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On a Leray-α model of turbulence

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In this paper we introduce and study a new model for three-dimensional turbulence, the Leray-α model. This model is inspired by the Lagrangian averaged Navier–Stokes-α model of turbulence (also known Navier–Stokes-α model or the viscous Camassa–Holm equations). As in the case of the Lagrangian averaged Navier–Stokes-α model, the Leray-α model compares successfully with empirical data from turbulent channel and pipe flows, for a wide range of Reynolds numbers. We establish here an upper bound for the dimension of the global attractor (the number of degrees of freedom) of the Leray-α model of the order of $(L/l_{\mathrm{d}})^{12/7}$, where L is the size of the domain and l_{d} is the dissipation length-scale. This upper bound is much smaller than what one would expect for three-dimensional models, i.e. $(L/l_{\mathrm{d}})^3$. This remarkable result suggests that the Leray-α model has a great potential to become a good sub-grid-scale large-eddy simulation model of turbulence. We support this observation by studying, analytically and computationally, the energy spectrum and show that in addition to the usual $k^{-5/3}$ Kolmogorov power law the inertial range has a steeper power-law spectrum for wavenumbers larger than $1/\alpha$. Finally, we propose a Prandtl-like boundary-layer model, induced by the Leray-α model, and show a very good agreement of this model with empirical data for turbulent boundary layers.

Keywords: subgrid scale models; turbulence models; Leray-α model

1. Introduction

The Navier–Stokes equations (NSEs) of viscous incompressible fluids subject to periodic boundary conditions, with a basic periodic box $\Omega = [0, 2\pi L]^3$, are given by the

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set of equations

$$\left.\begin{aligned} \frac{\partial}{\partial t}v - \nu\Delta v + (v\cdot\nabla)v + \nabla p = f,\\ \nabla\cdot v = 0,\\ v \text{ periodic, with periodic box } \Omega = [0, 2\pi L]^3, \end{aligned}\right\} \tag{1.1}$$

where v, the velocity, and p, the pressure, are the unknowns, f is a given body-forcing term and $\nu > 0$ is the viscosity. To prove the existence of solutions to the NSEs in $\mathbb{R}^n$, $n = 2, 3$, Leray (1934) considered the following regularization of the system (1.1):

$$\left.\begin{aligned} \frac{\partial}{\partial t}v^\alpha - \nu\Delta v^\alpha + (u^\alpha\cdot\nabla)v^\alpha + \nabla p^\alpha = f,\\ \nabla\cdot v^\alpha = 0,\\ u^\alpha = \phi_\alpha * v^\alpha, \end{aligned}\right\} \tag{1.2}$$

where ϕ_α is a smoothing kernel such that $u^\alpha \to v^\alpha$, in some sense, as $\alpha \searrow 0^+$. In particular, the system (1.2) converges to the NSEs (1.1) as $\alpha \searrow 0^+$.

In this paper we consider a special smoothing kernel, the one associated with the Green function of the Helmholtz operator:

$$u^\alpha - \alpha^2\Delta u^\alpha = v^\alpha,$$

where $\alpha > 0$ is a given length-scale. Dropping the α-dependence in the superscript we arrive at the following modification of the NSEs, which we will call the Leray-α model:

$$\left.\begin{aligned} \frac{\partial}{\partial t}v - \nu\Delta v + (u\cdot\nabla)v + \nabla p = f,\\ \nabla\cdot v = 0,\\ v = u - \alpha^2\Delta u,\\ v \text{ periodic, with periodic box } \Omega = [0, 2\pi L]^3. \end{aligned}\right\} \tag{1.3}$$

The above model is very similar to the Lagrangian averaged Navier–Stokes-alpha (LANS-α) model (also known as the Navier–Stokes-alpha (NS-α) or viscous Camassa–Holm equations)

$$\left.\begin{aligned} \frac{\partial}{\partial t}v - \nu\Delta v + (u\cdot\nabla)v + \sum_{j=1}^{3} v_j\nabla u_j + \nabla p = f,\\ \nabla\cdot v = 0,\\ v = u - \alpha^2\Delta u,\\ v \text{ periodic, with periodic box } \Omega = [0, 2\pi L]^3, \end{aligned}\right\} \tag{1.4}$$

which was introduced in Chen *et al.* (1998, 1999*a*,*b*) and Foias *et al.* (2002) as a closure model for the Reynolds averaged equations of the NSEs. The inviscid LANS-α model, i.e. the Lagrangian averaged Euler equations, may be derived using variational principles from a Lagrangian that has been averaged along fluid particle trajectories (see, for example, Chen *et al.* 1999*a*; Holm 2002*a*; Holm *et al.* 1998; Marsden &

Shkoller 2003). The LANS-α model is then obtained from the Lagrangian averaged Euler equations by adding a suitable viscous term. A general LANS-α model for anisotropic turbulence was derived in Holm (1999) and Marsden & Shkoller (2003). It is an open question whether the Leray-α model has a similar derivation to that of an averaged equation. In Constantin (2001), however, another approach connecting Lagrangian and Eulerian formulations for the Navier–Stokes equations was introduced. This exact connection between Lagrangian and Eulerian formulations gives another perspective for looking at the relation between the Navier–Stokes equations and the LANS-α and the Leray-α models.

The successful comparison with empirical data for time-averaged quantities in Chen *et al.* (1998, 1999*a*,*b*), for a wide range of Reynolds numbers in turbulent channel and pipe flows, led to further study of the LANS-α in the context of turbulence modelling (see, for example, Cheskidov 2002, 2004; Holm *et al.* 2003; Mohseni *et al.* 2003; Putkaradze & Weidman 2003). Analytical studies of the global existence, uniqueness and regularity of solutions to (1.4) and their connection to the NSEs are performed in Foias *et al.* (2002). Similar results are also established in Marsden & Shkoller (2001) for the same model subject to ad hoc Dirichlet-type boundary conditions. The energy spectrum of (1.4) was studied in Foias *et al.* (2001*a*), and semi-rigorous arguments, similar to those introduced in Foias (1997) (see also Foias *et al.* 2001*b*), suggest that the inertial range of (1.4) has two parts. The first part is the usual Kolmogorov $\kappa^{-5/3}$ power law of energy spectra up to a wavenumber of the order $1/\alpha$, then a faster drop in the energy spectrum with the power law κ^{-3} is shown. In addition, the Kármán–Howarth theorem for fluid turbulence obeyed by the LANS-α model was proved in Holm (2002*b*). This theorem rigorously proves the $k^{-5/3} \to k^{-3}$ spectral scaling transition in wavenumber as $k\alpha < 1$ passes to $k\alpha > 1$. This property of the energy spectrum (which also has been observed computationally (Chen *et al.* 1999*c*)) indicates that the LANS-α model is more reliably 'computable' in direct numerical simulations than the NSEs, and can be used as a sub-grid scale model in large-eddy simulations (LESs). The effectiveness of both the LANS-α and the Leray-α models as LES models will be discussed further below.

Inspired by the work done in association with the system (1.4), LANS-α, we will compare here the analogous results associated with (1.3), Leray-α. In particular, using the steady-state equations of (1.3), Leray-α, as a closure model for the averaged Reynolds equations in the turbulent channels and pipes, we reach exactly the same conclusions as those reported in Chen *et al.* (1998, 1999*a*,*b*) for (1.4), LANS-α. This is because, in channels and pipes under the corresponding special symmetries, the term

$$\sum_{j=1}^{3} v_j \nabla u_j$$

in the LANS-α will be a complete gradient. That is, the difference between (1.4) and (1.3) in the channels and pipes, subject to certain special symmetries, will be in the modified pressure and possibly in some of the associated Reynolds stresses. Therefore, the successful story of the LANS-α as a closure model in turbulent channels and pipes applies word for word to the Leray-α model (1.3). Whether this is a mere coincidence or there is something much deeper to understand is a subject of current and future investigation. It is worth mentioning that there is already a preliminary computational comparison study which indicates that the Leray-α model is a valid competitor to the LANS-α and other sub-grid scale models of turbulence. Indeed, the LES applications tests for turbulent mixing layers in Geurts & Holm (2002*a*), Geurts & Holm (2002*b*) and Geurts & Holm (2003) found that the Leray-α model predicted the resolved energy evolution properly, exhibiting both forward and backward transfer of energy. Further analysis showed accurate momentum-thicknesses and reliable levels of turbulence intensities. The computational overhead associated with the Leray model was lower than that of dynamic (mixed) models and no introduction of ad hoc parameters was required. The regularized dynamics showed an appealing robustness at high Reynolds numbers. In a geophysical application (Holm & Nadiga 2003) the LANS-α and Leray-α models both gave realistic simulations of mean motion in the double-gyre problem for simulating Gulf Stream eddies. Thus the main purpose of this paper is to show that certain simple models (see, for instance, Cao *et al.* (2005), based on Clark *et al.* (1979) and Leonard (1974)) compare favourably with empirical data for time-averaged fluid quantities as well as the NS-α model does. These models may be more phenomenological than the NS-α model, but their comparisons with empirical data are just as valid. These models are meant to approximate Eulerian average fluid quantities. And Eulerian averaging in general is not known to have either a variational principle or a circulation theorem.

In § 2 we introduce the functional setting of the Leray-α model and establish some *a priori* bounds which are useful for later sections. The global existence and regularity of the Leray-α model is a classical result and can be found in many textbooks on the mathematical theory of the NSEs. Therefore, we will omit it. In § 3 we provide explicit upper bounds for the dimension of the global attractor of the Leray-α model in terms of the relevant physical parameters. Specifically, we show that the number of degrees of freedom in the Leray-α model is of the order of

$$\left(\frac{L}{l_{\mathrm{d}}}\right)^{12/7}\left(1+\frac{L}{\alpha}\right)^{9/14},$$

where l_{d} is the small dissipation length-scale associated with this model. Note that the number of degrees of freedom here does not grow cubically with the size of the domain as would be expected for three-dimensional (3D) systems. This is a strong indication that the Leray-α model has a great potential as a sub-grid scale large-eddy simulation model. In § 4 we follow the work in Foias (1997) and Foias *et al.* (2001*a*) (see also Foias *et al.* 2001*b*) and derive, using physical arguments, power laws for the energy spectra of the Leray-α model. Specifically, we show that for very high Reynolds numbers the inertial range consists of two parts. In the first part when $\kappa\alpha \ll 1$ we find the usual Kolmogorov $\kappa^{-5/3}$ power law, and for $\kappa\alpha \gg 1$ we have a different, much steeper, power law. We derive different power laws depending on what one might use for a typical eddy turnover time. Since we have several options in this model, the power laws may vary. Computational studies, reported in § 5 indicate that around the wavenumber $\kappa = 1/\alpha$ the energy spectrum becomes steeper than $\kappa^{-5/3}$. Limited by the available computer power, we are unable to produce a wide enough inertial range to separate the two different parts of the energy spectra. It is worth adding that we have similar behaviour in the LANS-α, and intensive computational studies are being carried out by various groups to investigate this potential anomaly in the behaviour of the energy spectra of the LANS-α and Leray-α models. In § 6 we follow Cheskidov (2002, 2004) to develop a Leray-α Prandtl-like boundary-layer model. We study this model analytically as well as computationally. We noticed that it is much easier to study this model analytically than the corresponding LANS-α model studies in Cheskidov (2002, 2004). We tested this model successfully against the boundary layer empirical data. It is worth adding that other studies of boundary layer α-models have been reported in Holm *et al.* (2003) and Putkaradze & Weidman (2003).

2. *A priori* estimates

(*a*) *Functional setting*

First, let us introduce some notation and the functional setting. Recall the periodic box $\Omega = [0, 2\pi L]^3$ and fix a constant length-scale $\alpha > 0$. We denote by $(\cdot,\cdot)$ and $|\cdot|$ the L^2-inner product and the corresponding L^2-norm, respectively. We define

$$H = \left\{ u : u \in L^2(\Omega)^3,\ \nabla \cdot u = 0,\ u \text{ is periodic in periodic box } \Omega,\ \int_\Omega u \, \mathrm{d}x = 0 \right\},$$

and $V = H \cap H'(\Omega)^3$. Let $P_\sigma : L^2(\Omega)^3 \to H$ be the L^2-orthogonal projection, referred to as the Leray–Helmholtz projector. Denote by $A = -P_\sigma \Delta$ the Stokes operator with the domain $D(A) = (H^2(\Omega))^3 \cap V$. In the periodic case $A = -\Delta$. The Stokes operator is a self-adjoint positive operator with compact inverse. The eigenvalues of A are denoted by λ_j, so that

$$\frac{1}{L^2} = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

The inner product in V will be denoted by

$$((u, v)) := (A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \quad \|u\| := |A^{1/2}u|, \quad \text{for } u, v \in V.$$

Note that by the Poincaré inequality we have

$$|u|^2 \leqslant \frac{1}{\lambda_1}\|u\|^2 \tag{2.1}$$

for every $u \in V$. In order to have dimensionally homogeneous norms in $H^1(\Omega)^3$ and $H^2(\Omega)^3$, we will use the following inner products in these spaces, respectively:

$$\left.\begin{aligned} ((u, v))_{H^1} &:= \lambda_1[(u, v) + \alpha^2((u, v))], \\ ((u, v))_{H^2} &:= \lambda_1^2[(u, v) + 2\alpha^2((u, v)) + \alpha^4(Au, Av)]. \end{aligned}\right\} \tag{2.2}$$

Due to (2.2) we have

$$\lambda_1|v| \leqslant \|u\|_{H^2} \leqslant 2\lambda_1|v|, \quad \text{where } v = u - \alpha^2\Delta u, \tag{2.3}$$

i.e. the norm $\|u\|_{H^2}$ is equivalent to $\lambda_1|v|$, where $v = u - \alpha^2\Delta u$.

Following a well-accepted notation and well-established properties of the NSEs (see, for example, Constantin & Foias 1988; Foias *et al.* 2001*b*; Temam 1984, 1988 and references therein), we denote $B(u, v) := P_\sigma[(u \cdot \nabla)v] \in V'$ for all $u, v \in V$, where V' denotes the dual space of V. We denote by $\langle \phi, v\rangle_{V'}$ the dual action of $\phi \in V'$ on $v \in V$. The bilinear form B has the following property:

$$\langle B(u, v), w\rangle_{V'} = -\langle B(u, w), v\rangle_{V'}, \quad \text{for all } u, v, w \in V.$$

In particular,

$$\langle B(u,v),v\rangle_{V'}=0,\quad \text{for all } u,v\in V. \tag{2.4}$$

By analogy with the NSEs (see, for example, Constantin & Foias 1988; Foias *et al.* 2001*b*; Temam 1988 and references therein) the Leray-α model, system (1.3), in Ω is equivalent to the functional differential equation

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}v+\nu Av+B(u,v)=f,\\ u+\alpha^2Au=u-\alpha^2\Delta u=v,\\ u,v \text{ are periodic, with periodic box } \Omega\\ v|_{t=0}=v_0. \end{aligned}\right\} \tag{2.5}$$

For simplicity, we assume that the forcing term f does not depend on time.

As we have indicated in the introduction, Leray (1934) established the existence of solutions to the Navier–Stokes equations in $\mathbb{R}^n$, $n=2,3$. To accomplish this he introduced a modified system similar to (1.3), for which it was easier to establish the existence and uniqueness, and then by passing with the parameter $\alpha\searrow 0^+$ he could achieve existence of solutions to the Navier–Stokes equations. Indeed, the global existence of solutions to (1.3) in $\mathbb{R}^n$, $n=2,3$ follows from Leray's 1934 analysis. For the periodic case, similar arguments to those established for the 3D LANS-α model (see Foias *et al.* 2002) lead to the global existence and uniqueness of weak and strong solutions to the system (1.3) (equivalently (2.5)). Here, we will only state the theorem without a proof. However, we will formally establish *a priori* estimates on the solutions, which we will need later when we discuss global attractors for the system (2.5). Let us stress that all these estimates can be proved rigorously using, for instance, the Galerkin approximation procedure following, for instance, Foias *et al.* (2002).

Theorem 2.1 (Leray 1934). *Let $T>0$, $\nu>0$, $\alpha>0$ be given.*

(i) *If $f\in V'$ and $v_0\in H$, then the system (2.5) has a unique weak solution on $[0,T]$. That is, there is a unique function v such that*

$$v\in L^\infty((0,T);H)\cap L^2((0,T);V)\cap C([0,T];H\text{-weak})$$

with $(\mathrm{d}/\mathrm{d}t)v\in L^2((0,T);V')$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v,\phi\rangle_{V'}+\nu(A^{1/2}u,A^{1/2}\phi)+\langle B(u,v),\phi\rangle_{V'}=\langle f,\phi\rangle_{V'}$$

in $\mathcal{D}'((0,T))$, for every $\phi\in V$, where $u=(I+\alpha^2A)^{-1}v$ and $v(0)=v_0$.

(ii) *If $f\in H$, $v_0\in V$, then the unique weak solution $v(t)$ mentioned in (i) is a strong solution on $(0,T)$. That is, $v\in C([0,T];V)\cap L^2((0,T);D(A))$ with $(\mathrm{d}/\mathrm{d}t)v\in L^2((0,T);H)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(v,\phi)+\nu(A^{1/2}u,A^{1/2}\phi)+(B(u,v),\phi)=(f,\phi)$$

in $\mathcal{D}'((0,T))$, for every $\phi\in V$, where $u=(I+\alpha^2A)^{-1}v$ and $v(0)=v_0$.

Next, we will present formal *a priori* estimates for the solutions established in the above theorem. As we have mentioned before, these estimates can be obtained rigorously using the Galerkin procedure.

(*b*) L^2-*estimates*

Taking the inner product of (2.5) with v and using (2.4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 = (f, v).$$

By the Cauchy–Schwarz inequality and the Poincaré inequality (2.1), we reach

$$\begin{aligned}(f, v) \leqslant |f||v| &\leqslant \frac{|f|^2}{2\nu\lambda_1} + \frac{\nu\lambda_1}{2}|v|^2 \\ &\leqslant \frac{|f|^2}{2\nu\lambda_1} + \tfrac{1}{2}\nu\|v\|^2.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\|v\|^2 \leqslant \frac{|f|^2}{\nu\lambda_1}. \tag{2.6}$$

Using (2.1) one more time we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu\lambda_1|v|^2 \leqslant \frac{|f|^2}{\nu\lambda_1}.$$

Using Grönwall's inequality we conclude that

$$|v(t)|^2 \leqslant \mathrm{e}^{-\nu\lambda_1 t}|v(0)|^2 + \frac{(1-\mathrm{e}^{-\nu\lambda_1 t})|f|^2}{\nu^2\lambda_1^2} =: R(t), \tag{2.7}$$

and as result we have

$$\limsup_{t\to\infty} |v(t)| \leqslant R_0 := \frac{|f|}{\nu\lambda_1}.$$

Hence $B_1 = \{w \in H : |w| \leqslant R_0\}$ is an absorbing ball for the solution $v(t)$. Moreover, (2.3) implies

$$\limsup_{t\to\infty} \|u(t)\|_{H^2} \leqslant 2\lambda_1 R_0.$$

Therefore, $B_2 = \{w \in H : \|w\|_{H^2} \leqslant 2\lambda_1 R_0\}$ is an absorbing ball for the solution $u(t)$.

Furthermore, for every $T > 0$ we have, from (2.6),

$$|v(T)|^2 + \nu\int_0^T \|v(\tau)\|^2\,\mathrm{d}\tau \leqslant |v(0)|^2 + T\frac{|f|^2}{\nu\lambda_1}. \tag{2.8}$$

Thus $v \in L^2((0, T); V)$ for all $T > 0$.

(*c*) H^1-*estimates*

Taking the inner product of (2.5) with Av we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \nu|Av|^2 = (f, Av) - (B(u, v), Av).$$

Thus, by Cauchy–Schwartz, Young and Hölder inequalities we reach

$$\begin{aligned}|(f,Av)|+|(B(u,v),Av)| &\leqslant \frac{|f|^2}{\nu}+\tfrac{1}{4}\nu|Av|^2+|Av|\|v\|\|u\|_{L^\infty}\\ &\leqslant \frac{|f|^2}{\nu}+\tfrac{1}{4}\nu|Av|^2+\tfrac{1}{4}\nu|Av|^2+\frac{1}{\nu}\|v\|^2\|u\|_{L^\infty}^2\\ &\leqslant \frac{|f|^2}{\nu}+\tfrac{1}{2}\nu|Av|^2+\frac{1}{\nu}\|v\|^2\|u\|_{L^\infty}^2.\end{aligned}$$

Note that by the Sobolev inequality in three dimensions we have

$$\|u\|_{L^\infty}\leqslant \frac{c}{\lambda_1^{1/4}}\|u\|_{H^2},$$

for some dimensionless universal constant c. Therefore,

$$|(f,Av)|+|(B(u,v),Av)|\leqslant \frac{|f|^2}{\nu}+\tfrac{1}{2}\nu|Av|^2+\frac{c^2}{\nu\lambda_1^{1/2}}\|v\|^2\|u\|_{H^2}^2.$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\tfrac{1}{2}\nu|Av|^2\leqslant \frac{|f|^2}{\nu}+\frac{c^2}{\nu\lambda_1^{1/2}}\|v\|^2\|u\|_{H^2}^2.$$

We use (2.7) and (2.3) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+\|v\|^2)\leqslant K(t)(1+\|v\|^2),$$

where

$$K(t)=\max\left\{\frac{2|f|^2}{\nu},\frac{4c^2\lambda_1^{3/2}R^2(t)}{\nu}\right\}.$$

Now Grönwall's inequality implies that

$$1+\|v(t)\|^2\leqslant (1+\|v(s)\|^2)\exp\left\{\int_s^t K(\tau)\,\mathrm{d}\tau\right\},\quad t\geqslant s\geqslant 0.$$

Since for any $T>0$ $K(\tau)$ is integrable on $(0,T)$ (by (2.7)) and because of (2.8) we have that $v\in L^\infty([0,T];V)$, whenever $v_0\in V$, and $v\in L^\infty_{\mathrm{loc}}((0,T];V)$, whenever $v_0\in H$.

Denote by $S(t)$ the semi-group of the solution operators to the equation (2.5) corresponding to the unknown function $v(t)$, i.e. we have that $v(t)=S(t)v_0$. Following similar arguments as those well established for the two-dimensional (2D) NSEs (see, for example, Babin & Vishik 1992; Constantin & Foias 1985, 1988; Temam 1988), one can easily prove the following theorem.

Theorem 2.2. *Let $v_0, f\in H$. Then for any $T>0$ the semi-group $S(t)$ is compact and differentiable with respect to the initial data v_0 on the interval* $(0,T]$.

Since $S(t)$ is a compact semi-group and B_1 is an absorbing ball in H, the equation (2.5) has a unique global attractor

$$\mathcal{A}=\bigcap_{s>0}\bigcup_{t\geqslant s}S(t)B_1$$

(see, for example, Babin & Vishik 1992; Constantin & Foias 1985, 1988; Temam 1988).

3. Dimension of the attractor

Note that $|v|^2$ and $\nu\|v\|^2$ represent, in the Leray-α system (1.3), the kinetic energy and the rate of dissipation of energy, respectively. Therefore, by analogy with the conventional theory of turbulence à la Kolmogorov, the mean rate of dissipation of energy for the system (1.3) should be given by

$$\tilde{\epsilon}_{\text{Leray}} = \frac{\nu}{(2\pi L)^3} \langle \|v\|^2 \rangle,$$

where $\langle\cdot\rangle$ denotes an ensemble average. Influenced by the ergodic theorem of Birkhoff, people usually replace the ensemble average by the time average. In our case we will consider the worst scenario and define

$$\epsilon_{\text{Leray}} = \frac{\nu}{(2\pi L)^3} \sup_{v(0)\in\mathcal{A}} \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|v(\tau)\|^2 \, \mathrm{d}\tau \tag{3.1}$$

to be the mean rate of dissipation of energy for the system (1.3), which is finite because of (2.8) and the fact that we have a compact global attractor. Also by analogy with conventional theory of turbulence we set for the viscous dissipation length-scale

$$l_{\mathrm{d}} = \left(\frac{\nu^3}{\epsilon_{\text{Leray}}}\right)^{1/4},$$

which is supposed to represent the smallest scale that one needs to resolve in order to get a complete resolution for turbulent flows associated with the Leray-α model.

Theorem 3.1. *The Hausdorff and fractal dimensions of the global attractor of the Leray-α model satisfy*

$$d_{\mathrm{H}}(\mathcal{A}) \leqslant d_{\mathrm{F}}(\mathcal{A}) \leqslant c\left(\frac{L}{l_{\mathrm{d}}}\right)^{12/7}\left(1+\frac{L}{\alpha}\right)^{9/14},$$

for some universal constant c, which one can estimate explicitly.

Proof. We follow Constantin & Foias (1985) (see also Constantin & Foias 1988; Temam 1988, and references therein) and linearize the Leray-α model about a trajectory in the global attractor $v(t) = u(t) + \alpha^2 Au(t)$ to obtain

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\xi + \nu A\xi + B(u,\xi) + B(\eta, v) &= 0, \\ \eta + \alpha^2 A\eta &= \xi, \\ \xi(0) &= \xi^0. \end{aligned}\right\} \tag{3.2}$$

That is, the deviation $\xi(t)$, with initial deviation $\xi(0) = \xi^0$, evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi + \Lambda(t)\xi = 0,$$

where

$$\Lambda(t)\psi = A\psi + B(u(t),\psi) + B(\phi, v(t)), \qquad \phi + \alpha^2 A\phi = \psi, \quad u + \alpha^2 Au = v.$$

Let $\xi_j(t)$ be solutions of the above system with $\xi_j(0) = \xi_j^0$, $j = 1, \ldots, N$. Assume $\xi_1^0, \ldots, \xi_N^0$ are linearly independent. Let $Q_N(t)$ be the L^2-orthogonal projection from $L^2(\Omega)$ onto $\operatorname{span}\{\xi_1(t), \ldots, \xi_N(t)\}$. Then

$$\|(\xi_1\wedge\cdots\wedge\xi_N)(t)\|_{L^2}^2 = \|(\xi_1\wedge\cdots\wedge\xi_N)(0)\|_{L^2}^2 \exp\left\{-\int_0^t \operatorname{Tr}[Q_N(\tau)\circ\Lambda(\tau)\circ Q_N(\tau)]\,\mathrm{d}\tau\right\},$$

where $\operatorname{Tr}[\cdot]$ denotes the trace of a linear operator.

Now, let $\{\psi_1(t), \ldots, \psi_N(t)\}$ be an L^2-orthonormal basis of $\operatorname{span}\{\xi_1(t), \ldots, \xi_N(t)\}$, i.e. $(\psi_i, \psi_j) = \delta_{ij}$, and let $\phi_j = (I - \alpha^2\Delta)^{-1}\psi_j$. It is clear that $\psi_j \in H^1(\Omega)$, for $j = 1, 2, \ldots, N$. Recall that $(B(u, w), w) = 0$ (equation (2.4)) for all $u, w \in V$. Then we have

$$\begin{aligned}\operatorname{Tr}[Q_N(t)\circ\Lambda(t)\circ Q_N(t)] &= \sum_{j=1}^{N}(\Lambda(t)\psi_j, \psi_j)\\ &= \sum_{j=1}^{N}(\nu\|\psi_j\|^2 + (B(u, \psi_j), \psi_j) + (B(\phi_j, v), \psi_j))\\ &= \sum_{j=1}^{N}(\nu\|\psi_j\|^2 + (B(\phi_j, v), \psi_j))\\ &\geqslant \nu\sum_{j=1}^{N}\|\psi_j\|^2 - \left|\sum_{j=1}^{N}(B(\phi_j, v), \psi_j)\right|. \qquad (3.3)\end{aligned}$$

Note that

$$\begin{aligned}\left|\sum_{j=1}^{N}(B(\phi_j, v), \psi_j)\right| &= \left|\sum_{j=1}^{N}((\phi_j\cdot\nabla)v, \psi_j)\right|\\ &\leqslant \|v\|\|\rho_N\|_{L^\infty}\left(\sum_{j=1}^{N}\int_\Omega |\psi_j(x)|^2\,\mathrm{d}x\right)^{1/2}\\ &= \|v\|\|\rho_N\|_{L^\infty}N^{1/2}, \qquad (3.4)\end{aligned}$$

where

$$\rho_N^2(x) = \sum_{j=1}^{N}|\phi_j(x)|^2.$$

To finish the estimate for the $\operatorname{Tr}[Q_N(t)\circ\Lambda(t)\circ Q_N(t)]$ we need the following two propositions.

Proposition 3.2. *Let $\gamma = \alpha/L$. Then for every function $\phi \in H^2(\Omega)$*

$$\|\phi\|_{L^\infty} \leqslant C(\gamma)(2\pi L)^{-3/2}|(\phi + \alpha^2 A\phi)|,$$

where $C(\gamma)$ is given in equation (3.5).

Proof. We denote by

$$\hat{\phi}_\kappa = \frac{1}{(2\pi L)^3}\int_\Omega \phi(x)\mathrm{e}^{-\mathrm{i}(x/L)\kappa}\,\mathrm{d}x,$$

the Fourier coefficients of a function $\phi(x)$. Thus we have

$$|\phi| = \left(\sum_{\kappa\in\mathbb{Z}^3} |\hat{\phi}_\kappa|^2\right)(2\pi L)^3,$$

and

$$\begin{aligned}|\phi(x)| &= \left|\sum_{\kappa\in\mathbb{Z}^3} \hat{\phi}_\kappa \mathrm{e}^{\mathrm{i}(x/L)\kappa}\right| \leqslant \sum_{\kappa\in\mathbb{Z}^3} |\hat{\phi}_\kappa| \\ &\leqslant \left(\sum_{\kappa\in\mathbb{Z}^3} |\hat{\phi}_\kappa|^2(1+\gamma^2\kappa^2)^2\right)^{1/2}\left(\sum_{\kappa\in\mathbb{Z}^3}(1+\gamma^2\kappa^2)^{-2}\right)^{1/2}.\end{aligned}$$

It is obvious that there exists a universal constant $c_1 > 0$ (see, for example, Constantin & Foias (1988) or Temam (1988) for explicit bounds on c_1) such that

$$\sum_{\kappa\in\mathbb{Z}^3}(1+\gamma^2\kappa^2)^{-2} \leqslant \sum_{p=0}^{\infty}(1+c_1^2\gamma^2 p^{4/3})^{-2}.$$

Therefore,

$$\begin{aligned}\sum_{\kappa\in\mathbb{Z}^3}(1+\gamma^2\kappa^2)^{-2} &\leqslant \sum_{p=0}^{[(c_1\gamma)^{-3/2}]} 1 + \frac{1}{4} + \int_{(c_1\gamma)^{-3/2}}^{\infty} \frac{\mathrm{d}y}{c_1^4\gamma^4 y^{8/3}} \\ &\leqslant \left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} + \frac{5}{4} + \frac{3}{5}\left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} \\ &= \frac{5}{4} + \frac{8}{5}\left(\frac{1}{c_1^2\gamma^2}\right)^{3/4} =: C^2(\gamma). \qquad (3.5)\end{aligned}$$

Putting the above together we get

$$\|\phi\|_{L^\infty} \leqslant C(\gamma)\left(\sum_{\kappa\in\mathbb{Z}^3} |\hat{\phi}_\kappa|^2(1+\gamma^2\kappa^2)^2\right)^{1/2} = C(\gamma)(2\pi L)^{-3/2}|(\phi+\alpha^2 A\phi)|.$$

■

Proposition 3.3. *Let $\{\psi_1,\ldots,\psi_N\} \subset H^2(\Omega)$ be orthonormal in the L^2-inner product, i.e. $(\psi_k,\psi_l) = \delta_{kl}$. Let $\phi_\kappa = (I+\alpha^2 A)^{-1}\psi_\kappa$, $\kappa = 1,2,\ldots,N$. Let also*

$$\rho_N^2(x) = \sum_{j=1}^{N} |\phi_j(x)|^2.$$

There then exists a constant $C_{\mathrm{F}}(\gamma)$, independent of N, such that

$$\|\rho_N\|_{L^\infty} \leqslant C_{\mathrm{F}}(\gamma)(2\pi L)^{-3/2}. \qquad (3.6)$$

In fact, $C_{\mathrm{F}}(\gamma) \leqslant \sqrt{3}C(\gamma)$, where $C(\gamma)$ is given in (3.5).

Proof. Let $\theta_1, \ldots, \theta_N \in \mathbb{R}$ to be chosen later, such that $\sum_{\kappa=1}^N \theta_\kappa^2 = 1$. Then, by proposition 3.2,

$$\begin{aligned}\left|\sum_{\kappa=1}^N \theta_\kappa \phi_\kappa(x)\right| &\leqslant C(\gamma)(2\pi L)^{-3/2}\left|\sum_{\kappa=1}^N \theta_\kappa(\phi_\kappa + \alpha^2 A\phi_\kappa)\right| \\ &= C(\gamma)(2\pi L)^{-3/2}\left|\sum_{\kappa=1}^N \theta_\kappa \psi_\kappa\right| \\ &= C(\gamma)(2\pi L)^{-3/2}\left(\sum_{\kappa=1}^N |\theta_\kappa|^2\right)^{1/2} \\ &= C(\gamma)(2\pi L)^{-3/2},\end{aligned}$$

for all $x \in \Omega$, where we have used the orthogonality of $\{\psi_\kappa\}$. From the above we have

$$\left(\sum_{\kappa=1}^N \theta_\kappa \phi_\kappa^1(x)\right)^2 + \left(\sum_{\kappa=1}^N \theta_\kappa \phi_\kappa^2(x)\right)^2 + \left(\sum_{\kappa=1}^N \theta_\kappa \phi_\kappa^3(x)\right)^2 \leqslant C(\gamma)^2(2\pi L)^{-3}, \quad x \in \Omega.$$

Then we choose

$$\theta_\kappa = \frac{\phi_\kappa^1(x)}{(\sum_{\kappa=1}^N (\phi_\kappa^1(x))^2)^{1/2}},$$

and alternatively

$$\theta_\kappa = \frac{\phi_\kappa^2(x)}{(\sum_{\kappa=1}^N (\phi_\kappa^2(x))^2)^{1/2}} \quad \text{or} \quad \theta_\kappa = \frac{\phi_\kappa^3(x)}{(\sum_{\kappa=1}^N (\phi_\kappa^3(x))^2)^{1/2}}$$

to obtain

$$|\rho_N(x)|^2 \leqslant 3C(\gamma)^2(2\pi L)^{-3}, \quad x \in \Omega.$$

Hence, our estimate. ■

Now we go back to estimating $\mathrm{Tr}[Q_N(t) \circ \Lambda(t) \circ Q_N(t)]$. From (3.3), (3.4) and (3.6) we have

$$\begin{aligned}\mathrm{Tr}[Q_N(t) \circ \Lambda(t) \circ Q_N(t)] &\geqslant \nu \sum_{j=1}^N \|\psi_j\|^2 - \|v(t)\| \|\rho_N\|_{L^\infty} N^{1/2} \\ &\geqslant \nu \sum_{j=1}^N \lambda_j - \|v(t)\| C_{\mathrm{F}}(\gamma)(2\pi L)^{-3/2} N^{1/2}.\end{aligned}$$

Note that in the 3D case we have $\lambda_j \geqslant c_1 L^{-2} j^{2/3}$ for some positive universal constant c_1 (see, for example, Constantin & Foias 1988; Temam 1988). Therefore,

$$\mathrm{Tr}[Q_N(t) \circ \Lambda(t) \circ Q_N(t)] \geqslant c_2 \nu L^{-2} N^{5/3} - \|v(t)\| C_{\mathrm{F}}(\gamma)(2\pi L)^{-3/2} N^{1/2},$$

for some positive constant c_2. Hence,

$$\begin{aligned}\liminf_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{Tr}[Q_N(t)\circ\varLambda(t)\circ Q_N(t)]\,\mathrm{d}t \\ \geqslant c_2\nu L^{-2}N^{5/3} - C_{\mathrm{F}}(\gamma)(2\pi L)^{-3/2}N^{1/2}\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T \|v(t)\|^2\,\mathrm{d}t\right)^{1/2} \\ \geqslant c_2\nu L^{-2}N^{5/3} - C_{\mathrm{F}}(\gamma)(2\pi L)^{-3/2}N^{1/2}\left(\frac{L^3}{\nu}\epsilon_{\mathrm{Leray}}\right)^{1/2}.\end{aligned}$$

For $N \gg 1$, such that

$$N \geqslant \left(\frac{L}{l_{\mathrm{d}}}\right)^{12/7}\left(\frac{C_{\mathrm{F}}(\gamma)}{c_2}\right)^{6/7},$$

we have

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{Tr}[Q_N(t)\circ\varLambda(t)\circ Q_N(t)]\,\mathrm{d}t > 0.$$

Therefore, based on the trace formula (see Chepyzhov & Ilyin 2004; Constantin & Foias 1985, 1988; Temam 1988), this N is an upper bound for the dimension of the global attractor, i.e.

$$d_{\mathrm{H}}(\mathcal{A}) \leqslant d_{\mathrm{F}}(\mathcal{A}) \leqslant \left(\frac{L}{l_{\mathrm{d}}}\right)^{12/7}\left(\frac{C_{\mathrm{F}}(\gamma)}{c_2}\right)^{6/7}.$$

Since $C_{\mathrm{F}}(\gamma) \leqslant c_3(\gamma^{-3/4}+1)$ for some universal constant c_3, we have the following upper bound for the dimension of the global attractor:

$$d_{\mathrm{H}}(\mathcal{A}) \leqslant d_{\mathrm{F}}(\mathcal{A}) \leqslant c\left(\frac{L}{l_{\mathrm{d}}}\right)^{12/7}\left(1+\frac{L}{\alpha}\right)^{9/14},$$

for some universal constant c. This concludes the proof. ■

Remark 3.4. A heuristic physical argument in classical theory of turbulence suggests that the number of degrees of freedom for the 3D NSEs is proportional to $(L/l_{\mathrm{d}})^3$. This formula is still far from being reached rigorously for the 3D NSEs due to the lack of a proof for the global regularity of the 3D NSEs (see, however, Constantin *et al.* 1985). However, a similar formula has been shown to be correct for the LANS-α (NS-α or viscous Camassa–Holm) model (Foias *et al.* 2002). The above estimate, on the other hand, suggests that the number of degrees of freedom of the Leray-α model is much smaller than that of the NSEs or the LANS-α models. This remarkable result indicates that the Leray-α model might be much easier to compute with and that it lies, from the complexity point of view, between the 2D and 3D cases.

We observe that for $\gamma = \alpha/L$ large enough one can easily show, using energy estimates, that the dynamics of the Leray-α model is trivial and the attractor is a single stable steady state. Hence, the dimension of the global attractor tends to zero. While deriving the above estimate for the dimension of the attractor we assumed γ to be a positive finite number. In fact, we implicitly kept in mind that γ is a small number in order to stay 'close' to the 3D NSEs. See Ilyin & Titi (2003) for related results concerning the dependence of the global attractor on α for the 2D LANS-α or the NS-α model.

4. Energy spectrum

Following the work of Foias (1997) and Foias *et al.* (2001*a*) (see also Foias *et al.* 2001*b*) we provide here physical arguments for studying the energy spectrum of the Leray-α model, equations (1.3). Let

$$u_\kappa = \sum_{\kappa \leqslant |j| < 2\kappa} \hat{u}_j e^{ij\cdot x/L}, \qquad v_\kappa = \sum_{\kappa \leqslant |j| < 2\kappa} \hat{v}_j e^{ij\cdot x/L},$$

where again

$$\hat{\phi}_j = \frac{1}{(2\pi L)^3} \int_\Omega \phi(x) e^{-ij\cdot x/L}\, dx$$

denote the Fourier coefficients of the function $\phi(x)$. The energy balance for v_κ is given by

$$\frac{1}{2}\frac{d}{dt}(v_\kappa, v_\kappa) + \nu(-\Delta v_\kappa, v_\kappa) = T_\kappa - T_{2\kappa}, \tag{4.1}$$

where

$$T_\kappa = -((u_<^\kappa \cdot \nabla)v_\kappa, v_\kappa) + (((u_\kappa + u_>^\kappa)\cdot\nabla)(v_\kappa + v_>^\kappa), v_<^\kappa),$$

and

$$u_<^\kappa = \sum_{j<\kappa} u_j, \qquad u_>^\kappa = \sum_{j\geqslant 2\kappa} u_j.$$

Taking an ensemble average of (4.1), e.g. the long time average, we obtain

$$\nu\langle(-\Delta v_\kappa, v_\kappa)\rangle = \langle T_\kappa\rangle - \langle T_{2\kappa}\rangle.$$

In terms of the energy spectrum $E_\alpha^v(\kappa)$ of the variable v we have

$$\nu\kappa^3 E_\alpha^v(\kappa) \approx \nu\int_\kappa^{2\kappa} \eta^2 E_\alpha^v(\eta)\, d\eta \approx \langle T_\kappa\rangle - \langle T_{2\kappa}\rangle.$$

As long as

$$\nu\kappa^3 E_\alpha^v(\kappa) \ll \langle T_\kappa\rangle,$$

i.e. $\langle T_\kappa\rangle \approx \langle T_{2\kappa}\rangle$ (there is no leakage of energy due to dissipation), the wavenumber κ belongs to the inertial range.

As before, let $\tilde{\epsilon}_{\text{Leray}}$ represent the mean rate of dissipation of energy,

$$\tilde{\epsilon}_{\text{Leray}} := \left\langle \frac{\nu}{L^3}\int_\Omega (-\Delta v)\cdot v\, dx\right\rangle,$$

which in principle should be comparable with ϵ_{Leray}, which was introduced earlier, in equation (3.1). The average velocity of an eddy of spatial size of the order of $1/\kappa$ can be evaluated in three different ways:

$$U_\kappa^0 = \left\langle \frac{1}{L^3}\int_\Omega v_\kappa\cdot v_\kappa\, dx\right\rangle e^{1/2} = \left(\int_\kappa^{2\kappa} E_\alpha^v(\eta)\, d\eta\right)^{1/2} \sim \kappa^{1/2} E_\alpha^v(\kappa)^{1/2},$$

$$U_\kappa^1 = \left\langle \frac{1}{L^3}\int_\Omega u_\kappa\cdot v_\kappa\, dx\right\rangle e^{1/2} = \left(\int_\kappa^{2\kappa} \frac{E_\alpha^v(\eta)}{(1+\alpha^2\eta^2)}\, d\eta\right)^{1/2} \sim \frac{\kappa^{1/2}E_\alpha^v(\kappa)^{1/2}}{(1+\alpha^2\kappa^2)^{1/2}},$$

$$U_\kappa^2 = \left\langle \frac{1}{L^3}\int_\Omega u_\kappa\cdot u_\kappa\, dx\right\rangle e^{1/2} = \left(\int_\kappa^{2\kappa} \frac{E_\alpha^v(\eta)}{(1+\alpha^2\eta^2)^2}\, d\eta\right)^{1/2} \sim \frac{\kappa^{1/2}E_\alpha^v(\kappa)^{1/2}}{(1+\alpha^2\kappa^2)},$$

i.e.

$$U_\kappa^n \sim \frac{\kappa^{1/2} E_\alpha(\kappa)^{1/2}}{(1+\alpha^2\kappa^2)^{n/2}}, \quad n=0,1,2.$$

It is not clear, based on physical grounds, which one of these different expressions is the right one. As we see below, each expression will lead to a different power law in the energy spectrum. A careful study of the power laws in the energy spectra will shed some light on which of the above expressions is the right one, a subject of future and ongoing research. In the inertial range, according to the Kraichnan mechanism of energy cascade (Kraichnan 1972) (see also Foias 1997; Foias *et al.* 2001*a*,*b*), the turnover time of eddies of the spatial size $1/\kappa$ is the time it takes for the eddies of spatial size $1/\kappa$ to transfer their energy to the eddies of smaller size $1/(2\kappa)$, which is about

$$\tau_\kappa^n := \frac{1}{\kappa U_\kappa^n}, \quad n=0,1,2.$$

Then, for the different definitions of U_κ^n, $n=0,1,2$, we have

$$\tau_\kappa^n \approx \frac{(1+\alpha^2\kappa^2)^{n/2}}{\kappa^{3/2} E_\alpha^v(\kappa)^{1/2}}.$$

Therefore,

$$\tilde{\epsilon}_{\text{Leray}} = \frac{1}{\tau_\kappa^n} \int_\kappa^{2\kappa} E_\alpha^v(\eta)\,\mathrm{d}\eta \sim \frac{\kappa^{5/2} E_\alpha^v(\kappa)^{3/2}}{(1+\alpha^2\kappa^2)^{n/2}},$$

which implies the following spectral scaling law:

$$E_\alpha^v(\kappa) \sim (\tilde{\epsilon}_{\text{Leray}})^{2/3} \kappa^{-5/3} (1+\alpha^2\kappa^2)^{n/3}.$$

Consequently, the translational kinetic energy spectrum of the variable u is given by

$$E_\alpha^u(\kappa) = \frac{E_\alpha^v(\kappa)}{(1+\alpha^2\kappa^2)^2} \sim (\tilde{\epsilon}_{\text{Leray}})^{2/3} \kappa^{-5/3} (1+\alpha^2\kappa^2)^{(n-6)/3}.$$

Note that for $\alpha\kappa \ll 1$ the energy spectrum is the usual $\kappa^{-5/3}$ power law as for the Navier–Stokes equations. But for $\alpha\kappa \gg 1$ we have a steeper decaying power law $\kappa^{(2n-17)/3}$, for $n=0,1,2$. This indicates that the Leray-α model can serve as a very good sub-grid scale model. Similar results concerning the LANS-α (NS-α or viscous Camassa–Holm equations) have been reported in Foias *et al.* (2001*a*), based on the eddy turnover time τ_κ^2, i.e. $n=2$. It has been shown there that the power laws for the energy spectra in the initial range are $\kappa^{-5/3}$, for $\kappa\alpha \ll 1$, and κ^{-3} for $\kappa\alpha \gg 1$. Note that for the Leray-α model we also have $\kappa^{-5/3}$ for $\kappa\alpha \ll 1$, while we have $\kappa^{-13/3}$ for $\kappa\alpha \gg 1$, when we take $n=2$. Therefore, the Leray-α model decays even faster than the LANS-α (NS-α) model for $\kappa\alpha \gg 1$. Preliminary computational results which compare the energy spectra of the NSEs, LANS-α and the Leray-α support this observation (see figure 1).

5. Numerical simulations

Numerical simulations of flows with high-symmetry were conducted to compare the energy spectra of the Leray-α and LANS-α models to the incompressible Navier–Stokes equations. Flows with high-symmetry were first studied by Kida (1985). All

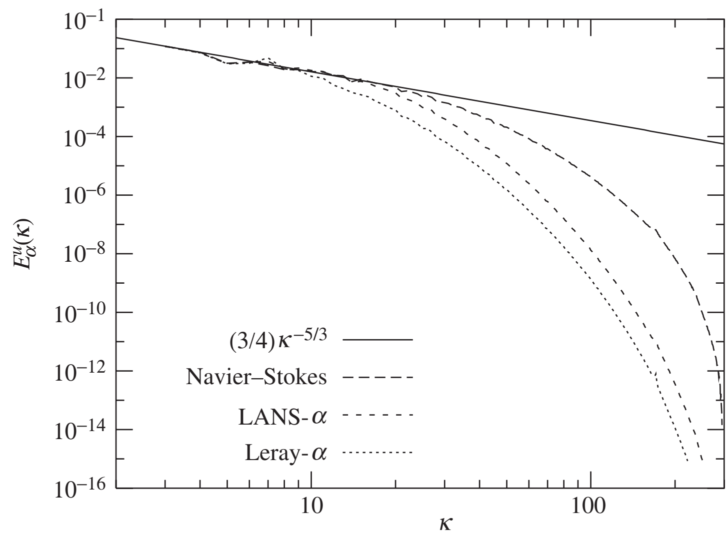

Figure 1. Comparison of the average energy spectra of the Navier–Stokes equations to the LANS and Leray models of turbulence for $\nu = 0.001$ and $\alpha = 0.05$.

computations were carried out using a modified version of the FORTRAN code of Nie & Tanveer (1999); see also Constantin *et al.* (1999). Changes were made to implement the Leray-α and LANS-α models. The actual calculations were made at the Department of Mathematics, University of California, Irvine, using Intel Xeon dual 1.8 Ghz P4 Beowulf compute nodes.

Fourier transforms were performed on a 128^3 grid using the $2/3$ rule to avoid aliasing. Due to the high-symmetry of the flow, the spatial resolution of our calculation is comparable with turbulence in a periodic box using Fourier transforms of size 512^3. Time was integrated using a second-order Adams–Bashforth method with a step size of 0.0005. We took viscosity $\nu = 0.001$ and $\alpha = 0.05$. The forcing function f was designed so that for $|k| \leqslant 4$ the Fourier modes $\hat{u}_k$ of the solution remained constant in time. The initial value was taken to be

$$u_0(x, y, z) = [U_0(x, y, z), U_0(y, z, x), U_0(z, x, y)],$$

where

$$\begin{aligned} U_0(x, y, z) = {} & 0.400\,312\,33 \sin x(\cos 3y \cos z - \cos y \cos 3z) \\ & + 0.222\,724\,69 \sin 3x(\cos 3y \cos z - \cos y \cos 3z) \\ & + 0.070\,431\,73 \sin 4x(\cos 2y + \cos 2z) \\ & - 0.140\,863\,46 \sin 2x(\cos 4y + \cos 4z). \end{aligned}$$

We calculated the translational energy spectrum $E^u_\alpha(\kappa)$ for the 3D Leray-α, LANS-α and incompressible Navier–Stokes equations by averaging in time from $t = 33$ to 100. It is evident from figure 1 that LANS-α has a more compact spectrum than the Navier–Stokes equations. This is consistent with results reported earlier in Chen *et al.* (1999*c*) and Mohseni *et al.* (2003). Note also that Leray-α has an even more compact spectrum than LANS-α. This is consistent with our analysis, which estimates a faster rate of decay for the energy spectrum of the Leray-α.

Our analytical estimate on the dimension of the global attractor indicates that the degrees of freedom of Leray-α is significantly less than would be expected for extensive 3D turbulence. Therefore, the relative compactness of the energy spectrum for Leray-α should increase at higher Reynolds numbers.

6. Boundary-layer approximation

Following Cheskidov (2002, 2004) we derive here a boundary-layer approximation of the Leray-α model for a stationary two-dimensional flow near a surface, and then reduce it to an extension of the Blasius equation in the case of a zero pressure gradient flow near a flat plate. Let x be the coordinate along the surface, y the coordinate normal to the surface, and $u = (U, V)$ the mean velocity of the flow.

Consider the stationary two dimensional Leray-α model,

$$\left.\begin{aligned}(u\cdot\nabla)v &= \nu\Delta v - \nabla p,\\ \nabla\cdot u &= 0,\end{aligned}\right\} \tag{6.1}$$

where $v = (\gamma, \tau) = u - \nabla\cdot(\alpha^2(x)\nabla u)$. We supplement system (6.1) with non-slip boundary conditions $u|_{y=0} = 0$, as well as

$$\lim_{y\to\infty} u(x,y) = (U_{\mathrm{e}}, 0)$$

for all $x > 0$, where $(U_{\mathrm{e}}, 0)$ is the mean external velocity of the flow. In addition, we assume here that $\alpha(\cdot)$ is a function of the x variable.

Let us fix l on the x-axis and define $\epsilon(l)$ in the following way:

$$\epsilon(l) := \frac{1}{\sqrt{R_l}} = \sqrt{\frac{\nu}{U_{\mathrm{e}} l}}.$$

We change variables:

$$x_1 = \frac{x}{l}, \qquad y_1 = \frac{y}{\epsilon l}, \qquad U_1 = \frac{U}{U_{\mathrm{e}}}, \qquad V_1 = \frac{V}{\epsilon U_{\mathrm{e}}}, \qquad p_1 = \frac{p}{U_{\mathrm{e}}^2}, \qquad \alpha_1 = \frac{\alpha}{\epsilon l}.$$

Note that the new variables are dimensionless. Recall that α_1 is a function of x only. Then we obtain

$$\begin{aligned}\frac{1}{U_{\mathrm{e}}}\gamma(x,y) &= U_1(x_1,y_1) - \epsilon^2\alpha_1^2\frac{\partial^2}{\partial x_1^2}U_1 - \alpha_1^2\frac{\partial^2}{\partial y_1^2}U_1 - \epsilon^2\frac{\partial}{\partial x_1}\alpha_1^2\frac{\partial}{\partial x_1}U_1,\\ \frac{1}{U_{\mathrm{e}}}\tau(x,y) &= \epsilon V_1(x_1,y_1) - \epsilon^3\alpha_1^2\frac{\partial^2}{\partial x_1^2}V_1 - \epsilon\alpha_1^2\frac{\partial^2}{\partial y_1^2}V_1 - \epsilon^3\frac{\partial}{\partial x_1}\alpha_1^2\frac{\partial}{\partial x_1}V_1.\end{aligned}$$

Neglecting the terms in equation (6.1) with high powers of ϵ, dropping subscripts and denoting

$$W = \left(1 - \alpha^2\frac{\partial^2}{\partial y^2}\right)U,$$

we arrive at the following Prandtl-like boundary-layer approximation of the Leray-α model:

$$\left.\begin{aligned} U\frac{\partial}{\partial x}W + V\frac{\partial}{\partial y}W &= \frac{\partial^2}{\partial y^2}W - \frac{\partial}{\partial x}p,\\ \frac{\partial}{\partial y}p &= 0,\\ \frac{\partial}{\partial x}U + \frac{\partial}{\partial y}V &= 0. \end{aligned}\right\} \tag{6.2}$$

For ϵ small enough we have

$$U(x,y) \approx U_{\mathrm{e}} U_\infty\left(\frac{x}{l}, \frac{y}{\sqrt{l\cdot l_{\mathrm{e}}}}\right), \qquad V(x,y) \approx \frac{U_{\mathrm{e}}}{\sqrt{R_l}} V_\infty\left(\frac{x}{l}, \frac{y}{\sqrt{l\cdot l_{\mathrm{e}}}}\right),$$

where l_{e} is a length associated with the external flow $l_{\mathrm{e}} = \nu/U_{\mathrm{e}}$ and (U_∞, V_∞) is a solution of (6.2).

Next we simplify (6.2) using Blasius's similarity variable in the case of a zero pressure gradient, i.e. we assume that

$$\frac{\partial}{\partial x}p = 0,$$

and the exterior velocity U_{e} is constant. We will study the flow near some fixed point x_0 on the plate. Let us choose the origin on the plate so that the point x_0 has the coordinates $(l, 0)$, where l is a parameter of the boundary layer. Now, we assume that α is proportional to $\sqrt{x}$, i.e.

$$\alpha = \sqrt{x}\beta,$$

where β is another parameter of the boundary layer. In addition, we will study the solutions (U_∞, V_∞) of (6.2) that on some adequate interval $l-\epsilon < x < l+\epsilon$ are of the form

$$U_\infty = f(\xi), \qquad V_\infty = \frac{1}{\sqrt{x}}g(\xi), \quad \xi = \frac{y}{\sqrt{x}}. \tag{6.3}$$

Now we obtain the following equations for f and g:

$$\begin{aligned} -\tfrac{1}{2}ff'\xi + \beta^2 f(\tfrac{1}{2}f'''\xi + f'') - \beta^2 ff'' + gf' - \beta^2 gf''' &= f'' - \beta^2 f^{\mathrm{iv}},\\ g' - \tfrac{1}{2}\xi f' &= 0. \end{aligned}$$

Let

$$h(\xi) = \int_0^\xi f(\eta)\,\mathrm{d}\eta.$$

Then $g = \frac{1}{2}\xi h' - \frac{1}{2}h$ and we have the following equation for h:

$$h''' + \tfrac{1}{2}hh'' - \beta^2(h^{\mathrm{v}} + \tfrac{1}{2}h^{\mathrm{iv}}) = 0. \tag{6.4}$$

The boundary condition $U|_{y=0} = 0$ requires $f(0) = 0$ and thus $h(0) = h'(0) = 0$. In addition, the physical interpretation of

$$\nu\frac{\partial}{\partial y}U \quad \text{for } y = 0$$

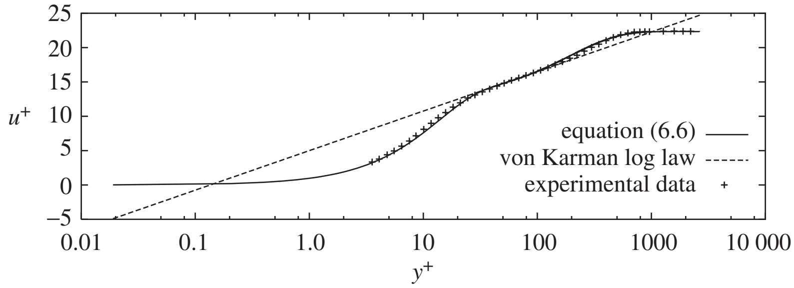

Figure 2. Comparison with experimental data from the Rolls-Royce Applied Science Laboratory, ERCOFTAC t3b test case for $c_{\mathrm{f}} = 0.004\,01$, $R_\theta = 1436$.

as the shear stress on the wall imposes the condition $f'(0) > 0$, that is, $h''(0) > 0$. Moreover, $U(x, y) \to U_{\mathrm{e}}$ as $y \to \infty$ requires that $h'(\xi) \to 1$ as $\xi \to \infty$.

Note that if $\hat{h}(\xi)$ is a solution of (6.4), then $h(x) := \beta\hat{h}(\beta x)$ is a solution of

$$-h^{\mathrm{v}} - \tfrac{1}{2}hh^{\mathrm{iv}} + h''' + \tfrac{1}{2}hh'' = 0. \tag{6.5}$$

This equation can be also written as

$$\left.\begin{aligned} m''' + \tfrac{1}{2}hm'' = 0,\\ m = h - h''. \end{aligned}\right\} \tag{6.6}$$

Here again $h(0) = h'(0) = 0$, $h''(0) > 0$. In addition, $U(x, y) \to U_{\mathrm{e}}$ as $y \to \infty$ requires that $h'(\xi) \to \beta^2$ as $\xi \to \infty$.

Note that equation (6.5) is the same as the corresponding equation for the LANS-α (NS-α) model. In Cheskidov (2004) it was proved that the solutions of this equation satisfying the above physical boundary conditions form a two-parameter family. These two parameters are the skin friction coefficient c_{f}, and the Reynolds number based on momentum thickness R_θ, and they determine the velocity profile for each horizontal coordinate. The family of velocity profiles $\{u_{R_\theta, c_{\mathrm{f}}}\}$ match experimental data for a wide range of Reynolds numbers (see figure 2). Another version on the boundary-layer approximation of the LANS-α (NS-α or viscous Camassa–Holm) model and its applications to turbulent jets and wakes is presented in Holm *et al.* (2003) and Putkaradze & Weidman (2003).

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