Applications of Idempotents

New idea: Go to ${}^{**}\mathbb{N}$, nonst. extension of ${}^{*}\mathbb{N}$.
In language for this extension, have symbol for $\mathbb{N}$, so can say

$${}^{*}\mathbb{N} \models (\forall x \in \mathbb{N})(\forall y \notin \mathbb{N})\ x<y \text{, so}$$

$${}^{**}\mathbb{N} \models (\forall x \in {}^{*}\mathbb{N})(\forall y \notin {}^{*}\mathbb{N})\ x<y \text{, so}$$

$${}^{*}\mathbb{N} < {}^{**}\mathbb{N} \setminus {}^{*}\mathbb{N}.$$

${}^{*}\mathbb{N} \subseteq {}^{**}\mathbb{N}$, so can add elements of ${}^{*}\mathbb{N}$ to elements of ${}^{**}\mathbb{N}$. Also, if $\alpha \in {}^{*}\mathbb{N} \setminus \mathbb{N}$, then ${}^{*}\alpha \in {}^{**}\mathbb{N} \setminus {}^{*}\mathbb{N}$, so $\alpha < {}^{*}\alpha$.

For $\alpha \in {}^{*}\mathbb{N}$, $\beta \in {}^{**}\mathbb{N}$, define
$\alpha \sim \beta$ if: for every $A \subseteq \mathbb{N}$, we have

$$\alpha \in {}^{*}A \iff \beta \in {}^{**}A.$$

Def $\alpha \in {}^*\mathbb{N}$ is *u-idempotent* if $\alpha + {}^*\alpha \sim \alpha$.

Suppose there is a u-idempotent α.

Suppose $A \subseteq \mathbb{N}$ and $\alpha \in {}^*A$.

Then $\alpha + {}^*\alpha \in {}^{**}A$.

So $\models (\exists x \in {}^*\mathbb{N})(x \in {}^*A \wedge x + {}^*\alpha \in {}^{**}A)$

$\therefore \models (\exists x \in \mathbb{N})(x \in A \wedge x + \alpha \in {}^*A)$, call it x_0.

$x_0 + \alpha \in {}^*A \Rightarrow x_0 + \alpha + {}^*\alpha \in {}^{**}A$ since

$\alpha \in {}^*\{y \in \mathbb{N} : x_0 + y \in A\}$.

$\models (\exists x \in {}^*\mathbb{N})(x \in {}^*A \wedge x + {}^*\alpha \in {}^{**}A \wedge x_0 + x \in {}^*A$

$\wedge\ x_0 + x + {}^*\alpha \in {}^{**}A \wedge x > x_0)$.

$\therefore \exists x_1 \in \mathbb{N}$, $x_1 \in A$, $x_1 + \alpha \in {}^*A$, $x_0 + x_1 \in A$,

$x_0 + x_1 + \alpha \in {}^*A$, $x_1 > x_0$.

Keep going... Sps have $x_0 < x_1 < \cdots < x_{n-1}$ s.t.

for all $F \subseteq \{0, \ldots, n-1\}$, setting $x_F := \sum_{i \in F} x_i$,

have $x_F \in A$, $x_F + \alpha \in {}^*A$.

Then $x_F + \alpha + {}^*\alpha \in {}^{**}A$, so can find $x_n > x_{n-1}$

s.t. $x_F + x_n \in A$ $\forall F \subseteq \{0, \dots, n-1\}$ ($F = \emptyset$ even), $x_F + x_n + \alpha \in {}^*A$. Keep going.

Def $A \subseteq \mathbb{N}$ is a **FS-set** if there is $(x_n)_{n \in \mathbb{N}}$ from A s.t. $x_F \in A$ $\forall F \subseteq \mathbb{N}$ finite.

Prop If $\alpha \in {}^*\mathbb{N}$ is u-idempotent and $\alpha \in {}^*A$, then A is a FS-set.

Thm u-idempotent elements of ${}^*\mathbb{N}$ exist.
(Come back to this...)

Cor (Hindman's Thm) If $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_n$ is a finite coloring of $\mathbb{N}$, then some A_i is a FS-set.

So why do u-idempotents exist?

Ultrafilters Define them on any set S.

If $s\in S$, then get principal ultrafilter $\mathcal{U}_s$.

If $\alpha\in {}^*S$, still get an ultrafilter $\mathcal{U}_\alpha := \{A\subseteq S : \alpha\in {}^*A\}$.

Say $\alpha\sim\beta \iff \mathcal{U}_\alpha=\mathcal{U}_\beta \iff \forall A\subseteq S\,[\alpha\in{}^*A \leftrightarrow \beta\in{}^*A]$

βS = set of ultrafilters, a compact space with base given by $U_A := \{\mathcal{U}\in\beta S : A\in\mathcal{U}\}$.

Exercise (Assume suff sat.) $\alpha\mapsto\mathcal{U}_\alpha : {}^*S\to\beta S$ onto, ${}^*S/\sim\ \to\beta S$ bijection.

u-top on ${}^*S$
Base ${}^*A$ for $A\subseteq S$

Now suppose S has a semigroup operation $\cdot$.

Define $\odot$ on βS by

$$A\in\mathcal{U}\odot\mathcal{V} \iff \{s\in S : s^{-1}\cdot A\in\mathcal{V}\}\in\mathcal{U}.$$

Here, $s^{-1}A := \{t\in S : s\cdot t\in A\}$.

Note: $\mathcal{U}_s\odot\mathcal{U}_t=\mathcal{U}_{s\cdot t}$ for $s,t\in S$.

What about $\mathcal{U}_\alpha\odot\mathcal{U}_\beta$ for $\alpha,\beta\in{}^*S$?

$$A\in\mathcal{U}_\alpha\odot\mathcal{U}_\beta \iff \{s\in S : s^{-1}\cdot A\in\mathcal{U}_\beta\}\in\mathcal{U}_\alpha$$
$$\iff \alpha\in{}^*\{s\in S : s^{-1}\cdot A\in\mathcal{U}_\beta\}$$

Now $s^{-1}\cdot A\in\mathcal{U}_\beta \iff \beta\in{}^*(s^{-1}\cdot A)$
$$\iff s\cdot\beta\in{}^*A$$

So $\alpha \in {}^*\{s \in S : s^{-1} \cdot A \in \mathcal{U}_\beta\} \iff \alpha \in {}^*\{s \in S : s \cdot \beta \in {}^*A\}$

$\iff \alpha \cdot^* \beta \in {}^{**}A$

So $A \in \mathcal{U}_\alpha \odot \mathcal{U}_\beta \iff \alpha \cdot^* \beta \in {}^{**}A \iff A \in \mathcal{U}_{\alpha \cdot^* \beta}$.

So $\mathcal{U}_\alpha \odot \mathcal{U}_\beta = \mathcal{U}_{\alpha \cdot^* \beta}$.

$\therefore$ $\alpha \in {}^*S$ is u-idemp iff $\alpha \cdot^* \alpha \sim \alpha$

iff $\mathcal{U}_{\alpha \cdot^* \alpha} = \mathcal{U}_\alpha$

iff $\mathcal{U}_\alpha \odot \mathcal{U}_\alpha = \mathcal{U}_\alpha$.

$\therefore$ $\alpha \in {}^*S$ is u-idemp iff $\mathcal{U}_\alpha$ is idempotent.

Thm Idempotent ultrafilters on S exist.

Pf $(\beta S, \odot)$ is a compact semitop semigroup (meaning $\mathcal{U} \mapsto \mathcal{U} \odot \mathcal{V}$ is cts $\forall \mathcal{V} \in \beta S$.)

A theorem of Ellis says any compact semi semi has an idempotent. ☒

If $T \subseteq \beta S$ is a closed nonempty subsemigroup, then T also has an idempotent.

Def $T \subseteq^* S$ is a $\mathcal{U}$-subsemigroup if $\forall \alpha, \beta \in T \ \exists \gamma \in T$ s.t. $\alpha \cdot^* \beta \sim \gamma$ (so image of T in βS is a subsemigroup).

Cor If $T \subseteq^* S$ is a closed, nonempty $\mathcal{U}$-subsemi, then T contains a $\mathcal{U}$-idempotent.

Many applications of these ideas. Here's one more:

Hales–Jewett Theorem

L finite set, $x \notin L$ (variable)

$W_L := L^{<\omega}$ (words) $W_{Lx} = (L \cup \{x\})^{<\omega} \setminus W_L$.

For $w \in W_{Lx}$, $a \in L$, $w[a] \in W_L$. $w[x] = w$

$v^\frown w$ concatenation. W_L, W_{Lx} semigroups

Def (w_n) sequence of variable words.

① $[(w_n)]_{W_L} = \{ w_{n_0}[a_0]^\frown \cdots ^\frown w_{n_{k-1}}[a_{k-1}] : \begin{matrix} n_0 < \cdots < n_{k-1} \\ a_0, \ldots, a_{k-1} \in L \end{matrix} \}$

partial subsemigroup (increasing indices)

② $[(w_n)]_{W_{Lx}}$ = same with $\lambda_0, \ldots, \lambda_{k-1} \in L$, some $\lambda_i = x$.

Infinite Hales-Jewett Theorem

For every finite coloring of $W_L \cup W_{Lx}$, there is (w_n) from W_{Lx} s.t. $[(w_n)]_{W_L}$ and $[(w_n)]_{W_{Lx}}$ are both monochromatic.

Powerful! Can derive finitary HJ $\rightsquigarrow$ vdW (Gallai)

Lemma 1 There are u-idempotents $\omega \in {}^*W_{Lx}$ and $\nu \in {}^*W_L$ s.t. ① $\omega ^\frown {}^*\nu \sim \nu ^\frown {}^*\omega \sim \omega$ and ② $\omega[a] \sim \nu$ $\forall a \in L$.

Lemma 2 Sps $A \subset W_L$, $B \subset W_{Lx}$ are s.t. $\nu \in {}^*A$, $\omega \in {}^*B$. Then there is (w_n) from W_{Lx} s.t. $[(w_n)]_{W_L} \subseteq A$, $[(w_n)]_{W_{Lx}} \subseteq B$.

(IHJ follows immediately from Lemma 2.)

Pf of Lemma 2 from Lemma 1 Set $C := A \cup B$.

If $a\in L$, then $w[a]\sim v\in^* B\subseteq {}^*C$, while

$$w[x]=w\in^* A\subseteq {}^*C.$$

Also, if $a,b\in L\cup\{x\}$, then

$$w[a]{}^\frown{}^* w[b]\sim\begin{cases} v{}^\frown{}^* v\sim v\in^* A & \text{if } a,b\in L\\ v{}^\frown{}^* w\sim w\in^* B & \text{if } a\in L, b=x\\ w{}^\frown{}^* v\sim w\in^* B & \text{if } a=x, b\in L\\ w{}^\frown{}^* w\sim w\in^* B & \text{if } a=b=x\end{cases}$$

So:

$\models(\exists w\in^* W_{Lx})(\forall a,b\in L\cup\{x\})(w[a]\in^* C\wedge w[a]{}^\frown{}^* w[b]\in^{**}C)$

So $(\exists w_0\in W_L)(\forall a,b\in L\cup\{x\})\ w_0[a]\in C\wedge w_0[a]{}^\frown w[b]\in^* C.$

Second condition $\Rightarrow w[a]{}^\frown w[b]{}^\frown{}^* w[c]\in^{**}C$

So get $w_1\neq w_0$ s.t. for all $a,b,c\in L\cup\{x\}$,

$w_1[a]\in C$, $w_0[a]{}^\frown w_1[b]\in C$,

$w_0[a]{}^\frown w_1[b]{}^\frown w[c]\in^* C$

Keep going... $\boxtimes$

Pf of Lemma 1 $L=\{a_1,..,a_m\}$. By recursion, define u-idempotents $w_1,..,w_m$ of ${}^*W_{Lx}$ and $v_1,...,v_m$ of ${}^*W_L$ s.t. for $1\le i\le j\le m$:

① $w_j[a_i]\sim v_j$

② $w_j \sim w_j \hat{}^* v_i \sim v_i \hat{}^* w_j$.

Then $w = w_m$ and $v = v_m$ work.

Start with w_0 any u-idemp in ${}^*W_{Lx}$.

Set $v_1 := w_0[a_1]$.

Note $v_1 \hat{}^* v_1 = w_0[a_1] \hat{}^* w_0[a_1] = [w_0 \hat{}^* w_0][a_1]$

$\sim w_0[a_1] = v_1$.

So v_1 is u-idempotent.

Take $\rho_1 \in {}^*W_{Lx}$ s.t. $w_0 \hat{}^* v_1 \sim \rho_1$.

Then $\rho_1[a_1] \sim w_0[a_1] \hat{}^* v_1 = v_1 \hat{}^* v_1 \sim v_1$.

$\rho_1 \hat{}^* v_1 \sim w_0 \hat{}^* v_1 \hat{}^{**} v_1 \sim w_0 \hat{}^* v_1 \sim \rho_1$

ultrafilter justifications

Let $T := \{ z \in {}^*W_{Lx} : z[a_1] \sim v_1,\ z \hat{}^* v_1 \sim z \}$.

Then $\rho_1 \in T$ (so $T \neq 0$).

If $z_1, z_2 \in T$, then $(z_1 \hat{}^* z_2)[a_1] = z_1[a_1] \hat{}^* z_2[a_1]$

$\sim v_1 \hat{}^* v_1 \sim v_1$

$(z_1 \hat{}^* z_2) \hat{}^{**} v_1 \sim z_1 \hat{}({}^* z_2 \hat{}^{**} v_1) \sim z_1 \hat{}^* z_2$.

$\therefore z_1 \hat{}^* z_2$ is $\sim$ to something in T.

Also, T is u-closed

$z[a_1] \sim \nu_1$ means $z[a_1] \in^* A$ whenever $\nu_1 \in^* A$,

so $z \in^* \{p \in W_{LX} : p[a_1] \in A\}$

Similar for $z ^{\frown*} \nu_1 \sim z$.

$\therefore$ there is a u-idemp $\beta_1 \in T$.

Take $\omega_1 \in^* W_{LX}$ s.t. $\nu_1 ^{\frown*} \beta_1 \sim \omega_1$.

Then $\omega_1 ^{\frown*} \omega_1 \sim \nu_1 ^{\frown*} \beta_1 ^{\frown**} \nu_1 ^{\frown***} \beta_1$

$\sim \nu_1 ^{\frown*} \beta_1 ^{\frown**} \beta_1$

$\sim \nu_1 ^{\frown*} \beta_1$ (since β_1 idemp)

$\sim \omega_1$, so ω_1 u-idemp.

Ler $[a_1] \sim \nu_1 ^{\frown*} \nu_1 \sim \nu_1$

$\omega_1 ^{\frown*} \nu_1 \sim \nu_1 ^{\frown*} \beta_1 ^{\frown**} \nu_1 \sim \nu_1 ^{\frown*} \beta_1 \sim \omega_1$

$\nu_1 ^{\frown*} \omega_1 \sim \nu_1 ^{\frown*} \nu_1 ^{\frown**} \beta_1 \sim \nu_1 ^{\frown*} \beta_1 \sim \omega_1$.

So (1) & (2) hold.

The inductive step is similar. Have ω_i, ν_i for $1 \le i \le k < m$.

$\nu_{k+1} = \omega_k[a_{k+1}]$. $\nu_{k+1} ^{\frown*} \nu_i \sim \nu_i ^{\frown*} \nu_{k+1} \sim \nu_{k+1}$

$p_{k+1} \sim \omega_k ^{\frown*} \nu_{k+1}$. Same as with p_{k+1}

$p_{k+1}[a_i] \sim \nu_{k+1}$ $1 \le i \le k+1$

$T = \{ z \in {}^*W_{LX} : z[a_i] \sim v_{k+1},\ z \,{}^\wedge{}^*\, v_i \sim z \quad 1 \le i \le k+1 \}$.

nonempty, closed u-subsemigroup, so contains u-idemp β_{k+1}.

Take $w_{k+1} \sim v_{k+1} \,{}^\wedge{}^*\, \beta_{k+1}$. $\boxtimes$