Jin's Sumset Theorem (Comb. Version)

Suppose $A, B \subseteq \mathbb{N}$ are s.t. $BD(A), BD(B) > 0$.
Then $A+B$ is piecewise syndetic.

Def Sps $A \subseteq \mathbb{N}$. $\Delta_n(A) = \max\left\{\frac{|A \cap I|}{n} : I \text{ int}, |I| = n\right\}$

$BD(A) = \lim_{n \to \infty} \Delta_n(A)$ (exists by Fekete's lemma)

$= \inf_n \Delta_n(A)$

exercise If $N > \mathbb{N}$, then
$BD(A) = \max\left\{ st\left(\frac{|{}^*A \cap I|}{N}\right) : I \subseteq {}^*\mathbb{N} \text{ int}, |I| = N \right\}$.

Def Sps $A \subseteq \mathbb{N}$. A is thick if A contains arb. long. intervals.

exercise A thick iff ${}^*A$ contains an inf. interval.

exercise A thick iff $BD(A) = 1$.

Def A is *piecewise syndetic* if $\exists F \subseteq \mathbb{N}$ finite s.t. $A+F$ thick.

Lemma A is p.w. syndetic $\iff$ there is infinite I s.t. ${}^*A$ has only finite gaps on I.

Pf: ($\implies$) ${}^*A + [0,m]$ contains an inf. interval I.

($\impliedby$) The assumption implies that $\exists m \in \mathbb{N}$ s.t. all gaps of ${}^*A$ on I are of size m.

This implies that $I \subseteq {}^*A + [-m,m]$, so $A + [-m,m]$ thick. $\boxtimes$

Rmk A is *syndetic* if $\mathbb{N} \setminus A$ not thick, i.e. $\exists k \in \mathbb{N}$ s.t. A has gaps of size $\leq k$ on $\mathbb{N}$.

A is p.w. synd if $\exists k \in \mathbb{N}$ s.t. on arb long intervals, A has gaps of size $\leq k$.

Jin's theorem looks similar to:

Steinhaus' Thm If $C, D \subseteq [0,1]$ are s.t. $\lambda(C), \lambda(D) > 0$, then $C+D$ contains an interval.

Both are instances of a general phenomenon:

Def A **cut** is an initial segment $U \subseteq {}^*\mathbb{N}_0$ s.t. $U+U \subseteq U$.

Exercise No proper cut is internal.

examples

① $\mathbb{N}$

② If $N > \mathbb{N}$, then $U_N = \{x \in {}^*\mathbb{N} : \frac{x}{N} \approx 0\}$.

Fix cut $U \subseteq {}^*\mathbb{N}$. Sps $U \subseteq [0, N)$.

For $x, y \in {}^*\mathbb{N}$, write $x \sim_U y$ iff $|x-y| \in U$.

equiv rel

$[x]_{U,N} = [x]_N$ equiv class

$\pi_U : [0, N) \to [0, N)/U$ quotient map get linear order, and $\therefore$ topology

Can push forward Loeb measure.

example Fix $N > \mathbb{N}$. $f : [0,N) \to [0,1]$

$f(x) := st\left(\frac{x}{N}\right)$ surjection.

$f(x) = f(y)$ iff $x \sim_{U_N} y$, so get

$f : [0,N)/U_N \to [0,1]$ bijection of lin ordered sets.

exercise Via f, Loeb measure on $[0,N)/U_N$ becomes Lebesgue measure!

Def $A \subseteq [0,N)$ is **U-nowhere dense** if $\pi_U(A)$ is nowhere dense in $[0,N)/U$:

given $a < b$ in $[0,N)$ with $b - a > U$, there is $[c,d] \subseteq [a,b]$ with $d - c > U$ & $[c,d] \subseteq [0,N) \setminus A$.

Back to above example

Sps $A, B \subseteq [0,N)$ internal are s.t. $\pi_{U_N}(A), \pi_{U_N}(B)$ have positive Loeb measure, i.e. $f(A), f(B) \subseteq [0,1]$ have positive Leb. measure. By Steinhaus,

$f(A)+f(B)$ contains an interval.

In particular, $A \oplus_N B$ is U_N-somewhere dense.

$\uparrow$ addition mod N

Jin's Sumset Thm (General version)

Suppose $U \subseteq [0,N)$ is a cut and $A, B \subseteq [0,N)$ are internal sets with positive Loeb measure. Then $A \oplus_N B$ is U-somewhere dense.

exercise Deduce Steinhaus from Jin. (Recall that Loeb measure sets can be approximated by internal sets.)

Pf of 1st Jin from 2nd Jin

$BD(A) = r > 0$, $BD(B) = s > 0$.

Fix $N > \mathbb{N}$. By above exercise, have $x, y \in {}^*\mathbb{N}$ s.t. $\frac{|{}^*A \cap [x, x+N)|}{N} \approx r$, $\frac{|{}^*B \cap [y, y+N)|}{N} \approx s$.

$C := {}^*A - x$, $D := {}^*B - y \subseteq [0, 2N)$.

$\mu(C) = \frac{r}{2}$, $\mu(D) = \frac{s}{2}$ (μ = Loeb meas on $[0,2N)$)

By 2nd Jin applied to $U = \mathbb{N}$,
$C \oplus_{2N} D = C + D$ is $\mathbb{N}$-somewhere dense.
So there is $[a,b] \subseteq [0, 2N)$, $b - a > \mathbb{N}$
s.t. all gaps of $C+D$ in $[a,b]$ belong to $\mathbb{N}$,
i.e. are finite. By overflow, there is
$m \in \mathbb{N}$ s.t. all gaps of $C+D$ on $[a,b]$
have size $\leq m$, so $C+D \supseteq [a,b] + [0,m]$.
$\therefore$ ${}^*A + {}^*B \supseteq x + y + [a,b] + [0,m]$
$\therefore$ $\underset{\overset{\parallel}{{}^*(A+B)}}{{}^*A + {}^*B}$ has only finite gaps on $[a,b]$.
$\therefore$ $A+B$ is p.w. synd. ∎

Pf of 2nd version Sps false for U.
If $H > U$ and $A, B \subseteq [0, H)$ are internal,
say (A,B) is (H,U)-bad if
$\mu_H(A), \mu_H(B) > 0$ yet $A \oplus_H B$ is U-n.w.d.

$r := \sup\{\mu_H(A) : \exists H > U, \exists B \subseteq [0,H)$ s.t. (A,B) is (H,U)-bad$\}$.

$r>0$ by assumption.

Fix $\varepsilon>0$ (TBD.)

$s := \sup\{\mu_H(B) : (A,B)$ is (H,U)-bad for some $H>U$, some $A \subseteq [0,H)$, $\mu_H(A)>r-\varepsilon\}$.

Then $s>0$.

By symm, $r \geq s$.

so $r \geq s$ and $r-\varepsilon \geq s-\varepsilon \geq \frac{1}{2}$

Claim 1 $s < \frac{1}{2}+\varepsilon$.

Pf Sps $s \geq \frac{1}{2}+\varepsilon$. Can find $H>N$, (H,U)-bad (A,B) s.t. $\mu_H(A), \mu_H(B) > \frac{1}{2}$.

$\therefore$ For any $x \in [0,H)$, have $A \cap (x \ominus_H B) \neq \emptyset$ (both have measure $> \frac{1}{2}$), whence $A \oplus_H B = [0,H)$ ↯.

Fix $\delta>0$ (TBD) and take $H>U$, (H,U)-bad (A,B) s.t. $\mu_H(A)>r-\varepsilon$, $\mu_H(B)>s-\delta$.

Goal: Find $K>U$, (K,U)-bad (A',B') s.t. $\mu_K(A')>r-\varepsilon$, $\mu_K(B')>s+\delta$; will ↯ def of s.

Claim 1.5 $\exists$ there is $K > 0$ with $\frac{K}{H} \approx 0$ and hyp int $I, J \subseteq [0,H)$ of length K s.t.

$$\mathrm{st}\left(\frac{|A \cap I|}{K}\right) > r - \varepsilon \qquad \mathrm{st}\left(\frac{|B \cap J|}{K}\right) > s + \delta.$$

Pf Write $I = [a, a+K)$, $J = [b, b+K)$

$A' = (A \cap I) - a$, $B' = (B \cap J) - b$.

$\mu_K(A') > r - \varepsilon$, $\mu_K(B') > s + \delta$.

$A \oplus_H B$ U-n.w.d $\Rightarrow (A \cap I) \oplus_H (B \cap J)$ U-n.w.d

$A' \oplus_H B' = ((A \cap I) \oplus_H (B \cap J)) \ominus (a+b)$, have

$A' \oplus_H B'$ U-n.w.d.

$\quad ||$

$A' \oplus_{2K} B'$ since $K/H \approx 0$.

But $A' \oplus_K B'$ is the union of two U-n.w. dense subsets of $[0, K)$, so also U-n.w. dense.

$\therefore (A', B')$ (K, U)-bad. $\boxtimes$

For $k \in U$, $A \oplus_H (B \oplus_H [-k, k]) = (A \oplus_H B) \oplus_H [-k,k]$ is U-n.w. dense. $\therefore \mu_H(B \oplus_H [-k,k]) \le s$.

$$\therefore \frac{|B \oplus_H [-k,k]|}{H} \leq s + \frac{\delta}{2} \quad \forall k \in U$$

U external $\Rightarrow \exists K > U$, $K/H \approx 0$ s.t.

$$\frac{|B \oplus_H [-K,K]|}{H} \leq s + \frac{\delta}{2}.$$

This K will work!

$$\mathcal{J} = \{ [iK, (i+1)K) : 0 \leq i \leq \tfrac{H}{K} - 1 \}$$

partition $[0, H-1)$ into intervals of length K

(negligible tail)

$$X = \{ i \in [0, \tfrac{H}{K} - 1] : [iK, (i+1)K) \cap B = \emptyset \}.$$

Claim 2 $\dfrac{|X|}{|\mathcal{J}|} > \dfrac{1}{3}$.

Pf Sps, TAC, $\dfrac{|X|}{|\mathcal{J}|} \leq \dfrac{1}{3}$.

If $i \notin X$, $x \in [iK, (i+1)K)$, $x = iK + j$, $j \in [0, K-1]$,

there is $\ell \in [0, K-1)$ s.t. $iK + \ell \in B$.

$\therefore$ $x = iK + \ell + (j - \ell) \in B \oplus_H [-K, K]$

$\therefore$

$$|B \oplus_H [-K,K]| \geq \sum_{i \notin X} K \geq \frac{2}{3}\left(\frac{H}{K}-1\right)K$$
$$= \frac{2}{3}H - \frac{2}{3}K$$

$$\therefore \frac{|B \oplus_H [-K,K]|}{H} \geq \frac{2}{3} - \frac{2}{3}\frac{K}{H} \approx \frac{2}{3}.$$

But $\frac{|B \oplus_H [-K,K]|}{H} \leq s + \frac{\delta}{2}$ ($\lightning$ if ε, δ small)

$\square$

$\mathcal{J}' = \{[iK, (i+1)K) : i \notin X\}.$

Claim 3 $\exists I, J \in \mathcal{J}'$ ($'$ missing in book).

$St\left(\frac{|A \cap I|}{K}\right) > r - \varepsilon$, $St\left(\frac{|B \cap J|}{K}\right) > s + \delta$.

<u>Pf</u> Do J: Sps no J exists. Then

$$s - \delta < \frac{|B \cap [0, H-1)|}{H} = \frac{1}{H} \sum_{J \in \mathcal{J}'} |B \cap [iK, (i+1)K)|$$

$\uparrow$ def of $\mathcal{J}'$

$$\leq \frac{1}{\cancel{H}} \cdot \underbrace{\frac{2}{3}\left(\frac{\cancel{H}}{\cancel{K}}\right)}_{\text{Claim 2}} \cdot \underbrace{(s+\delta)}_{\lightning\ \text{assumption}}\cancel{K}$$

$$= \frac{2}{3}(s+\delta)$$

If $\delta \leq \frac{s}{5}$, get $\lightning$. $\blacksquare$