Triangle Removal & Szemeredi Regularity

Def (G, E) finite graph. ← undirected, simple

$$e(G) = \frac{|E|}{|V|^2}$$ edge density (Rmk: a little off since not directed, simple)

$$t(G) = \frac{|\{(x,y,z) \in V^3 : (x,y), (y,z), (x,z) \in E\}|}{|V|^3}$$

Triangle Removal Thm $\forall \varepsilon \, \exists \delta \, \forall (G, E)$ finite
$t(G) < \delta \rightarrow \exists G' = (V, E') \subseteq G$ s.t. $t(G') = 0$
and $e(G \setminus G') \leq \varepsilon$.
"Remove few edges to get no triangles"

Application (Roth's Thm)
$\forall \varepsilon \, \exists n_0 \, \forall n \geq n_0 \, \forall A \subseteq [1, n] \left(\frac{|A|}{n} \geq \varepsilon \rightarrow A \text{ contains 3-AP} \right)$

Pf Sketch $V = 3$ copies of $[1, 3n]$, V_1, V_2, V_3
For $(v, w) \in (V_1 \times V_2) \cup (V_2 \times V_3)$, then $(v, w) \in E \iff w - v \in A$
For $(v, w) \in V_1 \times V_3$, $(v, w) \in E \iff w - v \in 2A$.

Sps (v_1, v_2, v_3) triangle.

$a := v_2 - v_1$, $b := v_3 - v_2$, $c := v_3 - v_1/2$ all in A.

Then b, c, a is a 3-AP in A w/

difference $c - b = a - c$ except could be 0.

So call the Δ trivial if they are equal.

Given $a \in A$, $k \in [n]$, get a trivial Δ

$(k, k+a, k+2a)$; there are $|A| \cdot n$ such Δ's,

edge disjoint, so have to remove $\geq 3|A| \cdot n \geq 3\varepsilon n^2$

such edges to have no Δ.

Let δ correspond to 3ε in Δ-removal.

Follows $t(G) \geq \delta$, i.e. have $\geq 27\delta n^3$ triangles.

But but there are $\leq n^2$ trivial triangles.

So for $n \gg 0$, must be a nontrivial Δ. $\blacksquare$

Rmk Hyp. removal $\Rightarrow$ Full Szemeredi

Nonstandard Δ-removal V nonempty, hyperfinite.

$E_{12}, E_{23}, E_{13} \in \mathcal{L}_{V \times V}$ satisfy

$$\int_{V^3} 1_{E_{12}}(u,v) 1_{E_{23}}(v,w) 1_{E_{13}}(u,w) \, d\mu(u,v,w) = 0. \quad (+)$$

Then $\forall \varepsilon>0\ \forall (i,j)\ \exists$ hyp $F_{ij}\in V^2$ s.t.
$\mu_{V\times V}(E_{ij}\setminus F_{ij})<\varepsilon$ and
$1_{F_{12}}(u,v)\,1_{F_{23}}(v,w)\,1_{F_{13}}(u,w)=0\ \ \forall (u,v,w)\in V^3$.

<u>Nonst $\Rightarrow$ Stand</u> Sps $\exists \varepsilon$ s.t. no δ works.
Get finite $G_n=(V_n,E_n)$ s.t. $t(G_n)\le \frac{1}{n}$ and yet...
Note: $|V_n|\to\infty$.
Overflow $\Rightarrow$ $\exists$ hyp $G=(V,E)$ s.t. $t(G)\approx 0$ and yet...
Apply Nonst to V with each $E_{ij}=E$:
$(\dagger)$ holds since the integral $=\mathrm{st}(t(G))=0$.
Get F_{ij}'s corresponding to $\frac{\varepsilon}{6}$.
Symmetrize: $E' := E\cap \bigcap_{ij}(F_{ij}\cap F_{ij}^{-1})$.
Then $(V,E')\subseteq(V,E)$ is triangle-free.
$\mu(E\setminus E')<\varepsilon$, $\mathscr{L}$. property of G.

<u>Lemma</u> Sps $f\in L^2(\mathcal{L}_{V\times V})\ominus L^2(\mathcal{L}_V\otimes\mathcal{L}_V)$.
Then for any $g,h\in L^2(\mathcal{L}_{V\times V})$, we have
$$\int_{V\times V\times V} f(x,y)\,g(y,z)\,h(x,z)\,d\mu_{V\times V\times V}(x,y,z)=0.$$
By an appropriate Fubini

Pf Sketch $\overset{\text{By [illegible]}}{=} \int_V \left[\int_{V\times V} f(x,y) \underbrace{g_z(y) h_z(x)}_{\in L^2(\mathcal{L}_V \otimes \mathcal{L}_V)} \, d\mu_{V\times V}(x,y) \right] d\mu_V(z)$

$\boxtimes$

Pf of Nonst Δ-removal

Step 1 Reduce to the case $E_{ij} \in \mathcal{L}_V \otimes \mathcal{L}_V$.

Set $f_{ij} = \mathbb{E}[1_{E_{ij}} \mid \mathcal{L}_V \otimes \mathcal{L}_V]$

By lemma (3 times),

$$\int_{V\times V\times V} f_{12} f_{23} f_{13} \, d\mu_{V\times V\times V} = \int_{V\times V\times V} 1_{E_{12}} 1_{E_{23}} 1_{E_{13}} = 0.$$

$G_{ij} := \{(u,v) \in V\times V : f_{ij}(u,v) \geq \frac{\varepsilon}{2}\} \in \mathcal{L}_V \otimes \mathcal{L}_V$.

Then by above, $\int_{V\times V\times V} 1_{G_{12}} 1_{G_{23}} 1_{G_{13}} = 0$.

Note $\mu(E_{ij} \setminus G_{ij}) = \int_{V\times V} 1_{E_{ij}} (1 - 1_{G_{ij}}) = \int_{V\times V} f_{ij} (1 - 1_{G_{ij}})$

$\leq \frac{\varepsilon}{2}$.

If Thm true for elements of $\mathcal{L}_V \otimes \mathcal{L}_V$, get

hyp $F_{ij} \subseteq G_{ij}$, $\mu(G_{ij} \setminus F_{ij}) < \frac{\varepsilon}{2}$ s.t. ...

Since $\mu(E_{ij} \setminus F_{ij}) < \varepsilon$, done.

Step 2 Assume each $E_{ij} \in \mathcal{L}_V \otimes \mathcal{L}_V$.

By product measure, have elem sets H_{ij} s.t.

$\mu(E_{ij} \Delta H_{ij}) < \frac{\varepsilon}{6}$.

Bool. alg. gen. by sides of the boxes $\rightsquigarrow$

$V = V_1 \sqcup \cdots \sqcup V_n$ finitely many hyp sets s.t.

each H_{ij} is a union of boxes $V_k \times V_\ell$.

$F_{ij} := \bigcup \{ V_k \times V_\ell : V_k \times V_\ell \subseteq H_{ij},\ \mu(V_k \times V_\ell) > 0,$

$\mu(E_{ij} \cap (V_k \times V_\ell)) > \frac{2}{3} \mu(V_k \times V_\ell) \}$.

Certainly hypfinite.

Claim 1 $\mu(E_{ij} \setminus F_{ij}) < \varepsilon$.

Pf: ETS $\mu(H_{ij} \setminus F_{ij}) \le \frac{\varepsilon}{2}$.

$\mu((H_{ij} \setminus F_{ij}) \cap E_{ij}) \le \frac{2}{3} \mu(H_{ij} \setminus F_{ij})$

$\mu((H_{ij} \setminus F_{ij}) \setminus E_{ij}) \le \mu(H_{ij} \setminus E_{ij}) \le \frac{\varepsilon}{6}$

So $\mu(H_{ij} \setminus F_{ij}) \le \frac{2}{3} \mu(H_{ij} \setminus F_{ij}) + \frac{\varepsilon}{6}$ ✓

Claim (††) holds.

Sps, TAC, that there are (u,v,w) s.t.

$(u,v) \in F_{12}$, $(v,w) \in F_{23}$, $(u,w) \in F_{13}$.

Then $u \in V_k$, $v \in V_\ell$, $w \in V_m$.

$$(u,v) \in F_{12} \Rightarrow \mu(E_{12} \cap (V_k \times V_\ell)) > \tfrac{2}{3}\mu(V_k \times V_\ell)$$

$$\Rightarrow \mu(E_{12} \times V_m) > \tfrac{2}{3}\mu(V_k \times V_\ell \times V_m).$$

Similarly, $\mu(E_{23} \times V_k), \mu(E_{13} \times V_\ell) > \tfrac{2}{3} \cdots$

Then $\mu(\{(x,y,z) \in V_k \times V_\ell \times V_m : (x,y) \notin E_{12} \vee (y,z) \notin E_{23}$

$\vee (x,z) \notin E_{12}\}) < \mu(V_k \times V_\ell \times V_m)$,

contradicting assumption (†). ∎

Szemeredi Regularity

Def (V,E) finite graph, $X, Y \subseteq V$ nonempty.

$$d(X,Y) := \frac{|E \cap (X \times Y)|}{|X| \cdot |Y|} \quad \text{density}$$

(X,Y) are *ε-pseudorandom* if: whenever

$A \subseteq X$, $B \subseteq Y$, $|A| \geq \varepsilon|X|$, $|B| \geq \varepsilon|Y|$, have

$$|d(A,B)-d(X,Y)|<\varepsilon$$

Explain intuition

$V := V_1 \sqcup \cdots \sqcup V_m.$

$R = \{(i,j) \in [m]^2 : V_i, V_j\ \varepsilon\text{-pseudorandom}\}$

The partition is ε-regular if

$$\sum_{(i,j)\in R} \frac{|V_i|\cdot|V_j|}{|V|^2} > 1-\varepsilon.$$

"Most pairs of points are in ε-pseudo pairs"

Note: A partition into singletons is obviously ε-regular. But huge!

Szemeredi's Reg. Lemma $\forall \varepsilon\ \exists C(\varepsilon)\ \forall\ G=(V,E)$

G admits an ε-reg partition into $m \leq C(\varepsilon)$ pieces.

Nonst. equivalent $\forall \varepsilon\ \forall$ hyp (V,ε) $\exists$ finite part.

$V = V_1 \sqcup \cdots \sqcup V_m$ into $\underbrace{\text{internal}}_{\text{hyp}}$ sets and

$R \subseteq [m]^2$ s.t.

① $\forall (i,j) \in R$, V_i, V_j internally ε-pseudorandom

② $\sum_{(i,j)\in R} \frac{|V_i|\cdot|V_j|}{|V|^2} > 1-\varepsilon.$

St $\Rightarrow$ Nonst: transfer

Nonst $\Rightarrow$ St: overflow as before.

<u>Pf of Nonst version</u>: Fix ε, hyp (V, E).

Set $f := \mathbb{E}[\mathbb{1}_E \mid \mathcal{L}_V \otimes \mathcal{L}_V]$.

Useful calculation Sps $A, B \subseteq V$ hyp, $\frac{|A|}{|V|}, \frac{|B|}{|V|} \not\approx 0$.

Then $\int_{A\times B} f\, d(\mu_V \otimes \mu_V) = \int_{A\times B} \mathbb{1}_E\, d\mu_{V\times V}$ (def of f)

$$= st\left(\frac{|E\cap(A\times B)|}{|V|^2}\right)$$

$$= \underbrace{st\left(\frac{|E\cap(A\times B)|}{|A|\cdot|B|}\right)}_{st(d(A,B))} st\left(\frac{|A|\cdot|B|}{|V|^2}\right)$$

Fix $r>0$ TBD. Take $\mu_V \otimes \mu_V$-simple $g \le f$

s.t. $\int (f-g)\, d(\mu_V \otimes \mu_V) < r$.

$C := \{\omega \in V\times V : f(\omega) - g(\omega) \geq \sqrt{r}\} \in \mathcal{L}_V \otimes \mathcal{L}_V$.

Then $\mu_V \otimes \mu_V(C) < \sqrt{r}$.

Take elem. $D \supseteq C$ s.t. $\mu_V \otimes \mu_V(D) < \sqrt{r}$.

In a similar way, can assume the meas. sets in the def of g are elem.

Then have $V = V_1 \sqcup \cdots \sqcup V_m$ fin. part, hyp sets s.t. g, 1_D constant on each $V_i \times V_j$.

d_{ij} = const value of g on $V_i \times V_j$.

Claim! If $\mu(V_i), \mu(V_j) > 0$ & $(V_i \times V_j) \cap D = \emptyset$, then V_i, V_j $2\sqrt{r}$-pseudorandom. If $\omega \in V_i \times V_j$,

Pf: $d_{ij} \underset{g \leq f}{\leq} f(\omega) \underset{\omega \notin C}{<} d_{ij} + \sqrt{r}$

Integrate over $A \times B$, $A \subseteq V_i$, $B \subseteq V_j$, $|A| \geq 2\sqrt{r}\,|V_i|$, $|B| \geq 2\sqrt{r}\,|V_j|$:

$d_{ij}\, st\left(\frac{|A|\cdot|B|}{|V|^2}\right) \leq st(d(A,B))\, st\left(\frac{|A|\cdot|B|}{|V|^2}\right) \leq (d_{ij} + \sqrt{r})\left(\quad\right)$

Legit since $\frac{|A|}{|V|} = \frac{|A|}{|V_i|} \cdot \frac{|V_i|}{|V|} \neq 0$.

$\therefore |d(A,B) - d(V_i, V_j)| \leq |d(A,B) - d_{ij}| + |d_{ij} - d(V_i, V_j)|$

$< 2\sqrt{r}$ (use $A = V_i$, $B = V_j$) ∎

Take $r < \left(\frac{\varepsilon}{2}\right)^2$.

Claim 2 The partition is ε-regular.

Pf: $R := \{(i,j) : V_i, V_j \ \varepsilon\text{-pseudorandom}\}$.

Then st $\left(\sum_{(i,j)\in R} \frac{|V_i|\cdot|V_j|}{|V|^2}\right) = \mu_{V\times V}\left(\bigcup_{(i,j)\in R} V_i \times V_j\right)$

$\geq \mu_{V\times V}((V\times V)\setminus D)$ (Claim 1)

$> 1 - \sqrt{r}$

$> 1 - \varepsilon$ ∎