GEOMETRIC DECOMPOSITION OF SMOOTH FINITE ELEMENTS

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ABSTRACT. This chapter focuses on the geometric decomposition of the C^m-conforming finite elements on simplexes in arbitrary dimension constructed by Hu, Lin and Wu. The distance structure is introduced for the simplicial lattice to present a key non-overlapping decomposition of the simplicial lattice, in which each component will be used to determine the normal derivatives at each lower dimensional sub-simplex.

CONTENTS

In a recent work [6], Hu, Lin and Wu have constructed a C^m-conforming finite elements on simplexes in arbitrary dimension for any positive integer m. It unifies the scattered results [3, 10, 1] in two dimensions, [11, 12, 8] in three dimensions, and [13] in four dimensions. In this chapter, we use the simplicial lattice to give a geometric decomposition of the finite element spaces constructed in [6] and consequently give a different construction of HLW element.

Our approach is closely related to the multivariate splines on triangulations [5, 7]. For example, construction of C^m element in $d = 2, 3$, but not arbitrary $d \geq 2$, can be also found in the book [7, Section 8.1 for 2D and Section 18.11 for 3D]. A key distinction between the HLW element and multivariate splines lies in the choice of Degrees of Freedom (DoFs). In multivariate splines, DoFs are typically selected as the function and its derivative values at specific points, aligning with their primary focus on data interpolation. In contrast, HLW element [6] establishes DoFs through integral forms on sub-simplexes. This choice

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not only facilitates straightforward proof of unisolvence but also proves advantageous in constructing finite element de Rham complexes, as demonstrated in our recent work [4].

1. Distance and Derivative in Simplicial Lattice

In this section we introduce distance structure to the simplicial lattice and use it to study the vanishing order of derivatives.

1.1. **Simplicial lattices.** Recall that a simplicial lattice, also known as the principal lattice [9], of degree k and dimension d is a multi-index set of $d+1$ components and with fixed sum k, i.e.,

$$\mathbb{T}^d_k = \left\{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{0:d} : |\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_d = k\right\}.$$

Let a d-simplex T be generated by vertices $\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_d$. We let $\Delta(T)$ denote all the subsimplices of T, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension ℓ, for $0 \leq \ell \leq d$. For a sub-simplex $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, d-1$, we will overload the notation f for both the geometric simplex and the algebraic set of indices. Namely $f = \{f(0), \ldots, f(\ell)\} \subseteq \{0, 1, \ldots, d\}$ and

$$f = \mathrm{Convex}(\mathrm{v}_{f(0)}, \ldots, \mathrm{v}_{f(\ell)}) \in \Delta_\ell(T)$$

is the ℓ-dimensional simplex with vertices $\mathrm{v}_{f(0)}, \ldots, \mathrm{v}_{f(\ell)}$.

If $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, d-1$, then $f^* \in \Delta_{d-\ell-1}(T)$ denotes the sub-simplex of T opposite to f. When treating f as a subset of $\{0, 1, \ldots, d\}$, $f^* \subseteq \{0, 1, \ldots, d\}$ so that $f \cup f^* = \{0, 1, \ldots, d\}$, i.e., f^* is the complement of set f. Geometrically,

$$f^* = \mathrm{Convex}(\mathrm{v}_{f^*(1)}, \ldots, \mathrm{v}_{f^*(d-\ell)}) \in \Delta_{d-\ell-1}(T)$$

is the $(d-\ell-1)$-dimensional simplex spanned by vertices not contained in f. See Fig. 1 for examples of f and f^*.

For $\alpha \in \mathbb{T}^d_k(T)$ and $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, d-1$, we have the following direct decomposition

$$\alpha = \alpha_f + \alpha_{f^*}, \text{ and } |\alpha| = |\alpha_f| + |\alpha_{f^*}|,$$

where α_f only keeps the component in f and set others zero. For example, when $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_4)$ and $f = \{1, 3\}$, then $\alpha_f = (0, \alpha_1, 0, \alpha_3, 0, 0)$ and $\alpha_{f^*} = (\alpha_0, 0, \alpha_2, 0, \alpha_4)$.

In summary, by treating f as a set of indices, we can apply the operators $\cup, \cap, ^*, \backslash$ on sets. While treating f as a geometric simplex, $\partial f, \mathring{f}$ etc. can be applied. For example, $\mathbb{T}^\ell_k(\mathring{f})$ is the sub-set of lattice nodes of $\mathbb{T}^d_k(T)$ whose geometric coordinate is in the interior of an ℓ-dimensional $f \in \Delta_\ell(T)$.

1.2. **Distance.** Given $f \in \Delta_\ell(T)$, $\ell = 0, 1, \ldots, d-1$, we define the distance of a node $\alpha \in \mathbb{T}^d_k$ to f as

$$\mathrm{dist}(\alpha, f) := |\alpha_{f^*}| = \sum_{i \in f^*} \alpha_i. \tag{1}$$

We present a geometric interpretation of $\mathrm{dist}(\alpha, f)$. Set the vertex $\mathrm{v}_{f(0)}$ as the origin and embed the lattice to the scaled reference simplex $k\hat{T} = \mathrm{Convex}\{\mathbf{0}, k\boldsymbol{e}_1, \ldots, k\boldsymbol{e}_d\}$, where $\{\boldsymbol{e}_i, i = 1, \ldots, d\}$ is the canonical basis of $\mathbb{R}^d$. Then $|\alpha_{f^*}| = s$ becomes the linear equation

$$x_{f^*(1)} + x_{f^*(2)} + \ldots + x_{f^*(d-\ell)} = s,$$

which defines a hyper-plane in $\mathbb{R}^d$, denoted by $L(f,s)$, with a normal vector $\mathbf{1}_{f^*}$. The simplex f can be thought of as the convex combination of vectors $\{\boldsymbol{e}_{f(0)f(i)}\}_{i=1}^{\ell}$. Obviously $\mathbf{1}_{f^*}\cdot \boldsymbol{e}_{f(0)f(i)} = 0$ as the zero pattern is complementary to each other. So f is parallel to the hyper-plane $L(f,s)$. The distance $\operatorname{dist}(\alpha, f)$ for $\alpha \in L(f,s)$ is the intercept of the hyper-plane $L(f,s)$ and the basis vector $\boldsymbol{e}_{f(0)f(i)}$; see Fig. 1 for an illustration. In particular $f\in L(f,0)$ and

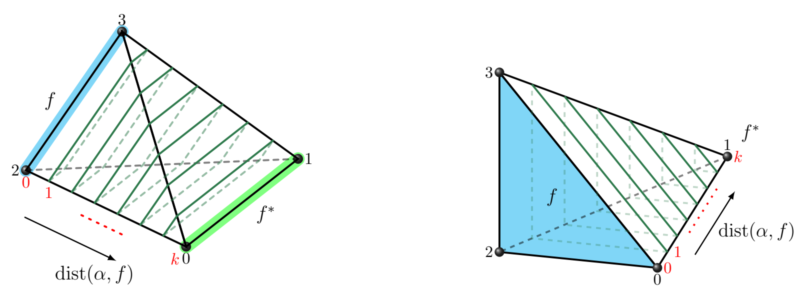
$$\lambda_i\mid_f = 0 \quad \text{if } i\notin f, \text{ i.e. } i\in f^*.$$

Indeed $f = \{x\in T : \lambda_i(x) = 0, i \in f^*\}$. See Chapter 2 (Lemma 2.3) for a proof.

We can extend the definition of distance to two sub-simplexes. For $e \in \Delta_\ell(T), f \in \Delta(T)$, define

$$\operatorname{dist}(e,f) := \min_{\alpha\in\mathbb{T}_k^\ell(e)} \operatorname{dist}(\alpha, f).$$

It is easy to verify that: for $e\in\Delta(f^*)$, $\operatorname{dist}(e,f) = k$ and for $e\in \Delta(f)$, $\operatorname{dist}(e,f)=0$.



(a) Distance to an edge. (b) Distance to a face.

FIGURE 1. Distance to a sub-simplex.

We define the lattice tube of f with distance r as

$$D(f,r) = \{\alpha\in\mathbb{T}_k^d, \operatorname{dist}(\alpha,f)\leq r\},$$

which contains lattice nodes at most r distance away from f. We overload the notation

$$L(f,s) = \{\alpha\in\mathbb{T}_k^d, \operatorname{dist}(\alpha,f) = s\},$$

which is defined as a plane early but here is a subset of lattice nodes on this plane. Then

$$D(f,r) = \cup_{s=0}^r L(f,s), \quad L(f,s) = L(f^*, k-s).$$

By definition $D(f,-1)=\varnothing$, $D(f,0) = L(f,0) = f$, and $L(f,k) = f^*$. See Fig. 1 for an illustration.

We have the following characterization of lattice nodes in $D(f,r)$.

Lemma 1.1. *For lattice node $\alpha\in\mathbb{T}_k^d$, and $f\in\Delta_\ell(T)$, $\ell = 0,1,\ldots,d-1$,*

$$\alpha\in D(f,r) \iff |\alpha_{f^*}|\leq r \iff |\alpha_f|\geq k-r,$$
$$\alpha\notin D(f,r) \iff |\alpha_{f^*}| > r \iff |\alpha_f|\leq k-r-1.$$

Proof. Use the definition of $\operatorname{dist}(\alpha,f)$ and the fact $|\alpha_f| + |\alpha_{f^*}| = k$. □

For each vertex $\mathrm{v}_i \in \Delta_0(T)$,

$$D(\mathrm{v}_i, r) = \{\alpha \in \mathbb{T}^d_k, |\alpha_{i^*}| \leq r\},$$

which is isomorphic to a simplicial lattice $\mathbb{T}^d_r$ of degree r; see the green triangle in Fig. 2 for $d = 2$, $k = 8$ and $r = 3$. For a $(d-1)$-face $f \in \Delta_{d-1}(T)$, $D(f, r)$ is a trapezoid of height r with base f. In general for $f \in \Delta_\ell(T)$, the hyper plane $L(f, r)$ will cut the simplex T into two parts, and $D(f, r)$ is the part containing f. See Fig. 1 for illustration.

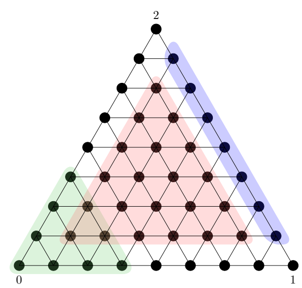


FIGURE 2. A simplicial lattice $\mathbb{T}^2_8(T)$ in two dimensions. The green triangle contains $D(\mathrm{v}_0, 3)$. The purple trapezoid is $D(\mathring{f}, 0)$. The red triangle is $\mathbb{T}^2_5(\mathring{T})$.

For two nodes $\alpha, \beta \in \mathbb{T}^d_k$, define the distance

$$\operatorname{dist}(\alpha, \beta) = \frac{1}{2}\|\alpha - \beta\|_{\ell_1}. \tag{2}$$

Two nodes $\alpha, \beta \in \mathbb{T}^d_k$ are adjacent if $\operatorname{dist}(\alpha, \beta) = 1$.

By assigning edges to all adjacent nodes, the simplicial lattice $\mathbb{T}^d_k$ becomes an undirected graph and denoted by $\mathcal{G}(\mathbb{T}^d_k)$. Obviously the graph $\mathcal{G}(\mathbb{T}^d_k)$ is connected. The distance of two nodes in the graph is the length of a minimal path connecting them, where the length of a path is defined as the number of edges in the path. Graph theory can be further applied for the study of the lattice $\mathbb{T}^d_k$ and in turn the polynomial space.

Define $\epsilon_i \in \mathbb{N}^{0:d}$ as $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ and $\epsilon_{ij} = \epsilon_i - \epsilon_j = (0, \ldots, 1, \ldots, -1, \ldots, 0)$.

Lemma 1.2. *For $\alpha, \beta \in \mathbb{T}^d_k$, it holds*

$$\operatorname{dist}(\alpha, \beta) = 1 \iff \beta = \alpha + \epsilon_{ij}, \quad \textit{for some } i, j \in [0, d], i \neq j.$$

Proof. Notice for two non-negative integers, if $\alpha_i \neq \beta_i$, then $|\alpha_i - \beta_i| \geq 1$. As $\alpha, \beta \in \mathbb{T}^d_k$, we have $\sum_{i=0}^d (\alpha_i - \beta_i) = 0$. The condition $\operatorname{dist}(\alpha, \beta) = 1$ means $\sum_{i=0}^d |\alpha_i - \beta_i| = 2$. So the only possibility is: $\alpha_i - \beta_i = -1$ and $\alpha_j - \beta_j = 1$ for some $0 \leq i, j \leq d$ and $i \neq j$. $\square$

Note that $\operatorname{dist}(\alpha, \beta)$ in (2) is defined for two nodes while $\operatorname{dist}(\alpha, f)$ in (1) is between a node and a sub-simplex. We show the two distance definitions are consistent.

Lemma 1.3. *For $\alpha \in \mathbb{T}^d_k$ and $f \in \Delta_\ell(T)$, for $\ell = 0, 1, \ldots, d-1$, it holds*

$$\operatorname{dist}(\alpha, f) = \min_{\beta_f \in \mathbb{T}^d_k(f)} \operatorname{dist}(\alpha, \beta_f).$$

Proof. For $\beta_f \in \mathbb{T}^d_k(f)$, since $|\beta_f| = k = |\alpha_f| + |\alpha_{f^*}|$, we have

$$\operatorname{dist}(\alpha, \beta_f) = \frac{1}{2}(\|\alpha_f - \beta_f\|_{\ell_1} + |\alpha_{f^*}|) \geq \frac{1}{2}(|\beta_f| - |\alpha_f| + |\alpha_{f^*}|) = |\alpha_{f^*}|.$$

Hence

$$\operatorname{dist}(\alpha, f) \leq \min_{\beta_f \in \mathbb{T}^d_k(f)} \operatorname{dist}(\alpha, \beta_f).$$

Then the equality holds by choosing $\beta_f = \alpha_f + |\alpha_{f^*}|\epsilon_{f(0)} \in \mathbb{T}^d_k(f)$. ☐

1.3. **Derivative and distance.** Recall that in [2] a smooth function u is said to vanish to order r on f if $D^\beta u|_f = 0$ for all $\beta \in \mathbb{N}^{1:d}, |\beta| < r$. The following result shows that the vanishing order r of a Bernstein polynomial λ^α on f is exactly the distance $\operatorname{dist}(\alpha, f)$.

Lemma 1.4. *Let $f \in \Delta_\ell(T)$ be a sub-simplex of T. For $\alpha \in \mathbb{T}^d_k, \beta \in \mathbb{N}^{1:d}$, then*

$$D^\beta \lambda^\alpha|_f = 0, \quad \textit{if } \operatorname{dist}(\alpha, f) > |\beta|,$$

Proof. For $\alpha \in \mathbb{T}^d_k$, we write $\lambda^\alpha = \lambda_f^{\alpha_f}\lambda_{f^*}^{\alpha_{f^*}}$. When $|\alpha_{f^*}| > |\beta|$, the derivative $D^\beta \lambda^\alpha$ will contain a factor $\lambda_{f^*}^\gamma$ with $\gamma \in \mathbb{N}^{1:d-\ell}$, and $|\gamma| = |\alpha_{f^*}| - |\beta| > 0$. Therefore $D^\beta \lambda^\alpha|_f = 0$ as $\lambda_i|_f = 0$ for $i \in f^*$. ☐

Consider the 1-D reference simplex $T = [0, 1]$ and $f = \{0\}$ the left vertex. Then $\lambda_0 = 1 - x$ and $\lambda_1 = x$. The result becomes $D^\beta((1-x)^{\alpha_f}x^{\alpha_{f^*}})\,|_{x=0} = 0$ if $|\alpha_{f^*}| > |\beta|$.

2. Smooth Finite Elements in Two Dimensions

We use two-dimensional case to explain the main idea of using the lattice decomposition to determine the derivative at sub-simplexes. Denote the smoothness vector $\boldsymbol{r} = (r_0, r_1)$ as the smoothness at edge r_1 and vertex r_0.

2.1. **Examples in two dimensions.** We present several decompositions of $\mathbb{T}^2_8$ for $\mathbb{P}_8$ in two dimensions. Fig. 3 (a) is the decomposition of the Lagrange element. In particular, at each vertex, function value is used as DoFs; see Chapter 2 for more details. In (b), the green triangle of vertex v is expanded so that the $D^\alpha u(\mathrm{v})$ can be determined for $|\alpha| \leq 2$. The vertex is for the function value. The 2 nodes in the second row are for 2 first derivatives, and the 3 nodes below are for 3 second derivatives. In general $D^\alpha u(\mathrm{v})$ can be identified with a lattice; see Lemma 2.1. In (c), the blue part is further expanded to include the normal derivative; see Lemma 2.4. The requirement $r_0 \geq 2r_1$ ensures the blue parts are non-overlapping; see Lemma 3.1.

2.2. **Derivatives at vertices.** Consider a function $u \in C^m(\Omega)$. The set of derivatives of order up to m can be written as

$$\{D^\beta u, \beta \in \mathbb{N}^{1:2}, |\beta| \leq m\}.$$

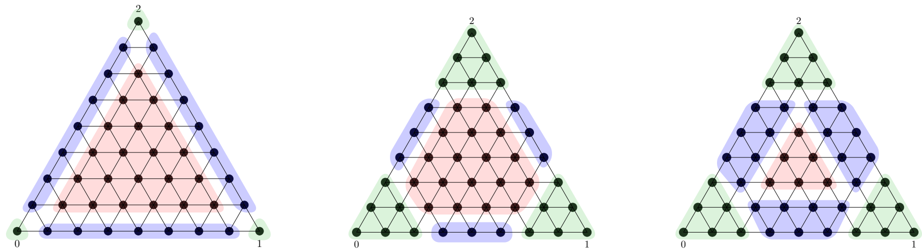
Notice that the multi-index $\beta \in \mathbb{N}^{1:2}$ not in $\mathbb{N}^{0:2}$. We can add a component with value $m - |\beta|$ to form a simplicial lattice $\mathbb{T}^2_m$ of degree m, which can be used to determine the derivatives at that vertex. See the green triangles in Fig. 3 (b) (c).

Lemma 2.1. *Let $i \in \{0, 1, 2\}$. The polynomial space*

$$\mathbb{P}_k(D(\mathrm{v}_i, m)) := \operatorname{span}\left\{\lambda^\alpha, \alpha \in \mathbb{T}^2_k, \operatorname{dist}(\alpha, \mathrm{v}_i) = |\alpha_{i^*}| \leq m\right\},$$

is uniquely determined by the DoFs

$$\{D^\beta u(\mathrm{v}_i), \beta \in \mathbb{N}^{1:2}, |\beta| \leq m\}. \tag{3}$$



(a) Lagrange element: $k = 8$, $m = 0, \boldsymbol{r} = (0, 0)$.

(b) Hermite-type element: $k = 8$, $m = 0, \boldsymbol{r} = (2, 0)$.

(c) C^1 element: $k = 8$, $m = 1, \boldsymbol{r} = (2, 1)$.

FIGURE 3. Geometric decompositions of Lagrange element, Hermite-type element, and a C^1 element.

Proof. Without loss of generality, consider v_0. Define map

$$\alpha = (\alpha_0, \alpha_1, \alpha_2) \to \beta = (\alpha_1, \alpha_2)$$

which induces a one-to-one map from $D(\mathrm{v}_0, m) = \left\{\alpha \in \mathbb{T}_k^2, \alpha_1 + \alpha_2 \leq m\right\}$ to $\{\beta \in \mathbb{N}^{1:2}, |\beta| \leq m\}$. So the dimension of $\mathbb{P}_k(D(\mathrm{v}_i, m))$ matches the number of DoFs (3). It suffices to show that for $u \in \mathbb{P}_k(D(\mathrm{v}_0, m))$ if DoFs (3) vanish, then $u = 0$.

When the triangle T is the reference triangle, v_0 is the origin and $\lambda_1 = x_1, \lambda_2 = x_2$. Recall the multivariate calculus result

$$D^\beta(x_1^{\alpha_1} x_2^{\alpha_2}) = \beta! \delta(\alpha, \beta) \text{ for } \alpha, \beta \in \mathbb{N}^{1:2}, |\alpha| = |\beta| = r \geq 0, \tag{4}$$

where $\delta(\alpha, \beta)$ is the Kronecker delta function. So we conclude that the homogenous polynomial space $\operatorname{span}\left\{x_1^{\alpha_1} x_2^{\alpha_2}, \alpha \in \mathbb{N}^{1:2}, |\alpha| = r\right\}$ is determined by DoFs $\{D^\beta u(\mathrm{v}_0), \beta \in \mathbb{N}^{1:2}, |\beta| = r\}$. Running $r = 0, 1, \ldots, m$, we finish the proof when the triangle is the reference triangle.

For a general triangle, instead of changing to the reference triangle, we shall use the barycentric coordinate. Clearly $\{\nabla \lambda_1, \nabla \lambda_2\}$ forms a basis of $\mathbb{R}^2$. Choose another basis $\{t^1, t^2\}$ of $\mathbb{R}^2$, being dual to $\{\nabla \lambda_1, \nabla \lambda_2\}$, i.e., $\nabla \lambda_i \cdot t^j = \delta_{i,j}$ for $i, j = 1, 2$. Indeed t^i is the edge vector e_{0i} as $\nabla \lambda_i$ is orthogonal to e_{0i} for $i = 1, 2$. We can express the derivatives in this non-orthogonal basis and denote by $D_n^\beta u := \frac{\partial^{|\beta|} u}{\partial (t^1)^{\beta_1} \partial (t^2)^{\beta_2}}$ with $\frac{\partial}{\partial t^i} = t^i \cdot \nabla$. By the duality $\nabla \lambda_i \cdot t^j = \delta_{i,j}$, $i, j = 1, 2$, we have the generalization of (4)

$$D_n^\beta(\lambda_1^{\alpha_1} \lambda_2^{\alpha_2}) = \beta! \delta(\alpha, \beta) \quad \text{for } \alpha, \beta \in \mathbb{N}^{1:2}, |\alpha| = |\beta| = r. \tag{5}$$

By the chain rule, it is easy to show that vanishing DoFs $\{D_n^\beta u(\mathrm{v}_0), \beta \in \mathbb{N}^{1:2}, |\beta| \leq m\}$ is equivalent to vanishing DoFs $\{D^\beta u(\mathrm{v}_0), \beta \in \mathbb{N}^{1:2}, |\beta| \leq m\}$.

A Bernstein basis of $\mathbb{P}_k(D(\mathrm{v}_0, m))$ is given by $\{\lambda_0^{k-|\alpha|} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}, \alpha \in \mathbb{N}^{1:2}, |\alpha| \leq m\}$. Assume $u = \sum_{\substack{\alpha \in \mathbb{N}^{1:2} \\ |\alpha| \leq m}} c_\alpha \lambda_0^{k-|\alpha|} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}$ with $c_\alpha \in \mathbb{R}$ and $D^\beta u(\mathrm{v}_0) = 0$ for all $\beta \in \mathbb{N}^{1:2}$ satisfying $|\beta| \leq m$. We shall prove $c_\alpha = 0$ by induction.

For $|\alpha| = 0$, as $c_{(0,0)} = u(\mathrm{v}_0) = 0$, we conclude $c_{(0,0)} = 0$. Assume $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^{1:2}$ satisfying $|\alpha| \leq r - 1$, i.e., $u = \sum_{\substack{\alpha \in \mathbb{N}^{1:2} \\ r \leq |\alpha| \leq m}} c_\alpha \lambda_0^{k-|\alpha|} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}$. By Lemma 1.4, the derivative $D^\beta(\lambda_0^{k-|\alpha|} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2})$ vanishes at v_0 for all $\beta \in \mathbb{N}^{1:2}$ satisfying $|\beta| < |\alpha|$. Hence, for $|\beta| = r$, using (5),

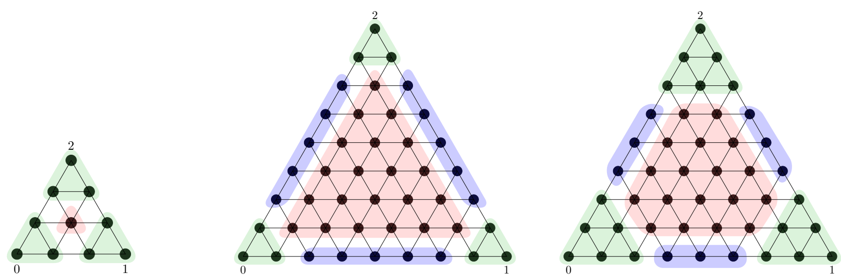
$$D_n^\beta u(\mathrm{v}_0) = D_n^\beta \Big(\sum_{\alpha\in\mathbb{N}^{1:2},|\alpha|=r} c_\alpha \lambda_0^{k-r}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\Big)(\mathrm{v}_0) = \beta! c_\beta = 0,$$

which implies $c_\beta = 0$ for all $\beta \in \mathbb{N}^{1:2}$, $|\beta| = r$. Running $r = 1, 2, \ldots, m$, we conclude $u = 0$. ☐

Such result can be generalized in the straight-forward way to arbitrary dimension $d \geq 1$. The set of derivatives of order up to m can be written as

$$\{D^\alpha u, \alpha \in \mathbb{N}^{1:d}, |\alpha| \leq m\}.$$

We add the component $\alpha_0 = m - |\alpha|$. Then the index set forms a simplicial lattice $\mathbb{T}_m^d$ of degree m. For each vertex, we can use the small simplicial lattice $\mathbb{T}_m^d \cong D(\mathrm{v}_i, m)$ to determine the derivatives at that vertex.



(a) $\boldsymbol{r} = (1, 0), k = 3$. (b) $\boldsymbol{r} = (1, 0), k = 8$. (c) $\boldsymbol{r} = (2, 0), k = 8$.

FIGURE 4. Lattice decompositions for the Hermite-type finite elements.

2.3. **Hermite-type finite elements.** Given an integer $m \geq 1$ and degree $k \geq 2m+1$, the Hermite-type finite elements are constructed based on the lattice decomposition

$$\mathbb{T}_k^d(T) = D(\Delta_0(T), m) \bigoplus_{\ell=1}^d \bigoplus_{f\in\Delta_\ell(T)} \left[\mathbb{T}_k^\ell(\mathring{f})\backslash D(\Delta_0(f), m)\right].$$

Consequently

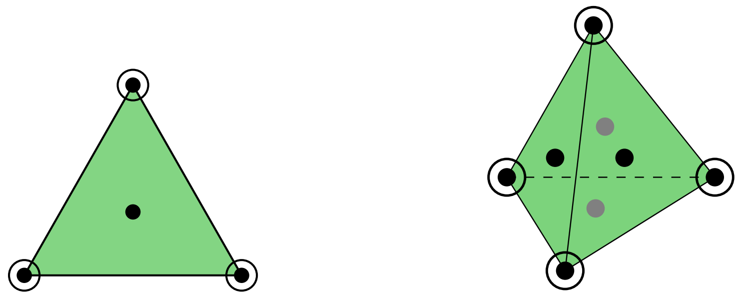
$$\mathbb{P}_k(T) = \mathbb{P}_k(D(\Delta_0(T), m)) \bigoplus_{\ell=1}^d \bigoplus_{f\in\Delta_\ell(T)} \mathbb{P}_k(\mathbb{T}_k^\ell(\mathring{f})\backslash D(\Delta_0(f), m)). \tag{6}$$

Classical Hermite element is $m = 1, k = 3$ and the general case will be called Hermite-type elements. Notice that when $d > 1$, the Hermite finite element space is C^0-conforming only but C^m continuous at vertices.

Lemma 2.2 (Hermite-type element in $\mathbb{R}^d$)**.** *Let $k \geq 2m+1$ and T be a d-dimensional simplex. The shape function space $\mathbb{P}_k(T)$ is determined by DoFs*

$$D^\alpha u(\mathrm{v}_i) \quad \alpha \in \mathbb{N}^{1:d}, |\alpha| \leq m, \mathrm{v}_i \in \Delta_0(T), i = 0, 1, \ldots, d, \tag{7}$$

$$\int_f u\, \lambda_f^{\alpha_f}\, \mathrm{d}s \quad \alpha_f \in \mathbb{T}_{k-(\ell+1)}^\ell, \alpha_f \leq k - m - 2, f \in \Delta_\ell(T), \ell = 1, \ldots, d. \tag{8}$$



(a) Hermite element in 2D (b) Hermite element in 3D

FIGURE 5. DoFs for the $\mathbb{P}_3$ Hermite element which is only C^0-conforming for dimension $d \geq 2$. Black dot represents moment DoF and the circle for the first derivative.

Proof. The proof is essentially the same as that for the Lagrange element (cf. See Chapter 2 (Theorem 3.2)) except using the decomposition (6) and Lemma 2.1 for lattice nodes in $D(\Delta_0(T), m)$. For a polynomial $u \in \mathbb{P}_k(\mathbb{T}_k^{\ell}(\mathring{f}) \backslash D(\Delta_0(f), m))$, as the distance of corresponding lattice nodes to all vertices are greater than m, (7) vanishes which means the DoF-Fun matrix is still block lower triangular.

The condition $k \geq 2m+1$ is required so that the disks $D(\mathrm{v}_i, m)$ are disjointed. The set

$$\mathbb{T}_k^{\ell}(\mathring{f}) \backslash D(\Delta_0(T), m) = \{\alpha_f \in \mathbb{T}_k^{\ell}(f), 1 \leq \alpha_f \leq k - m - 1\},$$

which can be verified as follows: for any $\mathrm{v} \in \Delta_0(f)$, $\alpha_{\mathrm{v}} \geq 1$ as $\alpha_f \in \mathbb{T}_k^{\ell}(\mathring{f})$. The condition $\alpha_f \notin D(\Delta_0(f), m)$ is equivalent to $\operatorname{dist}(\alpha_f, \mathrm{v}) = |\alpha_{\mathrm{v}^*}| > m$ which implies the upper bound $\alpha_{\mathrm{v}} \leq k - m - 1$.

Then we set $\tilde{\alpha}_f = \alpha_f - 1$ to get the lattice set in DoF (8). The degree of polynomial is reduced from k to $k - (\ell + 1)$ as $b_f = \lambda_f \in \mathbb{P}_{\ell+1}(f)$ is always positive in $\mathring{f}$. □

Exercise 2.3. *Prove that*

$$\mathbb{T}_k^{\ell}(\mathring{f}) \backslash D(\Delta_0(f), m) \cong \mathbb{T}_{k-(\ell+1)}^{\ell}(f) \backslash D(\Delta_0(f), m - \ell). \tag{9}$$

Geometrically we can consider the inner simplicial lattice and subtract vertex disks with a smaller radius.

Given a mesh $\mathcal{T}_h$, the decomposition can be naturally extended to the whole mesh

$$V^{\mathrm{H}}(\mathcal{T}_h) = \bigoplus_{\mathrm{v} \in \Delta_0(\mathcal{T}_h)} \mathbb{P}_k(D(\mathrm{v}, m)) \oplus \bigoplus_{\ell=1}^{d} \bigoplus_{f \in \Delta_\ell(\mathcal{T}_h)} \mathbb{P}_k(\mathbb{T}_k^{\ell}(\mathring{f}) \backslash D(\Delta_0(f), m)).$$

DoFs are single valued at each sub-simplex (symbolically change $\Delta_\ell(T)$ to $\Delta_\ell(\mathcal{T}_h)$). The obtained space $V^{\mathrm{H}}(\mathcal{T}_h)$ is C^0-conforming only for $d > 1$ but C^m continuous at vertices. The dimension of $V^{\mathrm{H}}(\mathcal{T}_h)$ is

$$\dim V^{\mathrm{H}}(\mathcal{T}_h) = |\Delta_0(\mathcal{T}_h)| \binom{d+m}{m} + \sum_{\ell=1}^{d} |\Delta_\ell(\mathcal{T}_h)| \left[\binom{k-1}{\ell} - (\ell+1) \binom{m}{\ell} \right].$$

When computing the dimension of $\mathbb{P}_k(\mathbb{T}_k^{\ell}(\mathring{f}) \backslash D(\Delta_0(f), m))$, it is easier to use the equivalent index set in (9).

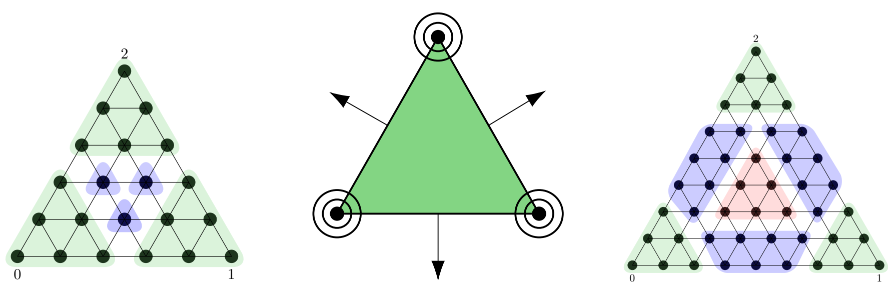
Compared with the Lagrange elements, more DoFs are accumulated to vertices and thus reduce the dimension of the global finite element space. For example, in two dimensions, moving edge-wise and element-wise DoFs to vertices will reduce the dimension of the finite element space around one half less, which is considered as an advantage of using Hermite elements vs Lagrange elements.

2.4. **Normal derivatives on edges.** Given an edge e, we identify lattice nodes to determine the normal derivative up to order m

$$\left\{\frac{\partial^\beta u}{\partial n_e^\beta}\,|_e, 0\le\beta\le m\right\}.$$

By Lemma 1.4, if the lattice node is r^e+1 away from the edge, then the corresponding Bernstein polynomial will have vanishing normal derivatives up to order r^e.

We have used lattice nodes $D(\Delta_0(e), r^{\mathrm{v}}) := \cup_{\mathrm{v}\in\Delta_0(e)} D(\mathrm{v}, r^{\mathrm{v}})$ to determine the derivatives at vertices. We will use $D(e, r^e)\backslash D(\Delta_0(e), r^{\mathrm{v}})$ for the normal derivative. We refer to Fig. 6 (a) (b) for an illustration of the lattice decomposition and DoFs for the C^1 Argyris element with $k=5$ and generalization to $k=8$ in (c).



(a) Lattice decomposition with $\boldsymbol{r}=(2,1)$ and $k=5$.

(b) DoFs: circles represent derivatives and arrows on the edges represent the normal derivative.

(c) Lattice decomposition with $\boldsymbol{r}=(2,1)$ and $k=8$.

FIGURE 6. Lattice decompositions and DoFs for the C^1 Argyris-type element.

Lemma 2.4. *Let $r^{\mathrm{v}}\ge r^e\ge 0$ and $k\ge 2r^{\mathrm{v}}+1$. Let $e\in\Delta_1(T)$ be an edge of a triangle T. The polynomial function space $\mathbb{P}_k(D(e,r^e)\backslash D(\Delta_0(e), r^{\mathrm{v}}))$ is determined by DoFs*

$$\int_e \frac{\partial^\beta u}{\partial n_e^\beta}\,\lambda_e^\alpha\,\mathrm{d}s,\quad \alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+\beta}, \beta=0,1,\dots,r^e.$$

Proof. Without loss of generality, take $e=e_{0,1}$. By definition $D(e,r^e)=\cup_{i=0}^{r^e}L(e,i)$, where recall that

$$L(e,i)=\{\alpha\in\mathbb{T}^2_k, \mathrm{dist}(\alpha,e)=i\}=\{\alpha\in\mathbb{T}^2_k, \alpha_0+\alpha_1=k-i\}$$

consists of lattice nodes parallel to e and with distance i. Define the map $(\alpha_0,\alpha_1,\alpha_2)\to(\alpha_0,\alpha_1)$ which is one-to-one between $L(e,i)$ and $\mathbb{T}^1_{k-i}(e)$.

Now we use the requirement $\alpha\notin D(\Delta_0(e), r^{\mathrm{v}})$ to figure out the bound of the components. Using Lemma 1.1, we derive from $\mathrm{dist}(\alpha,\mathrm{v}_0)>r^{\mathrm{v}}$ that $\alpha_0<k-r^{\mathrm{v}}$. Together

with $\alpha_0+\alpha_1=k-i$, we get the lower bound $\alpha_1\geq r^{\mathrm{v}}-i+1$. Similarly $\alpha_0\geq r^{\mathrm{v}}-i+1$. Therefore

$$L(e,i)\backslash D(\Delta_0(e),r^{\mathrm{v}})=\{(\alpha_0,\alpha_1,i),\alpha_0+\alpha_1=k-i,\min\{\alpha_0,\alpha_1\}\geq r^{\mathrm{v}}-i+1\}.$$

Define the one-to-one mapping

$$\begin{aligned}\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}&\to L(e,i)\backslash D(\Delta_0(e),r^{\mathrm{v}}),\\ (\alpha_0,\alpha_1)&\mapsto(\alpha_0+(r^{\mathrm{v}}-i+1),\alpha_1+(r^{\mathrm{v}}-i+1),i).\end{aligned}$$

With the help of this one-to-one mapping, we shall prove the polynomial function space $\mathbb{P}_k(L(e,i)\backslash D(\Delta_0(e),r^{\mathrm{v}}))$ is determined by DoFs

$$\int_e\frac{\partial^i u}{\partial n_e^i}\,\lambda_e^{\alpha}\,\mathrm{d}s,\quad \alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}. \tag{10}$$

Take a $u=\sum_{\alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}}c_\alpha\lambda_e^{\alpha+r^{\mathrm{v}}-i+1}\lambda_2^i\in\mathbb{P}_k(L(e,i)\backslash D(\Delta_0(e),r^{\mathrm{v}}))$ with coefficients $c_\alpha\in\mathbb{R}$. By the chain rule and the fact $\lambda_2|_e=0$, in the non-zero terms of $\frac{\partial^i u}{\partial n_e^i}\,|_e$, the derivative in $\frac{\partial^i u}{\partial n_e^i}|_e$ will all apply to λ_2^i, so

$$\frac{\partial^i u}{\partial n_e^i}\,|_e=i!(n_e\cdot\nabla\lambda_2)^i\lambda_e^{r^{\mathrm{v}}-i+1}\sum_{\alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}}c_\alpha\lambda_e^{\alpha}|_e.$$

Noting that $n_e\cdot\nabla\lambda_2$ is a constant and the bubble polynomial $\lambda_e^{r^{\mathrm{v}}-i+1}$ is always positive in the interior of e, the vanishing DoF (10) means $c_\alpha=0$ for all $\alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}$.

It follows from Lemma 1.4 that $\frac{\partial^\beta}{\partial n_e^\beta}(\lambda_e^\alpha\lambda_2^i)|_e=0$ for $\alpha\in\mathbb{T}^1_{k-i}(e)$ and $0\leq\beta<i\leq r^e$. That is the matrix

$$\left(\frac{\partial^\beta}{\partial n_e^\beta}(\lambda_e^\alpha\lambda_2^i)|_e\right)_{0\leq\beta\leq r^e,0\leq i\leq r^e,\alpha\in\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+i}(e)}$$

is lower block triangular as follows.

$\beta\backslash\ i$	0	1	$\cdots$	r^e-1	r^e
α	$\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)}$	$\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+1}$	$\cdots$	$\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+r^e-1}$	$\mathbb{T}^1_{k-2(r^{\mathrm{v}}+1)+r^e}$
0	$\square$	0	$\cdots$	0	0
1	$\square$	$\square$	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r^e-1	$\square$	$\square$	$\cdots$	$\square$	0
r^e	$\square$	$\square$	$\cdots$	$\square$	$\square$

Since we have proved each block matrix is invertible, then the whole lower block triangular matrix is invertible which is equivalent to the unisolvence. $\square$

Geometrically we push all lattice nodes in $D(e,r^e)\backslash D(\Delta_0(e),r^{\mathrm{v}})$ to the edge to determine normal derivatives on e up to order r^e.

3. Smooth Finite Elements in Arbitrary Dimensions

The smoothness at sub-simplexes is denoted by a vector $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$ which is exponentially increasing as the dimension decreases

$$r_{d-1} = m, \;\; r_\ell \geq 2r_{\ell+1} \;\text{ for } \ell = d-2, \ldots, 0.$$

The degree of polynomial $k \geq 2r_0 + 1 \geq 2^d m + 1$. The key of HLW element [6] is a non-overlapping decomposition of the simplicial lattice in which each component will be used to determine the normal derivatives on lower sub-simplexes. The distance introduced in Section 3 for the simplicial lattice helps to simplify the construction.

3.1. **Decompositions of the simplicial lattice.** We explain the requirement $r_{\ell-1} \geq 2r_\ell$. In Fig. 3, the blue part will be disjointed due to the requirement $r_0 \geq 2r_1$.

Lemma 3.1. *Let T be a d-dimensional simplex. For $\ell = 1, \ldots, d-1$, if $r_{\ell-1} \geq 2r_\ell$, the sub-sets $\{D(f, r_\ell)\backslash \left[\cup_{e\in\Delta_{\ell-1}(f)} D(e, r_{\ell-1})\right], f \in \Delta_\ell(T)\}$ are disjoint.*

Proof. Consider two different sub-simplices $f, \tilde{f} \in \Delta_\ell(T)$. The dimension of their intersection is at most $\ell-1$. Therefore $f \cap \tilde{f} \subseteq e$ for some $e \in \Delta_{\ell-1}(f)$. Then $e^* \subseteq (f\cap\tilde{f})^* = f^* \cup \tilde{f}^*$. For $\alpha \in D(f, r_\ell) \cap D(\tilde{f}, r_\ell)$, we have $|\alpha_{e^*}| \leq |\alpha_{f^*}| + |\alpha_{\tilde{f}^*}| \leq 2r_\ell \leq r_{\ell-1}$. Therefore we have shown the intersection region $D(f, r_\ell) \cap D(\tilde{f}, r_\ell) \subseteq \cup_{e\in\Delta_{\ell-1}(f)} D(e, r_{\ell-1})$ and the result follows. ∎

Next we remove $D(e, r_i)$ from $D(f, r_\ell)$ for all $e \in \Delta_i(T)$ and $i = 0, 1, \ldots, \ell-1$.

Lemma 3.2. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$ satisfy*

$$r_{d-1} = m, \;\; r_\ell \geq 2r_{\ell+1} \;\textit{ for } \ell = d-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^d m + 1$. For $\ell = 1, \ldots, d-1$, and any $f \in \Delta_\ell(T)$

$$D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r_i)\right] = D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(T)} D(e, r_i)\right]. \tag{11}$$

Proof. In (11), the relation $\supseteq$ is obvious as $\Delta_i(f) \subseteq \Delta_i(T)$.

To prove $\subseteq$, it suffices to show for $\alpha \in D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r_i)\right]$, it is not in $D(e, r_i)$ for $e \in \Delta_i(T)$ and $e \notin \Delta_i(f)$.

By definition,

$$|\alpha_{f^*}| \leq r_\ell, \;\; |\alpha_e| \leq k - r_i - 1 \;\text{ for all } e \in \Delta_i(f), i = 0, \ldots, \ell-1.$$

For each $e \in \Delta_i(T)$ but $e \notin \Delta_i(f)$, the dimension of the intersection $e \cap f$ is at most $i-1$. It follows from $r_j \geq 2r_{j+1}$ and $k \geq 2r_0 + 1$ that: when $i > 0$,

$$|\alpha_e| = |\alpha_{e\cap f}| + |\alpha_{e\cap f^*}| \leq k - r_{i-1} - 1 + r_\ell \leq k - r_i - 1,$$

and when $i = 0$,

$$|\alpha_e| = |\alpha_{e\cap f^*}| \leq r_\ell \leq k - r_i - 1.$$

So $|\alpha_{e^*}| > r_i$. We conclude that $\alpha \notin D(e, r_i)$ for all $e \in \Delta_i(T)$ and (11) follows. ∎

We are in the position to present a geometric decomposition of the simplicial lattice and polynomial spaces. It is a simplification of that in [6] using the distance structure introduced in Section 3 for the simplicial lattice.

Theorem 3.3. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$ satisfy*

$$r_{d-1} = m, \quad r_\ell \geq 2r_{\ell+1} \ \textit{for}\ \ell = d-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^d m + 1$. Then we have the following direct decomposition of the simplicial lattice on a d-dimensional simplex T:

$$\mathbb{T}^d_k(T) = \oplus_{\ell=0}^d \oplus_{f\in\Delta_\ell(T)} S_\ell(f), \tag{12}$$

where

$$S_0(\texttt{v}) = D(\texttt{v}, r_0),$$
$$S_\ell(f) = D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r_i)\right], \ \ell = 1, \ldots, d-1,$$
$$S_d(T) = \mathbb{T}^d_k(T)\backslash \left[\bigcup_{i=0}^{d-1} \bigcup_{f\in\Delta_i(T)} D(f, r_i)\right].$$

Consequently we have the following geometric decomposition of $\mathbb{P}_k(T)$

$$\mathbb{P}_k(T) = \oplus_{\ell=0}^d \oplus_{f\in\Delta_\ell(T)} \mathbb{P}_k(S_\ell(f)). \tag{13}$$

Proof. First we show that the sets $\{S_\ell(f), f \in \Delta_\ell(T), \ell = 0, \ldots, d\}$ are disjoint. Take two vertices $\texttt{v}_1, \texttt{v}_2 \in \Delta_0(T)$. For $\alpha \in D(\texttt{v}_1, r_0)$, we have $\alpha_{\texttt{v}_1} \geq k - r_0$. As $\texttt{v}_1 \subseteq \texttt{v}_2^*$ and $k \geq 2r_0 + 1$, $|\alpha_{\texttt{v}_2^*}| \geq \alpha_{\texttt{v}_1} \geq k - r_0 \geq r_0 + 1$, i.e., $\alpha \notin D(\texttt{v}_2, r_0)$. Hence $\{S_0(\texttt{v}), \texttt{v} \in \Delta_0(T)\}$ are disjoint and $\bigoplus_{\texttt{v}\in\Delta_0(T)} S_0(\texttt{v})$ is a disjoint union. By Lemma 3.1 and (11), we know $\{S_\ell(f), f \in \Delta_\ell(T), \ell = 0, \ldots, n\}$ are disjoint.

Next we inductively prove

$$\oplus_{i=0}^\ell \oplus_{f\in\Delta_i(T)} S_i(f) = \bigcup_{i=0}^{\ell} \bigcup_{f\in\Delta_i(T)} D(f, r_i) \quad \text{for } \ell = 0, \ldots, d-1. \tag{14}$$

Obviously (14) holds for $\ell = 0$. Assume (14) holds for $\ell < j$. Then

$$\begin{aligned}
\oplus_{i=0}^j \oplus_{f\in\Delta_i(T)} S_i(f) &= \oplus_{f\in\Delta_j(T)} S_j(f) \ \oplus \bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r_i) \\
&= \oplus_{f\in\Delta_j(T)} \left(D(f, r_j)\backslash \left[\bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r_i)\right]\right) \ \oplus \bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r_i) \\
&= \bigcup_{i=0}^{j} \bigcup_{f\in\Delta_i(T)} D(f, r_i).
\end{aligned}$$

By induction, (14) holds for $\ell = 0, \ldots, d-1$. Then (12) is true from the definition of $S_d(T)$ and (14). ∎

We can write out the inequality constraints in $S_\ell(f)$. For $\ell = 1, \ldots, d$,

$$S_\ell(f) = \{\alpha \in \mathbb{T}^d_k : |\alpha_{f^*}| \leq r_\ell, |\alpha_e| \leq k - r_i - 1, \forall e \in \Delta_i(f), i = 0, \ldots, \ell-1\}. \tag{15}$$

For $\alpha \in S_\ell(f)$, by Lemma 3.2 we also have $\alpha \notin D(\tilde{f}, r_\ell)$ for $\tilde{f} \in \Delta_\ell(T)\backslash\{f\}$, i.e.

$$|\alpha_{\tilde{f}}| \leq k - r_\ell - 1 \quad \forall\, \tilde{f} \in \Delta_\ell(T)\backslash\{f\}. \tag{16}$$

From the implementation point of view, the index set $S_\ell(f)$ can be found by a logic array and set the entry as true when the distance constraint holds. Explicit characterization of $S_\ell(f)$ is not necessary.

3.2. **Degrees of freedom.** Recall that $L(f,s) = \{\alpha \in \mathbb{T}_k^d, \operatorname{dist}(\alpha, f) = s\}$ consists of lattice nodes s away from f. The following result is a generalization of Lemma 2.4 in two dimensions.

Lemma 3.4. *Let $\ell = 0,\dots,d-1$ and $s \le r_\ell$ be a non-negative integer. Given $f \in \Delta_\ell(T)$, let $\{\boldsymbol{n}_f^1, \boldsymbol{n}_f^2, \dots, \boldsymbol{n}_f^{d-\ell}\}$ be $d-\ell$ vectors spanning the normal plane of f. The polynomial space $\mathbb{P}_k(S_\ell(f) \cap L(f,s))$ is uniquely determined by DoFs*

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s, \quad \alpha \in S_\ell(f), |\alpha_f| = k-s, \beta \in \mathbb{N}^{1:d-\ell}, |\beta| = s. \tag{17}$$

Proof. A basis of $\mathbb{P}_k(S_\ell(f) \cap L(f,s))$ is $\{\lambda^\alpha = \lambda_f^{\alpha_f}\lambda_{f^*}^{\alpha_{f^*}}, \alpha \in S_\ell(f), |\alpha_{f^*}| = s\}$ and thus the dimensions match (by mapping α_{f^*} to β).

We choose a basis of the normal plane $\{\boldsymbol{n}_f^1, \boldsymbol{n}_f^2, \dots, \boldsymbol{n}_f^{d-\ell}\}$ s.t. it is dual to the vectors $\{\nabla\lambda_{f^*(1)}, \nabla\lambda_{f^*(2)}, \dots,\}$, i.e., $\nabla\lambda_{f^*(i)} \cdot \boldsymbol{n}_f^j = \delta_{i,j}$ for $i,j = 1,\dots,d-\ell$. Then we have the duality

$$\frac{\partial^\beta}{\partial n_f^\beta}(\lambda_{f^*}^{\alpha_{f^*}}) = \beta!\delta(\alpha_{f^*}, \beta), \quad \alpha_{f^*}, \beta \in \mathbb{N}^{1:d-\ell}, |\alpha_{f^*}| = |\beta| = s, \tag{18}$$

which can be proved easily by induction on s. When T is the reference simplex $\hat{T}$, $\lambda_i = x_i$ and $\nabla\lambda_i = -\boldsymbol{e}_i$, (18) is the calculus result $D_{n_f}^\beta \boldsymbol{x}_{f^*}^{\alpha_{f^*}} = \beta!\delta(\alpha_{f^*}, \beta)$.

Assume $u = \sum c_{\alpha_f, \alpha_{f^*}} \lambda_f^{\alpha_f}\lambda_{f^*}^{\alpha_{f^*}} \in \mathbb{P}_k(S_\ell(f) \cap L(f,s))$. If the derivative is not fully applied to the component $\lambda_{f^*}^{\alpha_{f^*}}$, then there is a term $\lambda_{f^*}^\gamma$ with $|\gamma| > 0$ left and $\lambda_i^\gamma|_f = 0$ for $i \in f^*$. So for any $\beta \in \mathbb{N}^{1:d-\ell}$ and $|\beta| = s$,

$$\frac{\partial^\beta u}{\partial n_f^\beta}|_f = \beta! \sum_{\alpha \in S_\ell(f), |\alpha_f| = k-s} c_{\alpha_f,\beta}\lambda_f^{\alpha_f}.$$

The vanishing DoF (17) implies $\sum_{\alpha \in S_\ell(f), |\alpha_f| = k-s} c_{\alpha_f,\beta}\lambda_f^{\alpha_f}|_f = 0$. Hence $c_{\alpha_f,\beta} = 0$ for all $|\alpha_f| = k-s, \alpha \in S_\ell(f)$. As β is arbitrary, we conclude all coefficients $c_{\alpha_f,\alpha_{f^*}} = 0$ and thus $u = 0$. ☐

For $u \in \mathbb{P}_k(S_\ell(f) \cap L(f,s))$ and $\beta \in \mathbb{N}^{1:d-\ell}$ with $|\beta| < s$, by Lemma 1.4, $\frac{\partial^\beta u}{\partial n_f^\beta}|_f = 0$. Applying the operator $\frac{\partial^\beta (\cdot)}{\partial n_f^\beta}|_f$ to the direct decomposition $\mathbb{P}_k(S_\ell(f)) = \bigoplus_{s=0}^{r_\ell} \mathbb{P}_k(S_\ell(f) \cap L(f,s))$ will possess a block lower triangular structure and leads to the following unisolvence result.

Lemma 3.5. *Let $\ell = 0,\dots,d-1$. The polynomial space $\mathbb{P}_k(S_\ell(f))$ is uniquely determined by DoFs*

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s, \quad \alpha \in S_\ell(f), |\alpha_f| = k-s, \beta \in \mathbb{N}^{1:d-\ell}, |\beta| = s, s = 0,\dots,r_\ell.$$

Together with decomposition (13) of the polynomial space, we obtain the following result.

Theorem 3.6 (Theorem 1.1 in [6]). *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$ satisfy*

$$r_{d-1} = m, \quad r_\ell \geq 2r_{\ell+1} \text{ for } \ell = d-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^d m + 1$. Then the shape function $\mathbb{P}_k(T)$ is uniquely determined by the following DoFs

$$D^\alpha u(v), \quad \alpha \in \mathbb{N}^{1:d}, |\alpha| \leq r_0, v \in \Delta_0(T), \tag{19a}$$

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta} \lambda_f^{\alpha_f} \, \mathrm{d}s, \quad \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:d-\ell}, |\beta| = s, \tag{19b}$$
$$f \in \Delta_\ell(T), \ell = 1, \ldots, d-1, s = 0, \ldots, r_\ell,$$

$$\int_T u \lambda^\alpha \, \mathrm{d}x, \quad \alpha \in S_d(T). \tag{19c}$$

Proof. Thanks to the decomposition (13), the dimensions match. Take $u \in \mathbb{P}_k(T)$ satisfy all the DoFs (19) vanish. We are going to show $u = 0$.

For $\alpha \in S_\ell(f)$ and $e \in \Delta_i(T)$ with $i \leq \ell$ and $e \neq f$, by (15) and (16) we have $|\alpha_{e^*}| \geq r_i + 1$, hence $\frac{\partial^\beta \lambda^\alpha}{\partial n_e^\beta}|_e = 0$ for $\beta \in \mathbb{N}^{1:d-i}$ with $|\beta| \leq r_i$. Again this tells us that applying the operator $\frac{\partial^\beta (\cdot)}{\partial n_f^\beta}|_f$ to the direct decomposition $\mathbb{P}_k(T) = \bigoplus_{\ell=0}^d \bigoplus_{f \in \Delta_\ell(T)} \mathbb{P}_k(S_\ell(f))$ will produce a block lower triangular structure. Then apply Lemma 3.5, we conclude $u \in \mathbb{P}_k(S_d(T))$, which together with the vanishing DoF (19c) gives $u = 0$. $\square$

3.3. **Smooth finite elements in arbitrary dimension.** Given a triangulation $\mathcal{T}_h$, the finite element space is obtained by asking the DoFs depending on the sub-simplex f only.

Theorem 3.7. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, r_1, \cdots, r_{d-1})$ satisfy*

$$r_{d-1} = m, \quad r_\ell \geq 2r_{\ell+1} \text{ for } \ell = d-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^d m + 1$. The following DoFs

$$D^\alpha u(v) \quad \alpha \in \mathbb{N}^{1:d}, |\alpha| \leq r_0, v \in \Delta_0(\mathcal{T}_h), \tag{20a}$$

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta} \lambda_f^{\alpha_f} \, \mathrm{d}s \quad \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:d-\ell}, |\beta| = s, s = 0, \ldots, r_\ell, \tag{20b}$$
$$f \in \Delta_\ell(\mathcal{T}_h), \ell = 1, \ldots, d-1,$$

$$\int_T u \lambda^\alpha \, \mathrm{d}x \quad \alpha \in S_d(T), T \in \mathcal{T}_h, \tag{20c}$$

will define a finite element space

$$V_h = \{u \in C^m(\Omega) \mid \textit{DoFs } (20a)-(20b) \textit{ are single valued}, u|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}_h\}.$$

Proof. Restricted to one simplex T, by Theorem 3.6, DoFs (20) will define a function u s.t. $u|_T \in \mathbb{P}_k(T)$. We only need to verify $u \in C^m(\Omega)$.

It is apparent that $u \in C^0(\Omega)$ and $u\,|_F \in \mathbb{P}_k(F)$. Thus all tangential derivatives on F are also continuous. It suffices to prove $\frac{\partial^i u}{\partial n_F^i}|_F \in \mathbb{P}_{k-i}(F)$, for all $i = 0, \ldots, m$ and all $F \in \Delta_{d-1}(T)$, are uniquely determined by (20a)-(20b) on F.

Let $w = \frac{\partial^i u}{\partial n_F^i}|_F \in \mathbb{P}_{k-i}(F)$. Consider the modified index sequence $\boldsymbol{r}_F^i = (r_0 - i, r_1 - i, \ldots, r_{d-2} - i, 0)$ and degree $k^i = k - i$. Then $k^i, \boldsymbol{r}_F^i$ satisfies the condition in Theorem

3.6 and we obtain a direct decomposition of $\mathbb{T}^{d-1}_{k-i}(F) = \bigoplus_{\ell=0}^{d-1}\bigoplus_{f\in\Delta_\ell(F)} S_\ell^F(f)$, where

$$S_0^F(\mathrm{v}) = D(\mathrm{v}, r_0 - i) \cap \mathbb{T}^{d-1}_{k-i}(F),$$

$$S_\ell^F(f) = (D(f, r_\ell - i) \cap \mathbb{T}^{d-1}_{k-i}(F)) \backslash \left[\bigoplus_{i=0}^{\ell-1}\bigoplus_{e\in\Delta_i(F)} S_i^F(e)\right],\ \ell = 1, \ldots, d-2,$$

$$S_{d-1}^F(F) = \mathbb{T}^{d-1}_{k-i}(F) \backslash \left[\bigoplus_{\ell=0}^{d-2}\bigoplus_{f\in\Delta_\ell(F)} S_\ell^F(f)\right].$$

The DoFs (20a)-(20b) related to w are

$$D_F^\alpha w(\mathrm{v}) \quad \alpha \in \mathbb{N}^{1:d-1}, |\alpha| \leq r_0 - i, \mathrm{v} \in \Delta_0(F),$$

$$\int_f \frac{\partial^\beta w}{\partial n_{F,f}^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s \quad \alpha \in S_\ell^F(f), |\alpha_f| = k - i - s, \beta \in \mathbb{N}^{1:d-1-\ell}, |\beta| = s,$$

$$f \in \Delta_\ell(F), \ell = 1, \ldots, d-2, s = 0, \ldots, r_\ell - i,$$

$$\int_F w\lambda^\alpha\, \mathrm{d}x \quad \alpha \in S_{d-1}^F(F),$$

where $D_F w$ is the tangential derivatives of w, $n_{F,f}$ is the normal vector of f but tangential to F. Clearly the modified sequence $\boldsymbol{r}_i^F$ still satisfies constraints required in Theorem 3.6. We can apply Theorem 3.6 with the shape function space $\mathbb{P}_{k-i}(F)$ to conclude w is uniquely determined on F. □

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