# GEOMETRIC DECOMPOSITION OF LAGRANGE ELEMENTS

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ABSTRACT. This lecture note explores the geometric decomposition of Lagrange elements using barycentric coordinates and the simplicial lattice. The first section introduces the concept of barycentric coordinates, a powerful tool to represent points within a geometric simplex. The second section delves into the simplicial lattice, which is a fundamental concept in this study. The geometric decomposition of Lagrange elements, building upon the concepts of barycentric coordinates and the simplicial lattice, is presented in the third section.

## CONTENTS



## 1. BARYCENTRIC COORDINATES

Let $\mathrm{v}_i = (x_{1,i}, \ldots, x_{d,i})^{\intercal}$ for $i = 0, 1, \ldots, d$ be $d+1$ points in $\mathbb{R}^d$. We say $\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_d$ do not all lie in one hyper-plane if the $d$-vectors $\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}$ are linearly independent, where $\boldsymbol{t}_{i,j}$ is the edge vector from $\mathrm{v}_i$ to $\mathrm{v}_j$ for $0 \leq i, j \leq d$. This is equivalent to the matrix:

$$A = \begin{pmatrix} x_{1,0} & x_{1,1} & \ldots & x_{1,d} \\ x_{2,0} & x_{2,1} & \ldots & x_{2,d} \\ \vdots & \vdots & & \vdots \\ x_{d,0} & x_{d,1} & \ldots & x_{d,d} \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

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is non-singular. Given any point $\boldsymbol{x} = (x_1, \dots, x_d)^\intercal \in \mathbb{R}^d$, by solving the following linear system

$$A \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{d-1} \\ \lambda_d \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}, \tag{1}$$

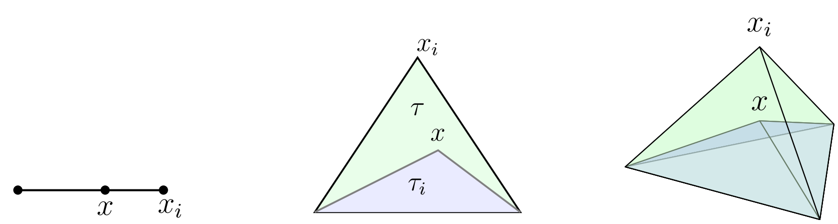
we obtain unique $d+1$ real numbers $\lambda_i(\boldsymbol{x}), 0 \le i \le d$, such that for any $\boldsymbol{x} \in \mathbb{R}^d$

$$\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x}) \mathrm{v}_i, \quad \text{with} \quad \sum_{i=0}^{d} \lambda_i(\boldsymbol{x}) = 1. \tag{2}$$

The *convex hull* of the $d+1$ points $\mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_d$ in $\mathbb{R}^d$

$$T := \{\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i \mathrm{v}_i \,|\, 0 \le \lambda_i \le 1, i = 0 : d, \sum_{i=0}^{d} \lambda_i = 1\} \tag{3}$$

is defined as a *geometric $d$-simplex* generated (or spanned) by the vertices $\mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_d$. For example, a triangle is a 2-simplex and a tetrahedron is a 3-simplex.




(a) 1d simplex (b) 2d simplex (c) 3d simplex

FIGURE 1. Geometric explanation of barycentric coordinates.

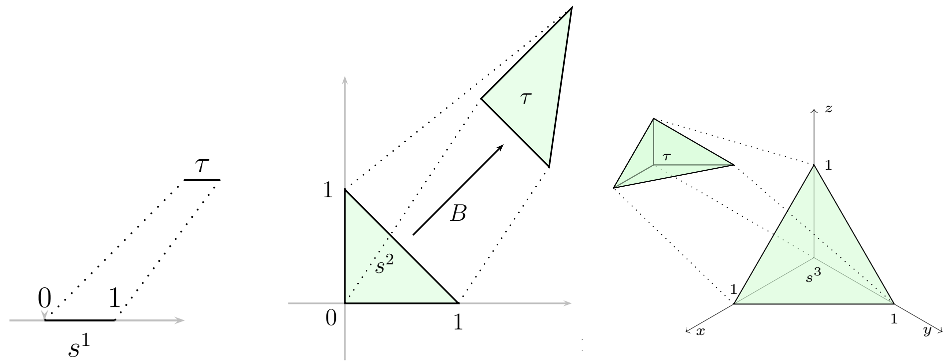
The numbers $\lambda_0(\boldsymbol{x}), \lambda_1(\boldsymbol{x}), \dots, \lambda_d(\boldsymbol{x})$ are called *barycentric coordinates* of $\boldsymbol{x}$ with respect to the $d+1$ points $\mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_d$. There is a simple geometric meaning of the barycentric coordinates. Given an $\boldsymbol{x} \in T$, let $T_i(\boldsymbol{x})$ be the simplex with vertex $\mathrm{v}_i$ replaced by $\boldsymbol{x}$. Then, by the Cramer's rule for solving (1),

$$\lambda_i(\boldsymbol{x}) = \frac{|T_i(\boldsymbol{x})|}{|T|}, \tag{4}$$

where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$, namely area in two dimensions and volume in three dimensions. See Fig. 1. Clearly, $\lambda_i(\mathrm{v}_j) = \delta_{ij}$ for $0 \le i, j \le d$, where $\delta_{ij}$ is the Kronecker delta function.

It is convenient to have a standard simplex $s^d \subset \mathbb{R}^d$ spanned by the vertices $\mathbf{0}, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_d$ where $\boldsymbol{e}_i = (0, \cdots, 1, \cdots, 0)^\intercal$. Then any $d$-simplex $\tau \subset \mathbb{R}^d$ can be thought as an image of $s^d$ through an affine map $B : s^d \to \tau$ with $B(\mathbf{0}) = \mathrm{v}_0$ and $B(\boldsymbol{e}_i) = \mathrm{v}_i$ for $i = 1, \dots, d$. See Fig. 2. The simplex $s^d$ is also often called the *reference simplex*. The explicit expression of $B$ is

$$B(\boldsymbol{x}) = \mathrm{v}_0 + (\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \dots, \boldsymbol{t}_{0,d})\boldsymbol{x} = \mathrm{v}_0 + \sum_{i=1}^{d} x_i \boldsymbol{t}_{0,i}.$$




(a) A standard simplex in $\mathbb{R}^1$ (b) A standard simplex in $\mathbb{R}^2$ (c) A standard simplex in $\mathbb{R}^3$

FIGURE 2. Reference simplexes in $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$.

In the traditional computation of finite elements, the process involves mapping back to the reference simplex. However, when dealing with vector and matrix functions, this transformation becomes intricate. To simplify this, we opt for expressing these functions in terms of barycentric coordinates. On the reference simplex, the barycentric coordinates $\lambda_i = x_i$ for $i = 1, \ldots, d$, and $\lambda_0 = 1 - \sum_{i=1}^d x_i$. The $d$ edge vectors $\{\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}\}$ of $s^d$ represent the canonical orthonormal basis of $\mathbb{R}^d$. This allows us to initially work with the familiar Cartesian coordinates $\{x_i, i = 1, \ldots, d\}$ and then symbolically transition to the barycentric coordinates $\{\lambda_i, i = 0, \ldots, d\}$.

For a general non-degenerated simplex, $\{\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}\}$ may not be orthonormal but still forms a basis of $\mathbb{R}^d$. We now figure out the dual basis of $\{\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}\}$. The $(d-1)$-dimensional face opposite to the vertex $\mathrm{v}_i$ will be denoted by $F_i$. Note that $\lambda_i(\boldsymbol{x})$ is an affine function of $\boldsymbol{x}$ and vanished on the face $F_i$. Therefore $\nabla\lambda_i$ will be orthogonal to $F_i$ the zero level-set of $\lambda_i$. Let $\boldsymbol{n}_T$ be the unit outward normal vector to $\partial T$, which will be abbreviated as $\boldsymbol{n}$. We have

$$\boldsymbol{n}|_{F_i} = -\nabla\lambda_i/|\nabla\lambda_i|. \tag{5}$$

By computing the constant directional derivative $\boldsymbol{t}_{i,j} \cdot \nabla\lambda_\ell$ by values on the two vertices, we have

$$\boldsymbol{t}_{i,j} \cdot \nabla\lambda_\ell = \delta_{j\ell} - \delta_{i\ell} = \begin{cases} 1, & \text{if } \ell = j, \\ -1, & \text{if } \ell = i, \\ 0, & \text{if } \ell \neq i, j. \end{cases} \tag{6}$$

Therefore we conclude that the $d$ scaled normal vectors

$$\{\nabla\lambda_1, \ldots, \nabla\lambda_d\} = \{-|\nabla\lambda_1|\boldsymbol{n}_1, \ldots, -|\nabla\lambda_d|\boldsymbol{n}_d\}$$

with $\boldsymbol{n}_i = \boldsymbol{n}|_{F_i}$ is the dual basis of $\{\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}\}$, i.e.,

$$\boldsymbol{t}_{0,i} \cdot (-|\nabla\lambda_j|\boldsymbol{n}_j) = \delta_{ij}, \quad 1 \leq i, j \leq d.$$

We refer to Fig. 3 for an illustration in 3-D.

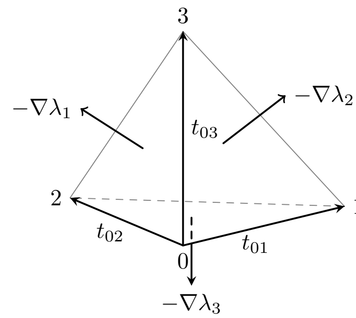
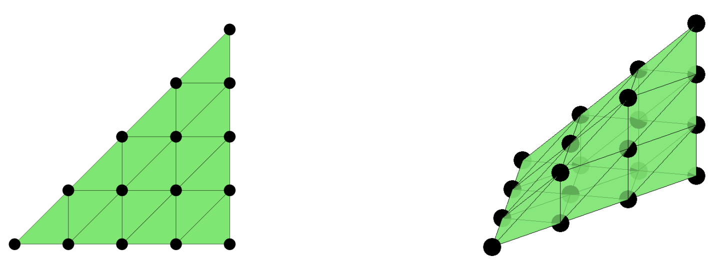



FIGURE 3. Basis $\{\boldsymbol{t}_{0,1}, \boldsymbol{t}_{0,2}, \ldots, \boldsymbol{t}_{0,d}\}$ and $\{\nabla\lambda_1, \ldots, \nabla\lambda_d\}$ for $d = 3$.

## 2. SIMPLICAL LATTICE AND BERNSTEIN BASIS

This section introduces the concept of a simplicial lattice, which is a multi-index set of nodes with fixed sum $k$. A geometric embedding of the lattice into the simplex is given by barycentric coordinates. Sub-simplicial lattices, formed by subsets of nodes, are also discussed. The Bernstein basis of polynomials on the simplex is used to provides an overview of how simplicial lattices and polynomials are interconnected and lays the foundation for studying polynomials through the simplicial lattice.

2.1. **The simplicial lattice and its geometric embeddings.** For two non-negative integers $\ell \leq m$, we will use the multi-index notation $\alpha \in \mathbb{N}^{\ell:m}$, meaning $\alpha = (\alpha_\ell, \ldots, \alpha_m)$ with integer $\alpha_i \geqslant 0$. The sum of a multi-index is $|\alpha| := \sum_{i=\ell}^m \alpha_i$ for $\alpha \in \mathbb{N}^{\ell:m}$. We can also treat $\alpha$ as a row vector with non-negative integer valued coordinates. We use the convention that: a vector $\alpha \geq c$ means $\alpha_i \geq c$ for all components $i = 0, 1, \ldots, d$.



(a) Simplicial lattice $\mathbb{T}_4^2(T)$ (b) Simplicial lattice $\mathbb{T}_3^3(T)$

FIGURE 4. Two examples of the simplicial lattices.

A simplicial lattice, also known as the principal lattice [3], of degree $k$ and dimension $d$ is a multi-index set of $d+1$ components and with fixed sum $k$, i.e.,

$$\mathbb{T}_k^d = \left\{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{0:d} : |\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_d = k\right\}.$$

An element $\alpha \in \mathbb{T}_k^d$ is called a node of the lattice. It holds that

$$|\mathbb{T}_k^d| = \binom{d+k}{k}.$$

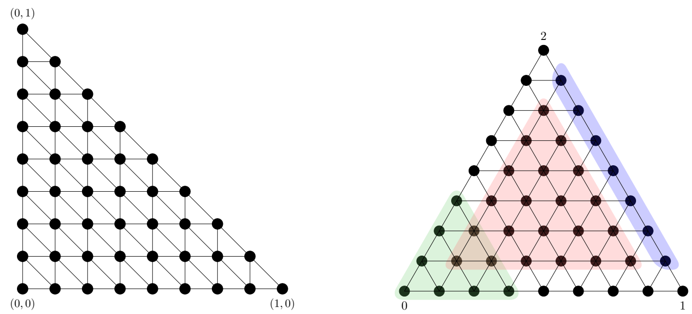



FIGURE 5. Two embeddings of the simplicial lattice $\mathbb{T}_8^2$ in two dimensions.

We can embed the simplicial lattice into a geometric simplex by using $\alpha/k$ as the barycentric coordinate of node $\alpha$. Given $\alpha \in \mathbb{T}_k^d$, the barycentric coordinate of $\alpha$ is given by

$$\lambda(\alpha) = (\alpha_0, \alpha_1, \dots, \alpha_d)/k.$$

Let $T$ be a simplex with vertices $\{\mathrm{v}_0, \mathrm{v}_1, \dots, \mathrm{v}_d\}$. The geometric embedding is

$$x : \mathbb{T}_k^d \to T, \quad x(\alpha) = \sum_{i=0}^{d} \lambda_i(\alpha)\mathrm{v}_i.$$

For a visual representation, please refer to Figs. 4 and 5.

The left figure of Fig. 5 illustrates the embedding of a two-dimensional simplicial lattice $\mathbb{T}_8^2$ within a reference triangle $\hat{T}$ with vertices $(0,0), (1,0), (0,1)$, while the right side shows the embedding of the same lattice into an equilateral triangle.

A simplicial lattice $\mathbb{T}_k^d$ is, by definition, an algebraic set. Through the geometric embedding $\mathbb{T}_k^d(T)$, we can use operators for the geometric simplex $T$ to study this algebraic set. For example, for a subset $S \subseteq T$, we use $\mathbb{T}_k^d(S) = \{\alpha \in \mathbb{T}_k^d, x(\alpha) \in S\}$ to denote the portion of lattice nodes whose geometric embedding is inside $S$. The superscript $^d$ will be replaced by the dimension of $S$ when $S$ is a lower dimensional sub-simplex.

**Exercise 2.1.** *Use the simplicial lattice to describe the uniform refinement of a $d$-simplex. That is: adding middle points of edges and connecting to form $2^d$ smaller simplices. The difficulty is to show the resulting smaller simplices will not degenerate when the process is repeated.*

2.2. **Sub-simplicial lattices.** Following [1], we let $\Delta(T)$ denote all the subsimplices of $T$, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension $\ell$, for $0 \le \ell \le d$. The cardinality of $\Delta_\ell(T)$ is $\binom{d+1}{\ell+1}$. Elements of $\Delta_0(T) = \{\mathrm{v}_0, \dots, \mathrm{v}_d\}$ are $d+1$ vertices of $T$ and $\Delta_d(T) = T$.

For a sub-simplex $f \in \Delta_\ell(T)$ with $\ell = 0, \dots, d-1$, we will overload the notation $f$ for both the geometric simplex and the algebraic set of indices. Namely $f = \{f(0), \dots, f(\ell)\} \subseteq \{0, 1, \dots, d\}$, and

$$f = \mathrm{Convex}(\mathrm{v}_{f(0)}, \dots, \mathrm{v}_{f(\ell)}) \in \Delta_\ell(T)$$

is the $\ell$-dimensional simplex spanned by the vertices $\mathrm{v}_{f(0)}, \dots, \mathrm{v}_{f(\ell)}$.

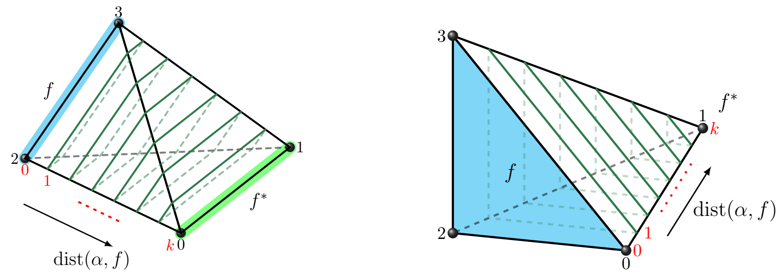



FIGURE 6. On the left, $f = \{2,3\}$ is an edge and $f^* = \{0,1\}$ is another edge. On the right, $f = \{0,2,3\}$ is a face and $f^* = \{1\}$ is a vertex.

If $f \in \Delta_\ell(T)$ with $\ell = 0,\dots,d-1$, then $f^* \in \Delta_{d-\ell-1}(T)$ denotes the sub-simplex of $T$ opposite to $f$. When treating $f$ as a subset of $\{0,1,\dots,d\}$, $f^* \subseteq \{0,1,\dots,d\}$ so that $f \cup f^* = \{0,1,\dots,d\}$, i.e., $f^*$ is the complement of set $f$. Geometrically,

$$f^* = \mathrm{Convex}(\mathrm{v}_{f^*(1)},\dots,\mathrm{v}_{f^*(d-\ell)}) \in \Delta_{d-\ell-1}(T)$$

is the $(d-\ell-1)$-dimensional simplex spanned by vertices not contained in $f$.

Given a sub-simplex $f \in \Delta_\ell(T)$ with $\ell = 0,\dots,d-1$, through the geometric embedding $f \hookrightarrow T$, we define the prolongation/extension operator $E : \mathbb{T}^\ell_k \to \mathbb{T}^d_k$ as follows:

$$E(\alpha)_{f(i)} = \alpha_i, i = 0,\dots,\ell, \quad \text{ and } E(\alpha)_j = 0, j \notin f.$$

For example, assume $f = \{1,3,4\}$, then for $\alpha = (\alpha_0,\alpha_1,\alpha_2) \in \mathbb{T}^\ell_k(f)$, the extension $E(\alpha) = (0,\alpha_0,0,\alpha_1,\alpha_2,\dots,0) \in \mathbb{T}^d_k(T)$. The geometric embedding $x(E(\alpha)) \in f$ justifies the notation $\mathbb{T}^\ell_k(f)$. On the other hand, for $\alpha \in \mathbb{T}^d_k(T)$ and $f \in \Delta_\ell(T)$, the restriction $\alpha_f \in \mathbb{N}^{0:\ell}$ is defined as $(\alpha_f)_i = \alpha_{f(i)}$ for $i = 0,\dots,\ell$. With a slight abuse of notation, for a node $\alpha_f \in \mathbb{T}^\ell_k(f)$, we still use the same notation $\alpha_f \in \mathbb{T}^d_k(T)$ to denote its extension $E(\alpha_f)$. Then for $\alpha \in \mathbb{T}^d_k(T)$ and $f \in \Delta_\ell(T)$ with $\ell = 0,\dots,d-1$, we have the following direct decomposition

$$\alpha = E(\alpha_f) + E(\alpha_{f^*}) = \alpha_f + \alpha_{f^*}, \text{ and } |\alpha| = |\alpha_f| + |\alpha_{f^*}|. \tag{7}$$

For example, when $\alpha = (\alpha_0,\alpha_1,\dots,\alpha_4)$ and $f = \{1,3\}$, then $\alpha_f = (0,\alpha_1,0,\alpha_3,0,0)$ and $\alpha_{f^*} = (\alpha_0,0,\alpha_2,0,\alpha_4)$.

In summary, by treating $f$ as a set of indices, we can apply the operators $\cup, \cap, ^*, \backslash$ on sets. While treating $f$ as a geometric simplex, $\partial f, \mathring{f}$ etc can be applied. For example, $\mathbb{T}^\ell_k(\mathring{f})$ is the sub-set of lattice nodes of $\mathbb{T}^d_k(T)$ whose geometric coordinate is in the interior of an $\ell$-dimensional $f \in \Delta_\ell(T)$.

2.3. **Polynomials and Bernstein basis.** For a domain $\Omega \subseteq \mathbb{R}^d$ and integer $k \geqslant 0$, $\mathbb{P}_k(\Omega)$ denotes the set of real valued polynomials defined on $\Omega$ of degree less than or equal to $k$. For simplicity, we let $\mathbb{P}_k = \mathbb{P}_k\left(\mathbb{R}^d\right)$. Hence, if $d$-dimensional domain $\Omega$ has nonempty interior, then $\dim \mathbb{P}_k(\Omega) = \dim \mathbb{P}_k = \binom{k+d}{d}$. When $\Omega$ is a point, $\mathbb{P}_k(\Omega) = \mathbb{R}$ for all $k \geqslant 0$. And we set $\mathbb{P}_k(\Omega) = \{0\}$ when $k < 0$.

For $d$-dimensional simplex $T$, the barycentric coordinates $\{\lambda_i, i = 0,1,\dots,d\}$ form a basis for $\mathbb{P}_1(T)$. The Bernstein representation of polynomial of degree $k$ on a simplex $T$ is

$$\mathbb{P}_k(T) := \{\lambda^\alpha := \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\dots\lambda_d^{\alpha_d}, \alpha \in \mathbb{T}^d_k\}.$$

With such one-to-one mapping between the lattice node $\alpha$ and the Bernstein polynomial $\lambda^\alpha$, we can study properties of polynomials through the simplicial lattice. Based on (7), we can write a Bernstein polynomial as

$$\lambda^\alpha = \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}},$$

where $\lambda_f = \lambda_{f(0)} \dots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f)$ is the bubble function on $f$.

The following integral formulae is quite useful in the computation

$$\int_T \lambda^\alpha \, \mathrm{d}x = \frac{\alpha! d!}{(|\alpha| + d)!} |T|, \quad \alpha \in \mathbb{T}^d_k. \tag{8}$$

For an elementary proof, we refer to [2, Exercise 1]. The normalized basis $\lambda^\alpha / \alpha!$ has the constant integral

$$\int_T \frac{1}{\alpha!} \lambda^\alpha \, \mathrm{d}x = \frac{d!}{(k + d)!} |T|, \quad \alpha \in \mathbb{T}^d_k. \tag{9}$$

For a subset $S \subseteq \mathbb{T}^d_k$, we define

$$\mathbb{P}_k(S) = \mathrm{span}\{\lambda^\alpha, \alpha \in S \subseteq \mathbb{T}^d_k\}.$$

In the Bernstein form, for an $f \in \Delta_\ell(T)$,

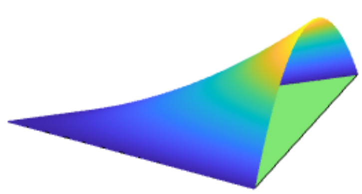
$$\mathbb{P}_k(f) = \{\lambda_f^\alpha = \lambda_{f(0)}^{\alpha_0} \lambda_{f(1)}^{\alpha_1} \dots \lambda_{f(\ell)}^{\alpha_\ell}, \alpha \in \mathbb{T}^\ell_k\}.$$

The barycentric coordinate $\lambda_{f(i)}(x)$ defined for $x \in f$ can be naturally extended to $x \in T$. By doing that, we can treat $\mathbb{P}_k(f) \subseteq \mathbb{P}_k(T)$.

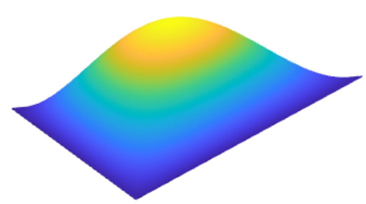
2.4. **Bubble polynomials.** For $\ell \geq 1$, the bubble polynomial of $f$ is a polynomial of degree $\ell + 1$:

$$b_f := \lambda_f = \lambda_{f(0)} \lambda_{f(1)} \dots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f).$$

To unify the notation, when $\ell = 0$, i.e. $f = \{i\}$ is a vertex, $b_f = \lambda_i$. It is called a bubble function as $b_f$ vanishes on the boundary of $f$ and the graph of $b_f$ is like a bubble; see Fig. 7. We shall show $b_f$ also vanishes on other sub-simplexes not containing $f$.



(a) Edge bubble $b_e$.



(b) Face bubble $b_f$.

FIGURE 7. Bubble functions in two dimensions.

We start from the following simple fact.

**Lemma 2.2.** *Let $f \in \Delta(T)$ be a sub-simplex of $T$. Then*

$$\lambda_i \mid_f = 0 \quad \text{if } i \notin f, \text{ equivalently } i \in f^*.$$

*Proof.* We know $\lambda_i \mid_{F_i} = 0$ where recall that $F_i$ is the $(d-1)$-face opposite to $i$-th vertex. If $i \notin f$, then $f \subseteq F_i$ and therefore $\lambda_i \mid_f = 0$. $\square$

Indeed $f$ can be identified as the intersection of the zero level sets.

**Lemma 2.3.** *Let $f \in \Delta(T)$ be a sub-simplex of $T$. Then*

$$f = \cap_{i\in f^*}\{\lambda_i = 0\} = \cap_{i\in f^*} F_i.$$

*Proof.* We prove the statement by treating the sub-simplex $f$ as a sub-set of vertex index. In this terminology, $F_i = \{i\}^*$ contains all indices except $i$ and $F_i^* = \{i\}$. Consequently

$$(\cap_{i\in f^*} F_i)^* = \cup_{i\in f^*} F_i^* = \cup_{i\in f^*}\{i\} = f^*.$$

□

**Lemma 2.4.** *Let $f, e \in \Delta(T)$. If $f \not\subseteq e$, then $b_f \mid_e = 0$.*

*Proof.* As $f = (f \cap e^*) \cup (f \cap e)$ and $f \not\subseteq e$, we conclude $f \cap e^* \neq \varnothing$. So $b_f$ contains $\lambda_i$ for some $i \in e^*$ and consequently $b_f|_e = 0$. □

In particular, $b_f$ vanishes at all sub-simplexes with dimensions $\leq \dim f$ and other than $f$, and higher dimensional sub-simplexes not containing $f$.

## 3. Geometric Decomposition of Lagrange Elements

This section introduces the geometric decomposition of Lagrange elements, expressing the simplicial lattice as a direct sum of spaces associated with vertices, edges, and all sub-simplexes. The discussion extends to the Lagrange finite element spaces on conforming meshes.

3.1. **Lagrange element.** We begin with the following decomposition of the simplicial lattice and refer to Fig. 3.1 for an illustration.

**Lemma 3.1.** *We have*

$$\mathbb{T}_k^d(T) = \bigoplus_{\mathrm{v}\in\Delta_0(T)} \mathbb{T}_k^0(\mathrm{v}) \oplus \bigoplus_{\ell=1}^d \bigoplus_{f\in\Delta_\ell(T)} \mathbb{T}_k^\ell(\mathring{f}). \tag{10}$$

*Proof.* As a geometric object $T = \partial T \cup \mathring{T}$. Then recursively apply this decomposition to each face in $\partial T$ to obtain

$$T = \cup_{\mathrm{v}\in\Delta_0(T)} \mathrm{v} \cup_{\ell=1}^d \bigoplus_{f\in\Delta_\ell(T)} \mathring{f}\ .$$

Restricting to the geometric coordinate of lattice nodes, we obtain (10). □

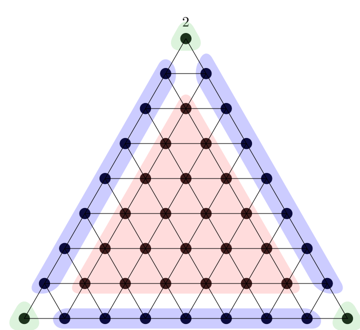



FIGURE 8. A geometric decomposition of the simplicial lattice $\mathbb{T}_8^2$.

The following geometric decomposition of Lagrange element is given in [1] without proofs. As it is the foundation of other geometric decompositions, we present it using our notation and provide a detailed proof. We refer to [1, Fig. 2.1] for an illustration of this geometric decomposition.

For $\ell = 0$, $\int_f v \,\mathrm{d}s := v(f)$, i.e. the function value at the vertex $f$. Here A polynomial $v \in \mathbb{P}_{k-(\ell+1)}(f)$ can be naturally extended to the element $T$ by the Bernstein form in the barycentric coordinate.

**Theorem 3.2** (Geometric decomposition of Lagrange element, (2.6) in [1])**.** *For the polynomial space $\mathbb{P}_k(T)$ with $k \geq 1$ on a $d$-dimensional simplex $T$, we have the following decomposition*

$$\mathbb{P}_k(T) = \bigoplus_{\ell=0}^{d} \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{k-(\ell+1)}(f). \tag{11}$$

*The function $u \in \mathbb{P}_k(T)$ is uniquely determined by DoFs*

$$\int_f u\, p \,\mathrm{d}s \quad p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, d. \tag{12}$$

*Proof.* We first prove the decomposition (11). Each component $b_f \mathbb{P}_{k-(\ell+1)}(f) \subset \mathbb{P}_k(T)$ and the sum is direct due to the property of $b_f$, cf. Lemma 2.4. Then count the dimensions to finish the proof.

To prove the unisolvence, we choose a basis $\{\phi_i\}$ of $\mathbb{P}_k(T)$ by the decomposition (11) and denote DoFs (12) as $\{N_i\}$. By construction, the dimension of $\{\phi_i\}$ matches the number of DoFs $\{N_i\}$. The square matrix $(N_i(\phi_j))$ is block lower triangular in the sense that for $\phi_f \in b_f \mathbb{P}_{k-(\ell+1)}(f)$,

$$\int_e \phi_f p \,\mathrm{d}s = 0, \qquad e \in \Delta_m(T) \text{ with } m \leq \ell \text{ and } e \neq f, p \in \mathbb{P}_{k-\dim e-1}(e)$$

due to the property of $b_f$ established in Lemma 2.4. Each diagonal block matrix is the Gram matrix

$$\int_f pq b_f \,\mathrm{d}x_f, \quad p, q \in \mathbb{P}_{k-(\ell+1)}(f),$$

in the measure $b_f \,\mathrm{d}x_f$ and thus symmetric and positive definite. In particular it is invertible. So the unisolvence follows from the inverstiablity of this lower triangular matrix; see below for an illustration.

$$\begin{array}{cc} N_f \backslash\, \phi_f & \begin{array}{ccccc} 0 & 1 & \cdots & d-1 & d \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ d-1 \\ d \end{array} & \left(\begin{array}{c|c|c|c|c} \square & 0 & \cdots & 0 & 0 \\ \hline \square & \square & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \square & \square & \cdots & \square & 0 \\ \hline \square & \square & \cdots & \square & \square \end{array}\right) \end{array}$$

$\square$

Consider a Hilbert space $V$ with the inner product $(\cdot,\cdot)$. Define $\mathcal{N} : V \to V^*$ as: for any $p \in V$, $\mathcal{N}(p) \in V^*$ is given by $\langle \mathcal{N}(p), \cdot \rangle = (\cdot, p)$. When restricted to face $f$, $\mathbb{P}_k(f)$ is a subspace of $L^2(f)$ and thus $\mathcal{N}(\mathbb{P}_k(f))$ consists of the degrees of freedom (DoFs):

$$\langle \mathcal{N}(p), \cdot \rangle = \int_f (\cdot)\, p \,\mathrm{d}s, \quad p \in \mathbb{P}_k(f).$$

For the 0-dimensional face, i.e., a vertex $\mathrm{v}$, we understand that $\langle \mathcal{N}(1), u\rangle = \int_{\mathrm{v}} u \,\mathrm{d}s = u(\mathrm{v})$ for $1 \in \mathbb{P}_k(\mathrm{v}) = \mathbb{R}$. Then (12) implies the decomposition of DoFs

(13)
$$\mathbb{P}_k^*(T) = \bigoplus_{\ell=0}^d \bigoplus_{f\in\Delta_\ell(T)} \mathcal{N}(b_f \mathbb{P}_{k-(\ell+1)}(f)) = \bigoplus_{\ell=0}^d \bigoplus_{f\in\Delta_\ell(T)} \mathcal{N}(\mathbb{P}_{k-(\ell+1)}(f)).$$

Notice that in the second decomposition, the bubble function $b_f$ is removed as $b_f \,\mathrm{d}x_f$ can be treat as a measure on $f$ and the test function in (12) is in $\mathbb{P}_{k-(\ell+1)}(f)$.

**Exercise 3.3.** *Define $\mathcal{X}_T = \{x(\alpha), \alpha \in \mathbb{T}_k^d\}$ and call it the set of interpolation nodes. Another DoFs are given by the function values on the interpolation nodes, i.e.*

$$N_\alpha(u) = u(x(\alpha)) \quad \alpha \in \mathbb{T}_k^d. \tag{14}$$

*Prove the unisolvence by finding a basis $\{\phi_\alpha, \alpha \in \mathbb{T}_k^d\}$ of $\mathbb{P}_k$ dual to $\{N_\alpha, \alpha \in \mathbb{T}_k^d\}$ of $\mathbb{P}_k^*$, i.e.,*

$$N_\alpha(\phi_\beta) = \delta(\alpha, \beta) \quad \alpha, \beta \in \mathbb{T}_k^d.$$

**Exercise 3.4.** *Find a basis of $\mathbb{P}_k$ dual to DoFs (12) normalized by $1/|f|$.*

3.2. **Lagrange finite element space.** Let $\{\mathcal{T}_h\}$ be a family of partitions of $\Omega$ into nonoverlapping simplexes with $h_T := \mathrm{diam}(T)$ and $h := \max_{T\in\mathcal{T}_h} h_T$. The mesh $\mathcal{T}_h$ is conforming in the sense that the intersection of any two simplexes is either empty or a common lower sub-simplex. Let $\Delta_\ell(\mathcal{T}_h)$ be the set of all $\ell$-dimensional faces of the partition $\mathcal{T}_h$ for $\ell = 0, 1, \ldots, d-1$. The Lagrange finite element space

$$S_h^k := \{v \in C(\Omega) : v|_T \in \mathbb{P}_k(T), \forall\, T \in \mathcal{T}_h,\ \text{DoFs (12) are single-valued}\}.$$

The following result is a direct corollary of Theorem 3.2.

**Corollary 3.5.** *The Lagrange finite element space has the geometric decomposition*

$$S_h^k = \bigoplus_{\ell=0}^d \bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)} b_f \mathbb{P}_{k-(\ell+1)}(f).$$

Here we extend the polynomial $b_f \mathbb{P}_{k-(\ell+1)}(f)$ to each element $T$ containing $f$ by the Bernstein form in the barycentric coordinate and thus it is a piecewise polynomial function and continuous in $\Omega$. Consequently $S_h^k \subset H^1(\Omega)$ and the dimension of $S_h^k$ is

$$\dim S_h^k = \sum_{\ell=0}^d |\Delta_\ell(\mathcal{T}_h)| \binom{k-1}{\ell},$$

where $|\Delta_\ell(\mathcal{T}_h)|$ is the cardinality, i.e., the number of $\ell$-dimensional simplexes in $\mathcal{T}_h$.

**Remark 3.6.** The degree $k$ could vary for different $f$. In $hp$ refinement, polynomial degree $k$ is a piecewise constant function on $\Delta_d(\mathcal{T}_h)$. Then set

$$k(f) = \max_{T\in\omega_f} k(T),$$

where $\omega_f := \{T \in \mathcal{T}_h, f \subseteq T\}$. Then the space

$$S_{hp} = \bigoplus_{\ell=0}^d \bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)} b_f \mathbb{P}_{k(f)-(\ell+1)}(f)$$

is a conforming finite element space with variable degree of polynomial.

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## References


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