

EFFECTIVE SUBFACTOR THEORY

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ABSTRACT. We characterize when a subfactor $N \subseteq M$ is oracle computable relative to a presentation of the ambient factor M in terms of computability of the Jones basic construction, in terms of computable Pismner-Popa bases, and in terms of computability of the conditional expectation map. We illustrate our main theorem with some examples.

1. INTRODUCTION

This paper contributes to the recent literature on effective structure theory as it applies to operator algebras [F, FGH23, G21, GH, GH21]. In particular, we study in a more systematic manner the question first raised in [G21], namely the complexity (in the sense of computability theory) of a subfactor inclusion.

More precisely, suppose that $N \subseteq M$ is a finite-index inclusion of II_1 factors¹; the rough question we wish to study is how much more complicated is the inclusion $N \subseteq M$ than just the complexity contained in M itself? In this paper, we introduce three natural notions of what it means for the inclusion $N \subseteq M$ to be no more complicated than M itself and show that these three notions coincide.

Before explaining these three approaches to the complexity of a subfactor inclusion $N \subseteq M$, we must briefly explain how we are to compare its complexity to the complexity of the ambient factor M . First, the entire discussion is relative to a given **presentation** $M^\#$ of M , which is simply a countable sequence from the unit ball of M that generates a SOT-dense $*$ -subalgebra of M . With a presentation $M^\#$ of M in hand, one can then reason about M from an algorithmic viewpoint. The most immediate question is how complicated (in the sense of Turing degree) must an algorithm be in order to calculate the $*$ -algebra operations and the 2-norm of M from the presentation $M^\#$. In general, we do not expect these operations to be computable (although this is the case for the standard presentation of the hyperfinite II_1 factor \mathcal{R}). Instead, our standing assumption is that the presentation $M^\#$ of M is \mathbf{D} -computable, where \mathbf{D} is some

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¹In this note, all II_1 factors are assumed to be separably acting.

(Turing) oracle that is knowledgeable enough to be able to carry out the aforementioned calculations.

We can now rephrase our main question: if the presentation $M^\#$ of M is \mathbf{D} -computable, what does it mean to say that the subfactor inclusion $N \subseteq M$ is also \mathbf{D} -computable?

One reasonable interpretation of this question involves the basic construction of Jones, which is the II_1 factor M_1 generated by M (viewed as concretely represented on $L^2(M)$ via its standard representation) and the orthogonal projection $e_N : L^2(M) \rightarrow L^2(N)$. The presentation $M^\#$ of M induces a natural presentation $M_1^\#$ of M_1 obtained by simply adding e_N to the presentation. One can then interpret our rough question from above as: when is $M_1^\#$ also \mathbf{D} -computable? (From the definition, $M_1^\#$ must be at least as complicated as $M^\#$.)

Another perspective on the question comes from considering Pimsner-Popa bases for the inclusion. A Pimsner-Popa basis for the inclusion $N \subseteq M$ leads to an elegant description of M as a finitely generated projective right N -module. It is thus reasonable to suspect that the subfactor is \mathbf{D} -computable if it has a \mathbf{D} -computable (with respect to $M^\#$) Pimsner-Popa basis and such that the inclusion of N^\dagger into $M^\#$ is \mathbf{D} -computable with respect to some presentation N^\dagger of N . Our main theorem asserts that (given a precise notion of \mathbf{D} -computable Pimsner-Popa basis), this notion coincides with the \mathbf{D} -computability of $M_1^\#$.

One final perspective on the \mathbf{D} -computability of subfactors comes from considering the \mathbf{D} -computability of the conditional expectation $E_N : M^\# \rightarrow M^\#$. It turns out that we were not able to show that this is equivalent to the previous two notions on its own but is equivalent to the previous two notions when combined with the assumption that the index $[M : N]$ is a \mathbf{D} -computable real number. In certain instances (such as when $[M : N] \leq 4$ or when M has finite depth), the index is simply computable and thus the \mathbf{D} -computability of E_N is equivalent to the previous two notions. We consider some examples of our main theorem in action in the final section of the paper.

It follows from our main result that if $M_1^\#$ is \mathbf{D} -computable, then so are the iterated basic constructions $M_n^\#$ comprising the Jones tower.

Two of the three equivalent conditions in our main theorem involve a conjunction of two statements. It is natural to wonder if either of those conjunctions can be simplified to one of their conjuncts. We conclude the paper with two partial results along these lines. We show that if E_N is \mathbf{D} -computable, then $M_1^\#$ is \mathbf{D}' -computable (so $M_1^\#$ is at most one “jump” of complexity higher than $M^\#$). We also show that if there is some presentation N^\dagger of N such that the inclusion $N^\dagger \hookrightarrow M^\#$ is \mathbf{D} -computable, then $M_1^\#$ is \mathbf{D}'' -computable (so at most two “jumps”

of complexity higher). It would be interesting to see if these jumps are actually necessary.

In future work, we plan on studying effective aspects of other objects related to a subfactor, such as its standard invariant and derived tower.

We assume that the reader is familiar with basic facts about tracial von Neumann algebras but not necessarily basic subfactor theory; we present what we need about subfactor theory in Section 4. We also assume that the reader is familiar with very basic computability theory; the uninitiated reader can consult [G21, Section 2], where it was presented with an eye towards the issues that concern us here. Nevertheless, we do discuss the basic notions concerning presentations of tracial von Neumann algebras in the next section.

We thank David Penneys for many useful discussions regarding this work.

2. GENERALITIES ON PRESENTATIONS

In this section, we discuss the notion of a presentation of a tracial von Neumann algebra and collect some basic facts to be used throughout the paper.

Definition 2.1. Given a tracial von Neumann algebra M , a **presentation** of M is a countable sequence $(a_n)_{n \in \mathbb{N}}$ of elements of the unit ball of M that generate a (SOT-) dense $*$ -subalgebra of M . The elements of the presentation are sometimes called the **special points** of the presentation.

We use letters like $M^\#$ and M^\dagger to denote presentations of M .

Definition 2.2. Given a presentation $M^\#$ of a tracial von Neumann algebra, the set of elements of M obtained by closing the elements of the presentation under addition, multiplication, adjoint, and multiplication by elements of $\mathbb{Q}(i)$ are referred to as the **rational points** of $M^\#$.

Throughout, we let $\mathcal{F}(k) = C^*\langle c_1, \dots, c_k : \|c_i\| \leq 1 \rangle$ be the universal contraction C^* -algebra on k generators.

Given a presentation $M^\#$ of M , there is an associated computable enumeration $(x_n)_{n \in \mathbb{N}}$ of the rational points of the presentation. Moreover, there is a computable functional \cdot^b on the rational points which computes a naïve upper bound on the operator norm. Specifically, if $x_n = p(a_1, \dots, a_k)$ is a rational point (so a_1, \dots, a_k are special points of $M^\#$ and $p(w_1, \dots, w_k)$ is a $*$ -polynomial with coefficients from $\mathbb{Q}(i)$), then we set

$$\begin{aligned} x_n^b &:= \|p(c_1, \dots, c_k)\|_{\mathcal{F}(k)} \\ &= \sup\{\|p(z_1, \dots, z_k)\|_{\mathcal{A}} : \mathcal{A} \text{ is a } C^*\text{-algebra, } \|z_i\|_{\mathcal{A}} \leq 1\}. \end{aligned}$$

In particular, $\alpha_n^b = 1$ for every special point α_n of the presentation. Since the standard presentation of $\mathcal{F}(k)$ is computable by [F, Corollary 3.8], it follows that α_n^b is a computable real number, uniformly in n .

A key consequence of our set-up is the following:

Lemma 2.3. *For any rational point x of $M^\#$ with $\|x\|_\infty \leq 1$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of rational points of $M^\#$ such that $\|x_n - x\|_\infty \rightarrow 0$ with $\alpha_n^b < 1$ for all n .*

Proof. Let $x = p(a_1, \dots, a_k)$. Let $\pi : \mathcal{F}(k) \rightarrow C^*\langle a_1, \dots, a_k \rangle \subseteq M$ be the natural surjection defined by $\pi(c_i) = a_i$. By [RLL00, 2.2.10], there exists $z \in \mathcal{F}(k)$ such that $\|z\| = \|x\|_\infty \leq 1$ and $\pi(z) = x$. Let (z_n) be a sequence of rational $*$ -polynomials in c_1, \dots, c_k such that $\|z_n - z\| \rightarrow 0$ and $\|z_n\| < 1$ for all n . Let $x_n = \pi(z_n)$, so x_n is a rational point of $M^\#$ and $\alpha_n^b = \|z_n\| < 1$. It remains to note that $\|x_n - x\|_\infty = \|\pi(z_n - z)\|_\infty \leq \|z_n - z\| \rightarrow 0$. \square

Corollary 2.4. *For any $x \in M$ with $\|x\|_\infty \leq 1$, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of rational points of $M^\#$ such that $\|x_n - x\|_2 \rightarrow 0$ with $\alpha_n^b < 1$ for all n .*

Proof. By Kaplansky density, there is a sequence of rational points y_n converging to x in the 2-norm, each with $\|y_n\|_\infty \leq 1$. For each rational point y_n , by Lemma 2.3, we can choose a rational point x_n close enough with $\alpha_n^b < 1$. \square

Throughout this paper, \mathbf{D} denotes a Turing oracle.

Definition 2.5. If $M^\#$ is a presentation of M , then $x \in M$ is a **\mathbf{D} -computable point of $M^\#$** if there is a \mathbf{D} -algorithm such that, upon input $k \in \mathbb{N}$, returns a rational point $p \in M^\#$ with $\|x - p\|_2 < 2^{-k}$.

The following lemma is clear.

Lemma 2.6. *The \mathbf{D} -computable points of $M^\#$ are closed under addition, multiplication, adjoint, and multiplication by rational complex scalars.*

Definition 2.7. If $M^\#$ is a presentation of M and \mathbf{D} is an oracle, then $M^\#$ is a **\mathbf{D} -computable presentation** if there is a \mathbf{D} -algorithm such that, upon input rational point $p \in M^\#$ and $k \in \mathbb{N}$, returns a rational number q such that $\|p\|_2 - q < 2^{-k}$.

If $M^\#$ is a \mathbf{D} -computable presentation, we refer to any code for a \mathbf{D} -algorithm as in the definition as a **code for the presentation $M^\#$** .

Given a presentation $M^\#$ of M whose special points are enumerated by $(\alpha_n)_{n \in \mathbb{N}}$ and a projection $p \in M$, we let $(pM^\#p)^\#$ denote the canonical induced presentation of the compression $pM^\#p$ whose special points are $(p\alpha_n p)_{n \in \mathbb{N}}$. Note that there is a computable function which, upon input a code for a \mathbf{D} -computable

presentation $M^\#$ and a code for a \mathbf{D} -computable projection p in $M^\#$, returns a code for the canonical presentation $(pMp)^\#$. Note also that if $M^\#$ and p are \mathbf{D} -computable, then every \mathbf{D} -computable point of $(pMp)^\#$ is a \mathbf{D} -computable point of $M^\#$, uniformly in the code for the presentation and a code for the point.

The following lemma is clear:

Lemma 2.8. *There is a computable function $f_{\text{corner}} : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that if x is the code of a \mathbf{D} -computable presentation $M^\#$ of a tracial von Neumann algebra M and y is the code of a \mathbf{D} -computable projection of $M^\#$, then $f_{\text{corner}}(x, y)$ is the code of the canonical \mathbf{D} -computable presentation $(pMp)^\#$ of pMp .*

Definition 2.9. Suppose that M and N are tracial von Neumann algebras with presentations $M^\#$ and N^\dagger respectively. Further suppose that $f : M^m \rightarrow N$ is a Lipschitz map (say with respect to the maximum metric on M^m)². Then f is a **\mathbf{D} -computable map from $M^\#$ into N^\dagger** if there is a \mathbf{D} -algorithm such that, upon input a tuple of rational points $\vec{p} \in (M^\#)^m$ and $k \in \mathbb{N}$, returns a rational point $p' \in N^\dagger$ such that $\|f(\vec{p}) - p'\| < 2^{-k}$. (Here, we use an effective numbering of \mathbb{N}^m to effectively enumerate the m -tuples of rational points of $M^\#$.)

Throughout the paper, \mathbb{C} always has its standard presentation, that is, with $\mathbb{Q}(i)$ as its presentation.

3. USEFUL TECHNICAL LEMMAS

In this section, we collect a number of useful technical lemmas stating that certain types of elements (such as projections and partial isometries) can, under certain assumptions, be found effectively.

Fact 3.1. *Suppose $0 < \epsilon < 1$ and $0 < \delta < \epsilon^2/48$. Then for any tracial von Neumann algebra M and $x \in M$, if $\|x\|_\infty \leq 1$, $\|x - x^*\|_2 \leq \delta$, and $\|x - x^2\|_2 \leq \delta$, then there is a projection $p \in M$ such that $\|x - p\|_2 < \epsilon$.*

Proof. Let $z = x^*x$. Then $0 \leq z \leq 1$ and

$$\|x - z\|_2 \leq \|x - x^2\|_2 + \|x\|_\infty \|x - x^*\|_2 \leq 2\delta.$$

Hence,

$$\begin{aligned} \|z^2 - z\|_2 &\leq \|z^2 - zx\|_2 + \|zx - z\|_2 \\ &\leq \|z\|_\infty \|z - x\|_2 + \|x^*\|_\infty \|x^2 - x\|_2 \\ &< \epsilon^2/16. \end{aligned}$$

²The Lipschitz condition can be weakened to having a “computable modulus of uniform continuity” but we will not need this more general notion in this paper.

Thus, by [C75, Lemma 1.1.5], there is a projection p such that $\|z - p\|_2 \leq \epsilon/2$, whence $\|x - p\|_2 < \epsilon$. \square

Definition 3.2. Given a presentation $M^\#$ of a tracial von Neumann algebra M and rational $\epsilon \in (0, 1)$, we say a rational point x of $M^\#$ is an ϵ -**quasi-projection** if $x^b < 1$, $\|x - x^*\|_2 < \epsilon^2/48$, and $\|x - x^2\|_2 < \epsilon^2/48$.

Lemma 3.3. Fix a presentation $M^\#$ of a tracial von Neumann algebra M and rational $\epsilon \in (0, 1)$.

- (1) Every ϵ -quasi-projection is ϵ -close (in 2-norm) to an actual projection.
- (2) Every projection in M is ϵ -close (in 2-norm) to an ϵ -quasi-projection.
- (3) If $M^\#$ is \mathbf{D} -computable, then the set of ϵ -quasi-projections of $M^\#$ is \mathbf{D} -c.e.

Proof. Item (1) follows from Fact 3.1 while item (2) follows from Corollary 2.4. Item (3) is clear. \square

Lemma 3.4. Suppose that $M^\#$ is a \mathbf{D} -computable presentation of a tracial von Neumann algebra M . Then $(x, y) \mapsto \text{tr}(x^*y) : (M^\#)^2 \rightarrow \mathbb{C}$ is a \mathbf{D} -computable map, uniformly in a code for $M^\#$.

Proof. This follows from the polarization identity. \square

Lemma 3.5. Suppose that $M^\#$ is a \mathbf{D} -computable presentation of a tracial von Neumann algebra M . Then 1 is a \mathbf{D} -computable point of $M^\#$, uniformly in a code for $M^\#$.

Proof. Given $k \in \mathbb{N}$, search for a rational point x such that xx^*x is a 2^{-2k-3} -quasi-projection with $|\text{tr}(xx^*x) - 1| < 2^{-2k-3}$, where we know $x \mapsto \text{tr}(xx^*x)$ is \mathbf{D} -computable by Lemma 3.4. Such a rational point must exist by Corollary 2.4 when applied to 1 . By Fact 3.1, there is a projection p with $\|xx^*x - p\|_2 < 2^{-2k-3}$. Then

$$\begin{aligned} \|p - 1\|_2^2 &= 1 - \text{tr}(p) \\ &\leq |1 - \text{tr}(xx^*x)| + \|xx^*x - p\|_2 \text{ (by Cauchy-Schwarz)} \\ &< 2^{-2k-2}. \end{aligned}$$

Thus $\|xx^*x - 1\|_2 \leq \|xx^*x - p\|_2 + \|p - 1\|_2 < 2^{-k}$. \square

By the previous lemma, if $M^\#$ is a \mathbf{D} -computable presentation of a tracial von Neumann algebra M , then we can compute, uniformly in a code for $M^\#$, a \mathbf{D} -computable presentation of M with 1 as a special point which is \mathbf{D} -computably isomorphic to $M^\#$, uniformly in $M^\#$. As a result, with no loss of generality, **in the rest of the paper, we always assume that 1 is in fact a special point of $M^\#$.**

Corollary 3.6. If $M^\#$ is a \mathbf{D} -computable presentation of tracial von Neumann algebra M , then $\text{tr} : M^\# \rightarrow \mathbb{C}$ is a \mathbf{D} -computable map, uniformly in a code for $M^\#$.

Lemma 3.7. *There is a computable function $f_{\text{ap}} : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that if x is the code of a \mathbf{D} -computable presentation $M^\#$ of a II_1 factor M and r is the code of a \mathbf{D} -computable real number $\lambda \in [0, 1]$, then $f_{\text{ap}}(x, r, k)$ is the code of a \mathbf{D} -computable projection of $M^\#$ with trace within $1/2^k$ of λ .*

Proof. For $n \in \mathbb{N}$, let $\epsilon_n = 2^{-n-k-2}$. We effectively determine a sequence $(x_n)_{n \in \mathbb{N}}$ of rational points such that x_n is a ϵ_n -quasi-projection, $\|x_{n+1} - x_n\|_2 < \epsilon_{n+1} + \epsilon_n$ and $|\text{tr}(x_n) - \lambda| < \epsilon_n + 2 \sum_{i=1}^{n-1} \epsilon_i$.

First, search for the first ϵ_1 -quasi-projection x_1 such that $|\text{tr}(x_1) - \lambda| < \epsilon_1$. Such a rational point must exist by Corollary 2.4 since M has a projection with trace λ .

Assuming x_n has already been determined, search for the first ϵ_{n+1} -quasi-projection x_{n+1} such that $\|x_{n+1} - x_n\|_2 < \epsilon_{n+1} + \epsilon_n$ and $|\text{tr}(x_{n+1}) - \lambda| < \epsilon_{n+1} + 2 \sum_{i=1}^n \epsilon_i$. Such a rational point must exist by Corollary 2.4 since if q is one of the projections ϵ_n -close to x_n , then $|\text{tr}(q) - \lambda| < \epsilon_n + \epsilon_n + 2 \sum_{i=1}^{n-1} \epsilon_i$ by Cauchy-Schwarz, so any ϵ_{n+1} -quasi-projection ϵ_{n+1} -close to q suffices.

Now, observe that $(x_n)_{n \in \mathbb{N}}$ converges to a projection p and

$$|\text{tr}(p) - \lambda| \leq 2 \sum_{i=1}^{\infty} \epsilon_i < 2^{-k}.$$

□

Lemma 3.8. *There is a computable function $f_{\text{proj}} : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that if x is the code of a \mathbf{D} -computable presentation $M^\#$ of a II_1 factor M and r is the code of a \mathbf{D} -computable real number $\lambda \in (0, 1)$, then $f_{\text{proj}}(x, r)$ is the code of a \mathbf{D} -computable projection of $M^\#$ with trace exactly λ .*

Proof. We effectively determine a decreasing sequence $(p_n)_{n \in \mathbb{N}}$ of \mathbf{D} -computable projections of $M^\#$ such that $\text{tr}_M(p_n) \in (\lambda, \lambda + 2^{-n})$. Set $p_0 = 1$. Assuming p_n has already been determined, we find p_{n+1} . By Lemma 2.8, we can effectively find a code x_{p_n} for the \mathbf{D} -computable corner presentation $(p_n M p_n)^\#$. Find $k \in \mathbb{N}$ such that $k \geq n + 1$ and $\lambda / \text{tr}_M(p_n) + 2^{-k} < 1$. By Lemma 3.7, we can effectively find a \mathbf{D} -computable projection p_{n+1} of $(p_n M p_n)^\#$ such that

$$\text{tr}_{p_n M p_n}(p_{n+1}) \in (\lambda / \text{tr}_M(p_n), \lambda / \text{tr}_M(p_n) + 2^{-k}),$$

namely the projection determined by $f_{\text{ap}}(x_{p_n}, \lambda / \text{tr}_M(p_n) + 2^{-k-1}, k + 1)$. Thus $p_{n+1} \leq p_n$, p_{n+1} is a \mathbf{D} -computable point of $M^\#$, and $\text{tr}_M(p_{n+1}) \in (\lambda, \lambda + 2^{-(n+1)})$.

Observe

$$\|p_{n+1} - p_n\|_2^2 = \text{tr}(p_n) - \text{tr}(p_{n+1}) < 2^{-n},$$

so $(p_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of projections, whose limit we denote by p . Then p is a projection, $\text{tr}_M(p) = \lambda$, and $\|p_n - p\|_2 = \sqrt{\text{tr}(p_n) - \text{tr}(p)} < 2^{-n/2}$. Therefore p is a \mathbf{D} -computable projection of $M^\#$ of trace exactly λ . \square

Lemma 3.9. *There is a computable function $f_{\text{op}} : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that if x is the code of a \mathbf{D} -computable presentation of a II_1 factor M , r is the code for a \mathbf{D} -computable real number $\lambda \in (0, 1)$ and n is $\lfloor 1/\lambda \rfloor$, then $f_{\text{op}}(x, r, n)$ is the code for a sequence p_1, \dots, p_{n+1} of mutually orthogonal \mathbf{D} -computable projections of $M^\#$ with $\text{tr}(p_i) = \lambda$ for $i = 1, \dots, n$ and $\text{tr}(p_{n+1}) = 1 - n\lambda$.*

Proof. We effectively determine a sequence $(p_i)_{i=1}^n$ of mutually orthogonal \mathbf{D} -computable projections of $M^\#$ such that $\text{tr}(p_i) = \lambda$. By Lemma 3.8, we can effectively find a \mathbf{D} -computable projection p_1 in $M^\#$ of trace λ . Assuming p_k has already been determined, we find p_{k+1} . Let $q = 1 - \sum_{i=1}^k p_i$, so q is a \mathbf{D} -computable projection with $\text{tr}(q) \geq \lambda$ by Lemma 3.5. By Lemma 3.8, we can effectively find a \mathbf{D} -computable projection p_{k+1} in $(qMq)^\#$ with $\text{tr}_{qMq}(p_{k+1}) = \lambda / \text{tr}(q)$. It follows that p_{k+1} is orthogonal to p_i for $i = 1, \dots, k$ and $\text{tr}(p_{k+1}) = \lambda$. It remains to set $p_{n+1} = 1 - \sum_{i=1}^n p_i$. \square

As usual, we denote the fact that projections p and q are Murray von Neumann equivalent by writing $p \sim q$, while the notation $p \leq q$ signifies that $p \sim q'$ for some projection $q' \leq q$. If v is a partial isometry that satisfies $v^*v = p$ and $vv^* = q$, then we say that v is an **implement** of $p \sim q$.

A proof of the non-effective form of the following lemma is essentially contained in the proof of [C75, Proposition 1.1.3(b)].

Lemma 3.10. *Let M be a II_1 factor. Let $p, q \in M$ be projections such that $p \sim q$. Suppose $0 < \epsilon < 1$ and $0 < \delta < \epsilon^{16}/11^{16}$. Then for any $x \in M$, if $\|x\|_\infty \leq 1$, $\|x^*x - p\|_2 \leq \delta$, and $\|xx^* - q\|_2 \leq \delta$, then there is an implement $v \in M$ of $p \sim q$ with $\|x - v\|_2 < \epsilon$.*

Proof. Let $x \in M$ be as in the conclusion of the theorem and set $y := qxp$. Then

$$\|(1 - q)x\|_2^2 = \text{tr}((1 - q)(xx^* - q)(1 - q)) \leq \|xx^* - q\|_2 \leq \delta$$

and

$$\|x(1 - p)\|_2^2 = \text{tr}((1 - p)(x^*x - p)(1 - p)) \leq \|x^*x - p\|_2 \leq \delta,$$

so

$$\|x - y\|_2 \leq \|x - qx\|_2 + \|qx - y\|_2 \leq \|(1 - q)x\|_2 + \|x(1 - p)\|_2 \leq 2\delta^{1/2}.$$

Furthermore,

$$\|y^*y - p\|_2 \leq \|y^*y - y^*x\|_2 + \|y^*x - x^*x\|_2 + \|x^*x - p\|_2 \leq 5\delta^{1/2},$$

whence it follows that

$$\|(y^*y)^2 - y^*y\|_2 = \|(y^*y)^2 - y^*yp\|_2 \leq \|y^*y - p\|_2 \leq 5\delta^{1/2}.$$

Let $f : [0, 1] \rightarrow [0, 1]$ be defined by $f(a) = a^{-1/2}$ if $1 - \sqrt{5}\delta^{1/4} \leq a \leq 1$ and 0 otherwise. Set $z := yf(y^*y)$. Then z^*z and zz^* are projections satisfying

$$z^*z = y^*yf(y^*y)^2 \leq p \text{ and } zz^* = y(f^*y)^2y^* \leq q.$$

By construction, $(1 - y^*y)^2(1 - z^*z) \geq (5\delta^{1/2})(1 - z^*z)$ and $\text{tr}((y^*y)^2(1 - y^*y)^2) \leq (5)^2\delta$, whence

$$\begin{aligned} \|y(1 - z^*z)\|_2^4 &= \text{tr}(y^*y(1 - z^*z))^2 \\ &\leq \text{tr}((y^*y)^2(1 - z^*z)) \\ &\leq (5\delta^{1/2})^{-1} \text{tr}((y^*y)^2(1 - y^*y)^2(1 - z^*z)) \\ &\leq 5\delta^{1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \|y - z\|_2 &\leq \|y - yz^*z\|_2 + \|yz^*z - z\|_2 \\ &\leq \|y(1 - z^*z)\|_2 + \|y^*yf(y^*y)^2 - f(y^*y)\|_\infty \\ &\leq \sqrt[4]{5}\delta^{1/8} + \sqrt{5}\delta^{1/4} \\ &\leq 5\delta^{1/8}. \end{aligned}$$

Furthermore,

$$\|z^*z - p\|_2 \leq \|z^*z - z^*y\|_2 + \|z^*y - y^*y\|_2 + \|y^*y - p\|_2 \leq 15\delta^{1/8}.$$

Observe that $\text{tr}(q) - \text{tr}(zz^*) = \text{tr}(p) - \text{tr}(z^*z) = \|p - z^*z\|_2 \leq 15\delta^{1/8}$. Let w be a partial isometry such that $w^*w = p - z^*z$ and $ww^* = q - zz^*$. Then

$$\|w\|_2^2 = \text{tr}(p) - \text{tr}(z^*z) \leq 15\delta^{1/8},$$

and so $\|w\|_2 \leq 4\delta^{1/16}$. Finally, set $v := z + w$. Then v is an implement of $p \sim q$ and

$$\|x - v\|_2 \leq \|x - y\|_2 + \|y - z\|_2 + \|w\|_2 \leq 11\delta^{1/16} < \epsilon.$$

□

Definition 3.11. Given a presentation $M^\#$ of a II_1 factor M , projections p and q in M with $p \sim q$, and rational $\epsilon \in (0, 1)$, we say that a rational point x of $M^\#$ is an ϵ -quasi-implement of $p \sim q$ if $x^b < 1$, $\|x^*x - p\|_2 < \epsilon^{16}/11^{16}$, and $\|xx^* - q\|_2 < \epsilon^{16}/11^{16}$.

Lemma 3.12. Fix a presentation $M^\#$ of a II_1 factor M , projections p and q in M with $p \sim q$, and rational $\epsilon \in (0, 1)$.

- (1) Every ϵ -quasi-implement of $p \sim q$ is ϵ -close (in 2-norm) to an actual implement of $p \sim q$.
- (2) Every implement of $p \sim q$ is ϵ -close (in 2-norm) to an ϵ -quasi-implement of $p \sim q$.
- (3) If $M^\#$ is \mathbf{D} -computable and p, q are \mathbf{D} -computable points of $M^\#$, then the set of ϵ -quasi-implements of $p \sim q$ is \mathbf{D} -c.e.

Proof. Item (1) follows from Lemma 3.10 while item (2) follows from Corollary 2.4. Item (3) is clear. \square

Lemma 3.13. *Suppose that $M^\#$ is a \mathbf{D} -computable presentation of a II_1 factor M and p, q are \mathbf{D} -computable projections of $M^\#$.*

- (1) *If $p \sim q$, then there is a \mathbf{D} -computable implement of $p \sim q$, which can furthermore be computed from codes for $M^\#, p$ and q .*
- (2) *If $p \preceq q$, then there a \mathbf{D} -computable partial isometry v of $M^\#$ such that $v^*v = p$ and $vv^* \leq q$, which can furthermore be computed from codes for $M^\#, p$ and q .*

Proof. For (1), fix $n \in \mathbb{N}$ and set $\epsilon_n := 2^{-n}$. We effectively determine a sequence $(x_n)_{n \in \mathbb{N}}$ of rational points of $M^\#$ such that x_n is an ϵ_n -quasi-implement of $p \sim q$ and $\|x_{n+1} - x_n\|_2 < \epsilon_{n+1} + \epsilon_n$. First, search for an ϵ_1 -quasi-implement x_1 of $p \sim q$. Assuming x_n has already been determined, search for an ϵ_{n+1} -quasi-implement x_{n+1} of $p \sim q$ such that $\|x_{n+1} - x_n\|_2 < \epsilon_{n+1} + \epsilon_n$. Such a rational point must exist since if v is an implement of $p \sim q$ which is ϵ_n -close to x_n , then any ϵ_{n+1} -quasi-implement x_{n+1} of $p \sim q$ which is ϵ_{n+1} -close to v suffices. It remains to observe that $(x_n)_{n \in \mathbb{N}}$ converges to an implement of $p \sim q$.

To prove (2), by (1) it suffices to show that we can find a \mathbf{D} -computable projection $q' \leq q$ such that $p \sim q'$, or, equivalently, that $\text{tr}(q') = \text{tr}(p)$. Since p and q are \mathbf{D} -computable elements of $M^\#$, $\text{tr}(p)$ and $\text{tr}(q)$ are \mathbf{D} -computable real numbers by Corollary 3.6, whence by Lemma 3.8, we can find a \mathbf{D} -computable projection $q' \in (qMq)^\#$ whose corner trace is $\text{tr}(p)/\text{tr}(q)$, and thus $\text{tr}(q') = \text{tr}(p)$, as desired. \square

4. PRELIMINARIES ON SUBFACTOR THEORY

In this section, we present the basic facts from subfactor theory from [J83, PP86] needed in the rest of the paper. Another good reference for this material is [P94].

Throughout, by a **subfactor** we mean an inclusion of II_1 factors $N \subseteq M$. We let $e_N : L^2(M) \rightarrow L^2(N)$ (or simply e if the context is clear) denote the canonical orthogonal projection map, noting that the restriction E_N of e_N to M takes values in N ; the linear map $E_N : M \rightarrow N$ is called the **conditional expectation map** from M onto N . We note the following properties of E :

- $E_N(x) = x$ for all $x \in N$.
- $E_N(xyz) = xE_N(y)z$ for all $x, z \in N$ and $y \in M$.
- $E_N(x^*) = E_N(x)^*$ for all $x \in M$.
- $E_N(x)$ is the unique $y \in N$ such that $\text{tr}((x - y)z) = 0$ for all $z \in N$.
- $E_N(x)$ is the unique $y \in N$ such that $e_N x e_N = y e_N$.
- E_N is trace-preserving and contractive with respect to both the operator norm and the 2-norm.

An important invariant of the subfactor $N \subseteq M$ is the **index** $[M : N]$ of N in M . In order to avoid introducing modules over a II_1 factor, we give a definition of the index in terms of the conditional expectation map due to Pimsner and Popa [PP86], namely

$$[M : N]^{-1} := \max\{\lambda \in \mathbb{R}_+ : E_N(x) \geq \lambda x \text{ for all } x \geq 0\}.$$

This can alternatively be described by

$$[M : N]^{-1} := \inf \left\{ \frac{\|E_N(x)\|_2^2}{\|x\|_2^2} : x > 0 \right\}.$$

The terminology is inspired by the fact that if $L(H) \subseteq L(G)$ is the subfactor induced by a subgroup $H \leq G$, then $[L(G) : L(H)] = [G : H]$. However, unlike the case of groups, the index does not necessarily have to be an integer. A landmark result of Jones states that $[M : N]$ must always be an element of the set $\{4 \cos^2(\frac{\pi}{n}) : n = 3, 4, 5, \dots\} \cup [4, \infty]$. Moreover, all of these values can be realized as indices of subfactors of \mathcal{R} .

Associated to the subfactor $N \subseteq M$ is the **Jones basic construction**, which is the von Neumann algebra $M_1 := (M \cup \{e_N\})'' \subseteq \mathcal{B}(L^2(M))$ generated by M and e_N . We recall the following basic facts about M_1 :

- $M + Me_N M$ is a weakly dense $*$ -subalgebra of M_1 .
- M_1 is a II_1 factor if and only if $[M : N] < \infty$.

Assuming $[M : N] < \infty$, we also have:

- $[M_1 : M] = [M : N]$.
- For $x \in M$, we have $\text{tr}(x e_N) = [M : N]^{-1} \text{tr}(x)$. In particular, $\text{tr}(e_N) = [M : N]^{-1}$ and $\|x e_N\|_2^2 = [M : N]^{-1} \|x\|_2^2$ for any $x \in M$.
- $E_M(e_N) = \text{tr}(e_N) \cdot 1 = [M : N]^{-1} \cdot 1$.
- For $x \in M_1$, $y = [M : N] E_M(x e_N)$ is the unique element of M such that $x e_N = y e_N$. In particular, $M_1 = Me_N M := \text{span}\{x e_N y : x, y \in M\}$.

Since $M \subseteq M_1$ is a subfactor, one can consider the associated basic construction, which is denoted M_2 . Continuing this process leads to the **Jones tower**

$$N \subseteq M_0 := M \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots .$$

Suppose that $[M : N] < \infty$ has integer part n . Then there exist elements m_1, \dots, m_{n+1} of M with the following properties:

- $E_N(m_j^* m_k) = 0$ for all distinct $j, k \in \{1, \dots, n+1\}$.
- $E_N(m_j^* m_j) = 1$ for all $j \in \{1, \dots, n\}$.
- $E_N(m_{n+1}^* m_{n+1})$ is a projection of trace $[M : N] - n$.

(By convention, $m_{n+1} = 0$ if $[M : N] \in \mathbb{N}$.) Any such family of elements is called a **Pimsner-Popa basis of M over N** . The terminology is inspired by the fact that if m_1, \dots, m_{n+1} is a Pimsner-Popa basis of M over N , then every $x \in M$ can be written uniquely as $x = \sum_{j=1}^{n+1} m_j x_j$ with each $x_j \in N$ for $j = 1, \dots, n$ and $x_{n+1} \in E_N(m_{n+1}^* m_{n+1})N$. Other important properties of a Pimsner-Popa basis are the following:

- $\sum_{j=1}^{n+1} m_j e_N m_j^* = 1$.
- $\sum_{j=1}^n m_j m_j^* = [M : N]$.

Pimsner-Popa bases are unique in a sense made precise in [PP86]. Relevant for us is how they can be constructed: Let g_1, \dots, g_{n+1} be orthogonal projections in M_1 with $\text{tr}(g_i) = [M : N]^{-1}$ for $i = 1, \dots, n$ and $\text{tr}(g_{n+1}) = 1 - n[M : N]^{-1}$. For $j = 1, \dots, n+1$, take partial isometries $v_j \in M_1$ such that $v_j v_j^* = g_j$, $v_j^* v_j = e_N$ for $j = 1, \dots, n$, and $v_{n+1}^* v_{n+1} \leq e_N$. A Pimsner-Popa basis is then obtained by letting m_j be the unique element of M such that $v_j = v_j e_N = m_j e_N$.

Moving forward, if the inclusion $N \subseteq M$ is clear from context, we simply write e for the Jones projection e_N .

Although we will not study computability of standard invariants in this article, the notion of the standard invariant will arise in examples below. Jones' **standard invariant** consists of:

- The two towers of higher relative commutants/centralizer algebras

$$(M'_0 \cap M_n)_{n \geq 0} \text{ and } (M'_1 \cap M_{n+1})_{n \geq 0},$$

which are finite dimensional by [J83];

- the Jones projections $e_1 \in N' \cap M_1$, $e_2 \in M' \cap M_2 \subset N' \cap M_2$, $e_3 \in M'_1 \cap M_3 \subset M' \cap M_3 \subset N' \cap M_3$, etc., which satisfy the Temperley-Lieb-Jones relations (see Example 8.4 below); and
- the Markov trace tr on $M_\infty = \varinjlim M_n$ restricted to the two towers of centralizer algebras.

The standard invariant, and thus the subfactor itself, is called **finite depth** if there is a global bound on the dimensions of the centers of the higher relative commutants.

5. THE INDUCED PRESENTATION ON THE JONES BASIC CONSTRUCTION

In this short subsection, we describe how a presentation $M^\#$ of the ambient factor M of a subfactor $N \subseteq M$ naturally induces a presentation $M_1^\#$ of the Jones basic construction M_1 . We then present a few lemmas that will be used in the next section when proving the main theorem of this paper.

Definition 5.1. Fix a subfactor $N \subseteq M$ and a presentation $M^\#$ of M . The **induced presentation** $M_1^\#$ of M_1 has as its special points the special points of $M^\#$ as well as the element e .

Note that $M_1^\#$ is indeed a presentation of M_1 as the von Neumann algebra generated by the special points is contained in M_1 and contains both e and the elements from M .

Definition 5.2. Given a subfactor $N \subseteq M$ and a presentation $M^\#$ of M , the **induced presentation of N** , denoted $N^\#$, is the presentation of N whose special points are those of the form $E_N(x)$, where x is either a special point of $M^\#$ or a rational point of $M^\#$ with $x^b < 1$.

Note that $N^\#$ is indeed a presentation of N as any point in the unit ball of N is the 2-norm limit of rational points from $M^\#$ with $b < 1$ and E_N is continuous with respect to the 2-norm.

Definition 5.3. We say that a rational point of $M_1^\#$ is in **syntactic normal form** if it is of the form $a + \sum_i b_i e c_i$, where a is a rational point of $M^\#$, each b_i is a rational point of $N^\#$, and each c_i is a rational point of $M^\#$.

Lemma 5.4. *There is a computable function which takes as input a rational point of $M_1^\#$ and returns an equivalent rational point in syntactic normal form.*

Proof. For each term, apply $exe = E_N(x)e$ ($x \in M$) from the left as necessary until at most a single factor of e remains. Use the linearity of E_N and b to write the factors on the left-hand side of e as a rational point of $N^\#$. \square

Lemma 5.5. *Suppose $M^\#$ is a \mathbf{D} -computable presentation, $[M : N] < \infty$ is \mathbf{D} -computable, and $E_N : M^\# \rightarrow M^\#$ is a \mathbf{D} -computable map. Then there is a \mathbf{D} -computable algorithm (uniform in codes for $M^\#$, $[M : N]$, and E_N) which, when given a \mathbf{D} -computable point v of $M_1^\#$ and $k \in \mathbb{N}$, returns a rational point w of $M^\#$ such that $\|w - y\|_2 < 2^{-k}$, where y is the unique element of M for which $ve_N = ye_N$.*

Proof. Find a rational point x of $M_1^\#$ such that $\|x - v\|_2^2 < 2^{-2k-2}[M : N]^{-1}$. By Lemma 5.4, we can write x in syntactic normal form as $a + \sum_i b_i e c_i$. Setting $z := a + \sum_i b_i E_N(c_i)$, observe $ze = xe$ and z is a **D-computable** point of $M^\#$. Find a rational point w of $M^\#$ such that $\|w - z\|_2 < 2^{-k-1}$. Then

$$\begin{aligned} \|z - y\|_2^2 &= [M : N] \|(z - y)e\|_2^2 \\ &= [M : N] \|xe - ve\|_2^2 \\ &\leq [M : N] \|x - v\|_2^2. \end{aligned}$$

Hence $\|w - y\|_2 \leq \|w - z\|_2 + \|z - y\|_2 < 2^{-k}$.

□

6. THE MAIN THEOREM

In this section, we state and prove the main result of this paper. Before doing so, we introduce some important terminology.

Definition 6.1. Suppose that $[M : N] < \infty$ and let m_1, \dots, m_{n+1} be a Pimsner-Popa basis of M over N . Let $M^\#$ be a presentation of M . We say that m_1, \dots, m_{n+1} is a **D-computable Pimsner-Popa basis of $M^\#$ over N** if

- (1) each m_j is a **D-computable** point of $M^\#$,
- (2) each $E_N(m_j)$ is a **D-computable** point of $M^\#$, and
- (3) $E_N(m_{n+1}^* m_{n+1})$ is a **D-computable** point of $M^\#$.

Remark 6.2. This definition formalizes the properties of a particular Pimsner-Popa basis used in [G21, Proposition 2.9]. However, that proof was incorrect as it did not include the condition (3) in the previous definition. We remedy this error in the proof of our main theorem below.

Definition 6.3. Suppose that N is a subfactor of M , and N and M are equipped with presentations N^\dagger and $M^\#$ respectively. We say that N^\dagger is **D-computably embedded** in $M^\#$ if $M^\#$ is **D-computable** and the inclusion map $\iota : N^\dagger \hookrightarrow M^\#$ is **D-computable** (whence N^\dagger is **D-computable**).

Recalling the definition of the induced presentation of a subfactor (Definition 5.2), the following lemma is clear:

Lemma 6.4. *If $M^\#$ is **D-computable**, then $E_N : M^\# \rightarrow M^\#$ is **D-computable** if and only if N^\dagger is **D-computably embedded** in $M^\#$.*

We can now state our main theorem:

Theorem 6.5. *Suppose that N is a subfactor of M such that $[M : N] < \infty$. Fix a presentation $M^\#$ of M and let $M_1^\#$ be the induced presentation of the Jones basic construction M_1 . Let \mathbf{D} be a Turing oracle such that $M^\#$ is \mathbf{D} -computable. Then the following are equivalent:*

- (1) $M_1^\#$ is \mathbf{D} -computable.
- (2) The following two statements hold:
 - (a) $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable.
 - (b) $[M : N]$ is a \mathbf{D} -computable real number.
- (3) The following two statements hold:
 - (a) There is a presentation N^\dagger of N which is \mathbf{D} -computably embedded in $M^\#$.
 - (b) There is a \mathbf{D} -computable Pimsner-Popa basis of $M^\#$ over N .

Proof. We first prove the equivalence of (1) and (2). Suppose that $M_1^\#$ is \mathbf{D} -computable. Since e is a special point of $M_1^\#$, we have $[M : N]^{-1} = \text{tr}(e) = \|e\|_2^2$ is \mathbf{D} -computable, and thus so is $[M : N]$. Furthermore, $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable: given a rational point x of $M^\#$ and $k \in \mathbb{N}$, search for a rational point y of $M^\#$ such that $\|ye - exe\|_2^2 < [M : N]^{-1}2^{-2k}$, in which case

$$\|y - E_N(x)\|_2^2 = [M : N] \|(y - E_N(x))e\|_2^2 = [M : N] \|ye - exe\|_2^2 < 2^{-2k}.$$

Conversely, suppose that $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable and $[M : N]$ is \mathbf{D} -computable. Given a rational point x of $M_1^\#$, by Lemma 5.4 we can write x^*x in syntactic normal form as $a + \sum_i b_i e c_i$. Then

$$\|x\|_2^2 = \text{tr}(x^*x) = \text{tr}(a) + \sum_i \text{tr}(b_i e c_i) = \text{tr}(a) + [M : N]^{-1} \sum_i \text{tr}(b_i c_i),$$

which is \mathbf{D} -computable by Corollary 3.6 and Lemma 6.4.

We now prove that (1) implies (3). Suppose (1), equivalently (2), holds. Trivially then, $N^\#$ is \mathbf{D} -computably embedded in $M^\#$ by Lemma 6.4. To finish proving (3), we apply Lemma 3.9 to find \mathbf{D} -computable projections g_1, \dots, g_{n+1} of $M_1^\#$ with

$$\text{tr}(g_i) = [M : N]^{-1} \text{ for } i = 1, \dots, n, \quad \text{tr}(g_{n+1}) = 1 - n[M : N]^{-1}.$$

By Lemma 3.13, we effectively find \mathbf{D} -computable partial isometries v_1, \dots, v_{n+1} of $M_1^\#$ such that $v_j v_j^* = g_j$ for $j = 1, \dots, n+1$, $v_j^* v_j = e$ for $j = 1, \dots, n$, and $v_{n+1}^* v_{n+1} \leq e$. By Lemma 5.5, we can effectively find \mathbf{D} -computable points m_1, \dots, m_{n+1} of $M^\#$, where each m_j is the unique element of M that satisfies $v_j e = m_j e$. It follows that m_1, \dots, m_{n+1} is a \mathbf{D} -computable Pimsner-Popa basis of $M^\#$ over N .

Finally, we prove that (3) implies (2). Let N^\dagger be \mathbf{D} -computably embedded in $M^\#$ and let m_1, \dots, m_{n+1} be a \mathbf{D} -computable Pimsner-Popa basis of $M^\#$ over N . The

statement that $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable is [G21, Proposition 2.9], but the argument there is incorrect as it is missing the third item in our definition of a \mathbf{D} -computable Pimsner Popa basis of $M^\#$ over N . For that reason, we repeat the argument again. Fix a rational point x of $M^\#$ and $k \in \mathbb{N}$. By our assumptions, we can find rational points y_1, \dots, y_{n+1} of N^\dagger such that $\|x - y\|_2 < 2^{-k}$, where $y = \sum_{j=1}^n m_j y_j + m_{n+1} E_N(m_{n+1}^* m_{n+1}) y_{n+1}$. Then

$$\|E_N(x) - E_N(y)\|_2 \leq \|x - y\|_2 < 2^{-k},$$

where

$$E_N(y) = \sum_{j=1}^n E_N(m_j) y_j + E_N(m_{n+1}) E_N(m_{n+1}^* m_{n+1}) y_{n+1}$$

is a \mathbf{D} -computable point of $M^\#$, uniformly in y . Thus E_N is \mathbf{D} -computable. It remains to show that $[M : N]$ is \mathbf{D} -computable. However, this follows from the fact that $[M : N] = \text{tr}(\sum_{j=1}^{n+1} m_j m_j^*)$. \square

Definition 6.6. Suppose that N is a subfactor of M with $[M : N] < \infty$ and $M^\#$ is a \mathbf{D} -computable presentation of M . We say that N is a **\mathbf{D} -computable subfactor** of $M^\#$ if the equivalent conditions of Theorem 6.5 hold.

Remark 6.7. Note that statement (b) in item (2) of the main theorem automatically holds if $[M : N] \in [1, 4]$ (where it equals the computable real number $4 \cos^2(\frac{\pi}{n})$ for some integer $n \geq 3$) or if $N \subset M$ has finite depth, since then it is a cyclotomic integer [ENO05].

Example 6.8. By Theorem 6.5, if $N \subseteq M$ is a subfactor and $[M : N]$ is not \mathbf{D} -computable, then N is not a \mathbf{D} -computable subfactor of M^\dagger for any presentation M^\dagger of M . In particular, \mathcal{R} has subfactors of any possible index, but only for countably many of those subfactors is there a presentation \mathcal{R}^\dagger of \mathcal{R} for which it is a \mathbf{D} -computable subfactor. On the other hand, it is known there are uncountably many nonisomorphic subfactors $(N_\alpha)_{\alpha < 2^\omega}$ of \mathcal{R} of index 6 [BNP07].³ If we consider the standard presentation $\mathcal{R}^\#$ of \mathcal{R} , which is computable, then only countably many of the conditional expectation maps $E_{N_\alpha} : \mathcal{R}^\# \rightarrow \mathcal{R}^\#$ can be \mathbf{D} -computable, so for all but countably many $\alpha < 2^\omega$, the subfactor N_α is not a \mathbf{D} -computable subfactor of $\mathcal{R}^\#$ even though its index is computable.

We can iterate Theorem 6.5 to conclude the \mathbf{D} -computability of all the factors appearing in the Jones tower from the \mathbf{D} -computability of the basic construction. In what follows, we recursively equip the iterated basic constructions M_n with their canonical induced presentations $M_n^\#$.

³More is true: the non-isomorphic subfactors of index 6 from [BNP07] all share the same standard invariant. The article [BV15] constructed a family of subfactors at index 6, all with the same standard invariant $A_3 * D_4$, which cannot be classified by countable structures.

Corollary 6.9. *Suppose that $N \subseteq M^\#$ is a \mathbf{D} -computable subfactor. Then $M_n \subseteq M_{n+1}^\#$ is a \mathbf{D} -computable subfactor for all $n \geq 0$. In other words, if $M_1^\#$ is \mathbf{D} -computable, then $M_n^\#$ is \mathbf{D} -computable for all $n \geq 1$.*

Proof. Suppose M_{n-1} is a \mathbf{D} -computable subfactor of $M_n^\#$ for some $n \geq 0$, where we set $M_{-1} := N$. We show M_n is a \mathbf{D} -computable subfactor of $M_{n+1}^\#$. By Theorem 6.5 and the fact that $[M_{n+1} : M_n] = [M : N]$ is \mathbf{D} -computable, it suffices to show that $E_{M_n} : M_{n+1}^\# \rightarrow M_{n+1}^\#$ is \mathbf{D} -computable. By Lemma 5.4, we can write any rational point x of $M_{n+1}^\#$ in syntactic normal form as $a + \sum_i b_i e c_i$, where a and each c_i are rational points of $M_n^\#$ and each b_i is a rational point of $M_{n-1}^\#$, the presentation on M_{n-1} induced by $M_n^\#$. Then $E_{M_n}(x) = a + [M : N]^{-1} \sum_i b_i c_i$, which is a \mathbf{D} -computable point of $M_{n+1}^\#$, uniformly in x , as $M_n^\#$ and $E_{M_{n-1}} : M_n^\# \rightarrow M_n^\#$ are \mathbf{D} -computable. \square

Remark 6.10. If $n \geq 1$, then M_n has two potential induced presentations: one induced by $M_{n-1}^\#$ from the Jones basic construction and one induced by $M_{n+1}^\#$ as its subfactor. While these presentations are not the same in general, they are \mathbf{D} -computably isomorphic when $M_{n+1}^\#$ and $E_{M_n} : M_{n+1}^\# \rightarrow M_{n+1}^\#$ are \mathbf{D} -computable.

7. IMPROVEMENTS TO THE MAIN RESULT?

It becomes natural to wonder if any of the conditions in the main theorem are redundant. While we are unable to accomplish this at the moment, we do mention some partial results in this regard.

Given item (2) in Theorem 6.5 above, one may ask whether or not $E_N : M^\# \rightarrow M^\#$ being \mathbf{D} -computable automatically implies that $[M : N]$ must be \mathbf{D} -computable. This is the best we are able to do at this point:⁴

Proposition 7.1. *If $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable, then $[M : N]$ is \mathbf{D}' -computable, whence $M_1^\#$ is \mathbf{D}' -computable and a \mathbf{D}' -computable Pimsner-Popa basis of $M^\#$ over N exists.*

Proof. By the equation $[M : N]^{-1} = \inf \{ \|E_N(x)\|_2^2 \|x\|_2^{-2} : x > 0 \}$, we can compute a decreasing sequence which converges to $[M : N]^{-1}$ from \mathbf{D} . Thus $[M : N]^{-1}$ is \mathbf{D}' -computable. \square

Question 7.2. If $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable, must $[M : N]$ be \mathbf{D} -computable? In other words, is $M_1^\#$ \mathbf{D} -computable if and only if $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable?

⁴Recall that \mathbf{D}' is the **Turing jump** of \mathbf{D} , which is the Turing degree which can compute the halting problem relative to \mathbf{D} .

One might also wonder if N^\dagger being \mathbf{D} -computably embedded in $M^\#$ for some presentation N^\dagger of N implies that $N^\#$ is \mathbf{D} -computably embedded in $M^\#$, that is, if $E_N : M^\# \rightarrow M^\#$ is \mathbf{D} -computable. In general, the following is always true:

Proposition 7.3. *If N^\dagger is \mathbf{D} -computably embedded in $M^\#$, then $E_N : M^\# \rightarrow N^\dagger$ (and hence $E_N : M^\# \rightarrow M^\#$) is \mathbf{D}' -computable.*

Proof. Given a rational point x of $M^\#$ and $k \in \mathbb{N}$, search for a rational point y of N^\dagger such that $|\operatorname{tr}((x - y)z)| < 2^{-2k}$ for all rational points z of N^\dagger ; this can be done effectively with \mathbf{D}' . Let $z = (E_N(x) - y)^*$ and observe

$$\begin{aligned} \|E_N(x) - y\|_2^2 &= \operatorname{tr}((E_N(x) - y)z) \\ &= \operatorname{tr}((E_N(x) - x)z) + \operatorname{tr}((x - y)z) \\ &= \operatorname{tr}((x - y)z) \\ &< 2^{-2k}. \end{aligned}$$

□

Remark 7.4. The main result of [G21] shows that, under certain extra assumptions, the fact that N^\dagger is \mathbf{D} -computably embedded in $M^\#$ implies that $E_N : M^\# \rightarrow N^\dagger$ is \mathbf{D} -computable. The context of that paper is quite different as the extra hypotheses there imply that the index is necessarily infinite.

Propositions 7.1 and 7.3 immediately imply:

Corollary 7.5. *If N^\dagger is \mathbf{D} -computably embedded in $M^\#$, then $[M : N]$ is always \mathbf{D}'' -computable (whence $M_1^\#$ is \mathbf{D}'' -computable and a \mathbf{D}'' -computable Pimsner-Popa basis of $M^\#$ over N exists).*

8. EXAMPLES

In this final section, we mention a few examples illustrating our main result.

Example 8.1. Let N be any II_1 factor and let $M := M_n(N) = N \bar{\otimes} M_n(\mathbb{C})$. Fix a presentation $N^\#$ of N and let $M^\#$ be the presentation of M given by matrices of rational points of $N^\#$. Note that if $N^\#$ is \mathbf{D} -computable, then so is $M^\#$. Then $[M : N] = n^2$ is clearly computable and the conditional expectation map $E_N : M^\# \rightarrow M^\#$ mapping a matrix to its diagonal part is clearly \mathbf{D} -computable. Thus N is a \mathbf{D} -computable subfactor of $M^\#$.

Example 8.2. Suppose that N is a II_1 factor and G is a finite group. Consider an action $G \curvearrowright^\alpha N$ by outer automorphisms. Set $M := N \rtimes_\alpha G$. Fix a presentation $N^\#$ of N and suppose that $N^\#$ is \mathbf{D} -computable. Let $M^\#$ be the presentation of M obtained by adding as special points the canonical unitaries u_g for $g \in G$. If,

in addition, each α_g is a \mathbf{D} -computable automorphism of $N^\#$, then the presentation $M^\#$ is \mathbf{D} -computable. Note also that the index $[M : N] = |G|$ is clearly computable and the conditional expectation map $E_N : M^\# \rightarrow M^\#$ mapping a sum $\sum_{g \in G} x_g u_g$ to x_e is \mathbf{D} -computable. Consequently, N is a \mathbf{D} -computable subfactor of $M^\#$.

Example 8.3. Suppose once again that we have an action $G \curvearrowright^\alpha M$ of a finite group by outer automorphisms. Let M^G be the fixed point subalgebra. The conditional expectation $E_{M^G} : M \rightarrow M^G$ is given by $E_{M^G}(x) := \frac{1}{|G|} \sum_{g \in G} \alpha_g(x)$. The index $[M : M^G] = |G|$ is clearly computable. Under what conditions is $E_{M^G} : M^\# \rightarrow M^\#$ \mathbf{D} -computable (that is, when is M^G a \mathbf{D} -computable subfactor of $M^\#$)? This is indeed the case when each $\alpha_g : M^\# \rightarrow M^\#$ is a \mathbf{D} -computable automorphism. Is the \mathbf{D} -computability of each α_g necessary? For example, if $G = \mathbb{Z}_2$ and α is the unique nontrivial automorphism of M , then the \mathbf{D} -computability of $E_{M^G} : M^\# \rightarrow M^\#$ clearly implies the \mathbf{D} -computability of $\alpha : M^\# \rightarrow M^\#$.

In regards to this question, we recall that, in the current context, $M_1 \cong M \rtimes G$, where the isomorphism is given by mapping M identically to itself and mapping e_N to the projection $e := \frac{1}{|G|} \sum_{g \in G} u_g$. Moreover, this map is computable when M_1 is equipped with its induced presentation and when $M \rtimes G$ is equipped with the presentation obtained by adding the above projection e to the presentation of M . Note that this presentation differs from the presentation of $M \rtimes G$ described in the previous example. Assuming that E_{M^G} is $M^\#$ -computable, it appears that the condition that each α_g be $M^\#$ -computable is equivalent to the above two presentations of $M \rtimes G$ being computably isomorphic.

Example 8.4. We consider the construction of subfactors of \mathcal{R} due to Jones [J83]. Fix a tracial von Neumann algebra M and suppose that $(e_i)_{i \geq 1}$ are projections in M satisfying:

- (1) $e_i e_{i \pm 1} e_i = t e_i$ for some $t \leq 1$.
- (2) $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.
- (3) $\text{tr}(w e_i) = t \text{tr}(w)$ for any word w on $1, e_1, \dots, e_{i-1}$.

Letting $P := \{e_i : i \geq 1\}''$ be the von Neumann subalgebra of M generated by the e_i 's for $i \geq 1$ and $P_t := \{e_i : i \geq 2\}''$ denote the subalgebra of P generated by the e_i 's for $i \geq 2$, we have that $P \cong \mathcal{R}$ and that P_t is a subfactor of P of index $[P : P_t] := \frac{1}{t}$. Moreover, this construction is only possible if $t \leq \frac{1}{4}$ or if $t = \frac{1}{4} \sec^2(\frac{\pi}{n})$ for $n = 3, 4, 5, \dots$

We equip P and P_t with the presentations $P^\#$ and $P_t^\#$ consisting of the set of the appropriate set of e_i 's. It is immediate from [J83, Lemma 4.1.6] that $P^\#$ and $P_t^\#$ are \mathbf{D} -computable, where \mathbf{D} is any oracle for which t is \mathbf{D} -computable.

We claim that, for any such \mathbf{D} , the subfactor P_t is a \mathbf{D} -computable subfactor of $P^\#$. Since t is \mathbf{D} -computable and $[P : P_t] = \frac{1}{t}$, we have that $[P : P_t]$ is \mathbf{D} -computable. It remains to see that $E_{P_t} : P^\# \rightarrow P^\#$ is \mathbf{D} -computable, which follows from the fact that $E_{P_t}(e_1) = t \cdot 1$.

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