Nonstandard Analysis Basics

Basic idea For any set X under investigation (e.g. $\mathbb{N}$, $\mathbb{R}$, graph $G=(V,E)$), we extend it to its nonstandard extension $^*X$, so $X \subseteq {^*X}$, which is a proper extension unless X is finite. Functions $f: X \to Y$ also get extended to functions $f: {^*X} \to {^*Y}$.

This extension is "logic-preserving" (Transfer Principle).

ex $\;{^*(A \cup B)} = {^*A} \cup {^*B}, \ldots$

ex $\;\mathbb{N} \subseteq {^*\mathbb{N}}$ draw picture

ex $\;\mathbb{R} \subseteq {^*\mathbb{R}}$ ordered field extension

Infinite elements, infinitesimal elements

Thm If $x \in {^*\mathbb{R}}$ is finite, there is a unique $y \in \mathbb{R}$ s.t. $x \approx y$; y is called the standard part of x, denoted $st(x)$.

Our nonstandard extensions need to be suff. "rich".
If X is a basic set, also assume $\mathcal{P}(X)$ is a basic set and $E \subseteq X \times \mathcal{P}(X)$.
Then ${}^*E \subseteq {}^*X \times {}^*\mathcal{P}(X)$.

Fact Can view ${}^*\mathcal{P}(X) \subseteq \mathcal{P}({}^*X)$ and
${}^*E$ = membership restricted to ${}^*X \times {}^*\mathcal{P}(X)$.
Internal sets

Similarly, have $\mathcal{P}_{fin}(X)$; elements of ${}^*\mathcal{P}_{fin}(X)$ are called **hyperfinite subsets of ${}^*X$**.
They are internal.
The map $|\cdot|: \mathcal{P}_{fin}(X) \to \mathbb{N}$ extends to
$|\cdot|: {}^*\mathcal{P}_{fin}(X) \to {}^*\mathbb{N}$, **internal cardinality**

If $B \subseteq A \subseteq {}^*X$ are internal, A hyp, then B is hyp. by transfer.
ex For $N \in {}^*\mathbb{N}$ have hyp int $[1,N]$, $|[1,N]| = N$.

Def Our nonst. ext. is **countably saturated** if: whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of int. sets w/ FIP, then $\bigcap_n A_n \neq \emptyset$.

$\aleph_1$-sat

example Sps $(B_n)_{n\in\mathbb{N}}$ is a collection of int. subsets of $^*X$. For $n\in\mathbb{N}$, set $A_n := \{ f : {}^*\mathbb{N} \to {}^*\mathcal{P}(X) : f(m) = A_m \text{ for } m \le n \}$.

Then A_n is internal (Internal definition principle) and has FIP ($A_n \subseteq A_{n+1}$ and $A_n \neq \emptyset$ since we can just set $f(m) = {}^*X \ \forall m > n$).

By countable sat, $\exists f \in \bigcap_n A_n$.

Write it suggestively as $(A_n)_{n\in{}^*\mathbb{N}}$.

Overflow If $B \subseteq {}^*\mathbb{N}$ internal & $B \cap \mathbb{N}$ infinite, then $B \cap ({}^*\mathbb{N} \setminus \mathbb{N}) \neq \emptyset$: If not B bounded above, so $\max(B)$ exists.

Underflow If $B \subseteq {}^*\mathbb{N}$ int. contains arb small elts of ${}^*\mathbb{N} \setminus \mathbb{N}$, then $B \cap \mathbb{N} \neq \emptyset$.