Loeb measure

X hyperfinite set, $\mathcal{A}$ = algebra of internal subsets
$\mu : \mathcal{A} \to [0,1]$, $\mu(Y) := st\left(\frac{|Y|}{|X|}\right)$.
Check μ is finitely additive.

Lemma μ is a premeasure: if $A_1, A_2, \ldots \in \mathcal{A}$ are p.w. disjoint and $\bigcup_1^\infty A_n \in \mathcal{A}$, then
$\mu(\bigcup_1^\infty A_n) = \sum_1^\infty \mu(A_n)$.

Pf If $\bigcup_1^\infty A_n \in \mathcal{A}$, then by sat there is $m \in \mathbb{N}$ s.t. $\bigcup_1^\infty A_n = \bigcup_1^m A_n$ (and rest are empty); now use fin. add. $\boxtimes$

By Carathéodory extension, there is a σ-algebra $\mathcal{L}_X \supseteq \mathcal{A}$ and a measure μ_X on $\mathcal{L}_X$ s.t. $\mu_X | \mathcal{A} = \mu$; call $(X, \mathcal{L}_X, \mu_X)$ the associated Loeb measure space.

Lemma If $B \in \mathcal{L}_X$, then

$$\mu_X(B) = \inf\{\mu_X(A) : A \text{ \underline{internal}},\ B \subseteq A\}.$$

Pf: Fix $\varepsilon > 0$. Have internal $A_1 \subseteq A_2 \subseteq \ldots$

s.t. $B \subseteq \bigcup_n A_n$ and $\mu_X(A_n) < \mu_X(B) + \varepsilon$ $\forall n$.

$(A_n)_{n \in \mathbb{N}} \rightsquigarrow (A_n)_{n \in {}^*\mathbb{N}}$ (saturation)

For $k \in \mathbb{N}$, have

$$(\forall n \in {}^*\mathbb{N})\left(n \leq k \to A_n \subseteq A_k,\ \frac{|A_n|}{|X|} < \mu_X(B) + \varepsilon\right).$$

By overflow, have $K > \mathbb{N}$ s.t. $B \subseteq \bigcup_{n \in \mathbb{N}} A_n \subseteq A_K$

and $\mu_X(A_K) \leq \mu_X(B) + \varepsilon$. ∎

Product measure

Sps X, Y two hyp. sets.

Then can consider the Loeb spaces

$(X, \mathcal{L}_X, \mu_X)$, $(Y, \mathcal{L}_Y, \mu_Y)$ and the product

$(X \times Y, \mathcal{L}_X \otimes \mathcal{L}_Y, \mu_X \otimes \mu_Y)$.

Also, $X \times Y$ is hyp. so have

$(X \times Y, \mathcal{L}_{X \times Y}, \mu_{X \times Y})$.

If $A \subseteq X$, $B \subseteq Y$ are int, so is $A \times B \subseteq X \times Y$ and

$$\mu_{X\times Y}(A\times B) = st\left(\frac{|A\times B|}{|X\times Y|}\right) = st\left(\frac{|A|}{|X|}\right)\cdot st\left(\frac{|B|}{|Y|}\right)$$
$$= (\mu_X \otimes \mu_Y)(A\times B).$$

This can be used to prove that $\mathcal{L}_X \otimes \mathcal{L}_Y$ is a sub-σ-alg of $\mathcal{L}_{X\times Y}$ and $\mu_{X\times Y} \restriction \mathcal{L}_X \otimes \mathcal{L}_Y = \mu_X \otimes \mu_Y$.

Thm (Fubini for Loeb) $f : X\times Y \to \mathbb{R}$ bounded, $\mathcal{L}_{X\times Y}$-meas. $f_x : Y \to \mathbb{R}$, $f^y : X \to \mathbb{R}$.

Then

1. f_x is $\mathcal{L}_Y$-meas. for μ_X-a.e. $x \in X$
2. $f^y \ldots$
3. $$\int_{X\times Y} f(x,y)\, d\mu_{X\times Y}(x,y) = \int_X \left(\int_Y f_x(y)\, d\mu_Y(y) \right) d\mu_X(x)$$

$$= \ldots$$

Pf Sketch By usual shenanigans (MCT...) and above lemma, reduce to case $f(x,y) = \chi_E$ for $E \subseteq X \times Y$ internal.

Then $\int_Y \chi_E(x,y)\,d\mu_Y(y) = st\left(\frac{|E_x|}{|Y|}\right)$,

$E_x = \{y \in Y : (x,y) \in E\}$.

A calculation shows

$$\int_X st\left(\frac{|E_x|}{|Y|}\right) d\mu_X(x) \overset{\text{Thm 5.18}}{\approx} \frac{1}{|X|} \sum_{x \in X} \frac{|E_x|}{|Y|}$$

$$= \frac{|E|}{|X| \cdot |Y|} \approx \mu_{X \times Y}(E)$$

$$= \int_{X \times Y} \chi_E(x,y) \quad \boxtimes$$