contradicts fundamental work of Junge and Pisier.

A model-theoretic application

Definition

$\mathbb{GS}$ is the unique operator system with the property that, given any finite-dimensional 1-exact operator spaces $E \subseteq F$, $\epsilon > 0$, and unital complete isometry $\phi : E \to \mathbb{GS}$, there is an injective linear map $\psi : F \to \mathbb{GS}$ extending ϕ with $\|\psi\|_{\text{cb}} \cdot \|\psi^{-1}\|_{\text{cb}} \leq 1 + \epsilon$.

Theorem (G. and Lupini)

$\mathbb{GS}$ does not have quantifier-elimination. Consequently, the class of existentially closed operator systems is not axiomatizable.

The analog of the last statement for tracial vNas is due to myself, Hart, and Sinclair and for C* algebras is due to Eagle, Farah, Kirchberg, and Vignati. The statement is unkown for operator spaces.

References

- C. Eagle, I. Farah, E. Kirchberg, A. Vignati, *Quantifier elimination in C^* algebras*, preprint.
- I. Goldbring and T. Sinclair, *On Kirchberg's Embedding Problem*, to appear in the Journal of Functional Analysis. arXiv 1404.1861