BASICS ON FINITE ELEMENTS

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ABSTRACT. This lecture notes covers the construction of finite element spaces, focusing on simplicial triangulations. Examples of finite elements, such as the linear Lagrange element, Crouzeix-Raviart element, and higher-order Lagrange elements, are presented. The unisolvence property and continuity of these elements are discussed.

CONTENTS

1. AN EXAMPLE: THE LINEAR FINITE ELEMENT SPACE

A finite element space refers to a collection of piecewise polynomial functions with specific continuity properties defined on a mesh.

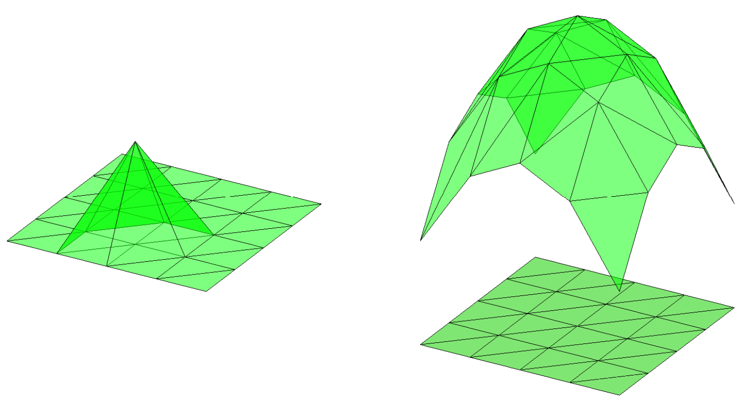

FIGURE 1. A linear finite element function on a triangulation.

Figure 1 illustrates a surface defined over a square domain on the right side. To approximate this surface, we divide the domain Ω into small triangles, forming a grid (mesh, or

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triangulation) denoted as $\mathcal{T}_h$. The parameter h represents the maximal diameter of these small triangles.

To construct the approximation, we evaluate the function value at each grid point and then connect these point values with triangles in space. This process yields an approximation of the surface. All such globally continuous and piecewise linear polynomial functions constitute a linear finite element space, commonly known as the linear Lagrange element or Courant element [5].

The linear finite element space is finite-dimensional, and its dimension equals the number of vertices in the mesh. Each vertex corresponds to a basis function, depicted on the left side of Figure 1. Due to its shape, this basis function is often referred to as a hat function. It has a range from 0 to 1 and exhibits a small support.

2. Simplicial Triangulations

In this section we give a mathematical definition of a conforming mesh. We restrict ourselves to simplicial triangulations. There are other type of meshes by partition the domain into quadrilaterals (in 2-D), cubes, prisms (in 3-D), or polytopes in general.

The *convex hull* of the $d+1$ points $\mathrm{v}_0, \mathrm{v}_1, \cdots, \mathrm{v}_d$ in $\mathbb{R}^d$

$$T := \{\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i \mathrm{v}_i \,|\, 0 \le \lambda_i \le 1, i = 0 : d, \sum_{i=0}^{d} \lambda_i = 1\} \tag{1}$$

is defined as a *geometric d-simplex* generated (or spanned) by the vertices $\mathrm{v}_0, \mathrm{v}_1, \cdots, \mathrm{v}_d$. For example, a triangle is a 2-simplex and a tetrahedron is a 3-simplex. An ℓ-dimensional sub-simplex/faces of T is generated by $\ell + 1$ vertices for $\ell < d$. Zero dimensional faces are vertices and one-dimensional faces are conventionally called edges of T.

Let Ω be a polyhedral domain in $\mathbb{R}^d, d \ge 1$. A geometric triangulation (also called mesh or grid) $\mathcal{T} = \{T\}$ of Ω is a set of d-simplices such that

$$\cup_{T \in \mathcal{T}} T = \overline{\Omega}, \quad \text{and} \quad \mathring{T} \cap \mathring{T}' = \varnothing, T, T' \in \mathcal{T}, T \neq T'.$$

There are two conditions that we shall impose on triangulations that are important in the finite element computation. The first requirement is a topological property. A triangulation $\mathcal{T}$ is called *conforming* or *compatible* if the intersection of any two simplexes T and T' in $\mathcal{T}$ is either empty or a common lower dimensional simplex.

(a) A triangulation with a hanging node

(b) A conforming triangulation

FIGURE 2. Two triangulations. The left is non-conforming and the right is conforming.

The second important condition depends on the geometric structure. A set of triangulations $\mathscr{T} = \{\mathcal{T}_h, h \in \mathcal{H}\}$ is called *shape regular* if there exists a constant c_0 such that

$$\max_{T \in \mathcal{T}_h} \frac{\operatorname{diam}(T)^d}{|T|} \le c_0, \quad \text{for all} \quad \mathcal{T}_h \in \mathscr{T}, \tag{2}$$

where $\operatorname{diam}(T)$ is the diameter of T and $|T|$ is the measure of T in $\mathbb{R}^d$. In two dimensions, it is equivalent to the minimal angle of each triangulation is bounded below uniformly in the shape regular class. We shall define $h_T = |T|^{1/d}$ for $T \in \mathcal{T}_h$. By (2), $h_T \eqsim \operatorname{diam}(T)$ represents the size of an element $T \in \mathcal{T}_h$ for a shape regular triangulation $\mathcal{T}_h \in \mathscr{T}$, where $h_T \eqsim \operatorname{diam}(T)$ means the ratio $h_T/\operatorname{diam}(T)$ has uniform upper and lower bounds with respect to T. Define the mesh size of $\mathcal{T}$ as $h_{\mathcal{T}} := \max_{T\in\mathcal{T}} h_T$. It is used to measure the approximation rate.

In addition to (2), if there exists a constant ρ such that

$$\frac{\max_{T\in\mathcal{T}_h}|T|}{\min_{T\in\mathcal{T}_h}|T|} \leq \rho, \quad \text{for all} \quad \mathcal{T} \in \mathscr{T}_h, \tag{3}$$

$\mathscr{T}$ is called *quasi-uniform*. For quasi-uniform grids, $h_{\mathcal{T}} \eqsim h_T$ for all $T \in \mathcal{T}_h$. In finite element literature, we often write a triangulation as $\mathcal{T}_h$.

3. Finite Element Triple

Let us now define finite elements mathematically. Ciarlet [4] defines a finite element as a triple: $(K, V, \mathcal{N})$

- K: element (triangle, square, tetrahedron, etc.)
- V: shape function space (usually polynomial space $\mathbb{P}_k$)
- $\mathcal{N}$: degrees of freedom (DoFs) (a set of basis for the dual space V')

With a triangulation $\mathcal{T}_h$, we can define a global finite element space

$$V_h := \{v \in L^2(\Omega) : v|_K \in V \text{ for } K \in \mathcal{T}_h, \text{ some DoFs in } \mathcal{N} \text{ are}$$
$$\text{single valued across interior lower-dimensional faces of } \mathcal{T}_h\}.$$

To be precise, if a DoF on a lower-dimensional faces f is single-valued, i.e., depending only on f not on the element T containing f, then we call it *global*. Otherwise we say it is *local* and might be different in different elements. We denote by $\mathcal{N}_h$ as the set of DoFs to define V_h.

The local DoFs $\mathcal{N}$ are usually well-defined for functions in a larger space U with $V \subseteq U$. The canonical interpolation operator $I : U \to V$ is defined by $N(Iu) = N(u)$ for all $N \in \mathcal{N}$ and abbreviate as $u_I := Iu$ We have $v_I = v$ for $v \in V$, i.e., the canonical interpolation is a projection.

For traditional finite elements, the shape function space is selected as a polynomial space, which is relatively easy. The key to constructing a finite element lies in finding a set of degrees of freedom $\mathcal{N}$ that satisfies two crucial conditions:

(U1) It is a basis of dual space V'. This ensures that each function in the shape function space can be uniquely represented by its DoF values.
(U2) The globally defined function space V_h has the desired continuity across the interior boundary of the elements.

The verification of Condition U1 is known as **unisolvence**. Let $\{\phi_1, \ldots, \phi_n\}$ be a set of basis of V with $n = \dim V$, and $v = c_1\phi_1 + \cdots + c_n\phi_n$ represents any function in V. Let $\mathcal{N} = \{N_1, \ldots, N_m\}$ be a set of DoFs. Define the following DoF-Basis matrix

$$\begin{pmatrix} N_1(\phi_1) & \cdots & N_1(\phi_n) \\ \vdots & & \vdots \\ N_m(\phi_1) & \cdots & N_m(\phi_n) \end{pmatrix}_{m\times n}. \tag{4}$$

The verification of unisolvence is indeed divided into two essential steps:

(1) **Count Dimensions**: In this step, we need to verify that the number of degrees of freedom is equal to the dimension of the shape function space. Each degree of freedom corresponds to one equation, and having the same number of equations as unknowns means that the DoF-Basis matrix in (4) is square.

Counting dimensions for scalar functions is relatively straightforward. However, for vector functions in high dimensions and for high degree of polynomials, the task becomes more challenging. When dealing with matrices or tensors, counting the number of DoFs is even more complex. The difficulty arises from the fact that these DoFs are assigned to sub-simplices of different dimensions (vertices, edges, faces, and volumes of tetrahedrons), and different components of vector and tensor functions are distributed differently.

(2) **Prove Full Rank**: In this step, we need to demonstrate that for any function in the shape function space, if all the degrees of freedom applied to this function are set to zero, then the function itself must be zero. Mathematically, the statement is:

$$\text{for any } v \in V, \text{ if } N_i(v) = 0 \text{ for } i = 1, \dots, n, \text{ then } v = 0. \tag{5}$$

This is equivalent to showing that the DoF-Basis matrix in (4) is full rank and therefore $\{N_1, \dots, N_n\}$ is linearly independent.

To ensure the desired continuity (Condition U2), we need to accurately characterize the trace of the corresponding Sobolev space using specific Green's identity. The finite element function is piecewise polynomial, but globally, it belongs to a certain Sobolev space. The piecewise polynomials are connected or "glued" together by the global DoFs.

For example, for a piecewise smooth function to belong to $H^1(\Omega)$, it must be globally continuous. Let $m \geq 0$ be a non-negative integer. For a function to be in $H^{m+1}(\Omega)$, it needs to belong to $C^m(\Omega)$, which means it is continuously differentiable up to order m.

Exercise 3.1. *Given a 2D triangular mesh $\mathcal{T}_h$, construct a piecewise polynomial and global $C^1(\Omega)$ finite element function space.*

Solving this problem is relatively straightforward in one dimension, and the resulting space is called a cubic Hermite spline. The one-dimensional result can be extended to higher dimensions through tensor product grids. However, when dealing with high-dimensional simplicial meshes, even a two-dimensional triangular mesh, the problem becomes more intricate.

In the future, we will delve into the construction of $C^m(\Omega)$ finite elements but now let us explore more examples on finite elements.

4. More Examples on Finite Elements

Example 4.1 (Lagrange linear element in two dimensions)**.** The shape function space is a linear (degree 1) polynomial space $\mathbb{P}_1$. In $\mathbb{R}^2$, its dimension is 3, which can be determined by the function values at the 3 vertices. See the left figure of Fig. 3 and three black dots in the middle of Fig. 3.

Example 4.2 (Crouzeix-Raviart linear element in two dimensions)**.** The shape function space is still $\mathbb{P}_1$. The DoFs are changed to the function values at the midpoints of edges. See the right figure of Fig. 3 and three red dots in the middle of Fig. 3. The function values at the midpoints of edges are equal to the integral average over edges $\frac{1}{|e|}\int_e v \, \mathrm{d}s$. The integral average DoFs require less regularity when defining the canonical interpolation.

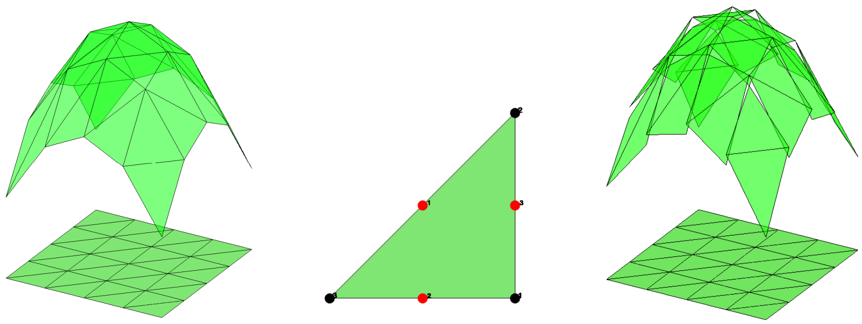


FIGURE 3. Left: Lagrange linear element is continuous at vertices and thus globally continuous. Right: Crouzeix-Raviart linear element is continuous at midpoints of edges only.

When restricted to each triangle, both choices of DoFs satisfy Condition U1, meaning that a linear polynomial can be uniquely determined by either function values at three vertices or function values at three midpoints of edges.

Fig. 3 illustrates the two choices of DoFs on a triangle. On the left, the Lagrange linear element is continuous at the vertices of the triangle, making it globally continuous as well. On the right, the Crouzeix-Raviart linear element is continuous only at the midpoint of each edge but not globally continuous. The Lagrange element function belongs to $H^1(\Omega)$, while the Crouzeix-Raviart element function does not.

Due to the difference in continuity properties, we classify the left choice as an H^1-conforming element and the right choice as an H^1-nonconforming element. Following the naming conventions of their discoverers, the left choice is known as the Lagrange element, while the right choice is known as the Crouzeix-Raviart (CR) element [6]. For an in-depth examination and review of the CR element, refer to [2].

While non-conforming elements can also be used to solve partial differential equations (PDEs), the theoretical analysis for them is more complicated [4, 3]. In this series, we focus on conforming finite elements.

Example 4.3 (Discontinuous linear element)**.** The shape function space is $\mathbb{P}_1$. There are still three DoFs but now they are local, i.e., depending on the element. Then no continuity is imposed when defining the global finite element space. The resulting space is a subspace of $L^2(\Omega)$. The local DoFs can be chosen as function values at the vertices or the midpoints of edges. A better choice is the moments over K: $\int_K vq \,\mathrm{d}x$ for $q \in \mathbb{P}_1$. The canonical interpolation operator defined by this DoF is the L^2-projection and usually denoted by Q_K.

Example 4.4 (Quadratic Lagrange element in two dimensions)**.** We then look at the quadratic element $\mathbb{P}_2$, whose dimension is 6. Then 6 DoFs are function values at 3 vertex values plus 3 edge midpoint values. This choice satisfies both Conditions U1 and U2.

The quadratic Lagrange element is illustrated on the left side of Fig. 4. It is defined on a triangle and has quadratic polynomials as shape functions. The right side of Fig. 4 shows the basis function corresponding to vertex 1, which is a quadratic polynomial that takes the value 1 at vertex 1 and 0 at all other nodes.

There are other choices of 6 DoFs for the quadratic element $\mathbb{P}_2$. However, not all choices of DoFs guarantee linear independence, which is crucial for satisfying Condition U1, i.e.,

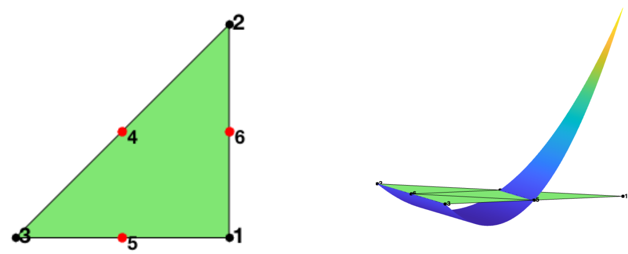


FIGURE 4. A basis function $\phi_1 = \lambda_1(2\lambda_1 - 1)$ of $\mathbb{P}_2$ Lagrange element.

forming a basis of the dual space V'. For instance, if we take two points on each edge of the triangle, the resulting DoFs may not be linearly independent.

To illustrate this, consider a configuration where these 6 points lie on a circle. In this case, the right-hand side in Fig. 5 will be a quadratic polynomial that vanishes at all 6 points but is non-zero itself. Consequently, the DoFs fail to form a linearly independent set, leading to the violation of Condition U1.

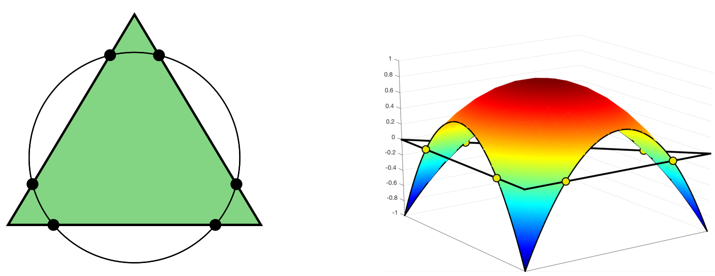


FIGURE 5. A quadratic polynomial $\phi(\lambda) = 2 - 3\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)$ vanishes at 6 nodes.

This example was introduced by Fortin and Soulie [8, 7] when attempting to generalize the CR linear non-conforming element to a quadratic non-conforming element. Constructing high-order non-conforming elements within the finite element framework is challenging. However, it becomes more feasible under the framework of Weak Galerkin [9] or Virtual Element spaces [1].

5. LAGRANGE ELEMENTS

The generalization of the Lagrange element from $\mathbb{P}_1$ to $\mathbb{P}_2$ and higher orders $\mathbb{P}_k$ for all integers $k \geq 1$ is straightforward [4]. In two dimensions, the triangular element is divided into four smaller triangles by connecting the midpoints of its edges, and this process is repeated $k - 1$ times to create the k-th order Lagrange element. DoFs are the function values at the vertices of all the small triangles formed after dividing the original triangle.

Exercise 5.1. *Prove the unisolvence of DoFs and that the Lagrange element function space is globally continuous.*

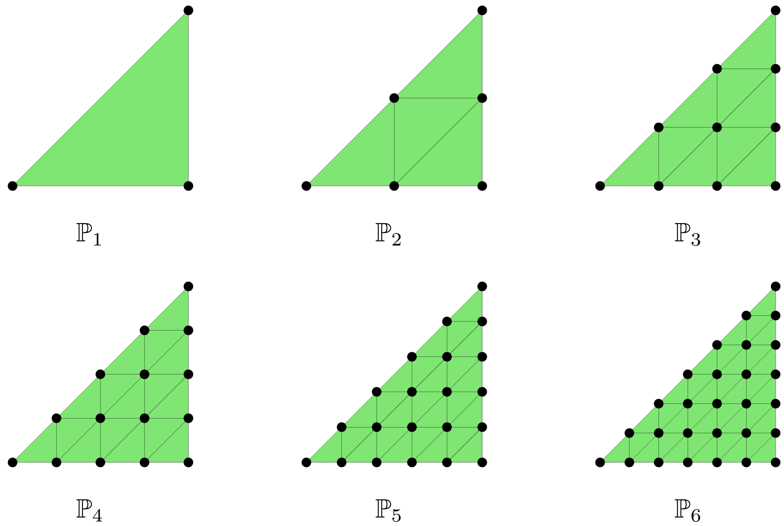
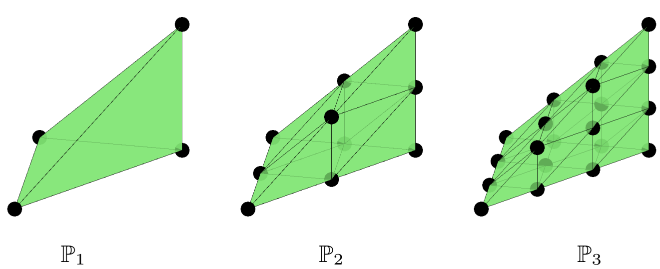


FIGURE 6. Lagrange elements in two dimensions.

Extending the Lagrange element to three dimensions requires a similar concept of dividing the tetrahedron repeatedly and using function values at the vertices of the small tetrahedra as DoFs. However, the process becomes less intuitive when transitioning to three-dimensions.

Let us start by considering the uniform refinement in two dimensions. We add the midpoints of the three edges of the triangle and then connect these midpoints to divide the triangle into four smaller triangles. These four small triangles are similar to the original one. This process allows us to uniformly refine the mesh while maintaining the similarity of triangles, ensuring that the small triangles formed do not degenerate.

Now, let us extend this process to three dimensions. In 3D, we add the midpoints of all six edges of the tetrahedron. To form smaller tetrahedra, we need to carefully connect these midpoints. The specific arrangement of connecting these midpoints can vary, and there are multiple valid configurations to create a total of eight small tetrahedra. These small tetrahedra are not all similar to the original tetrahedron. Only if the midpoints are connected appropriately, the resulting small tetrahedra will not degenerate when the process is repeated.



When programming this uniform refinement of 3D tetrahedral meshes, it is essential to handle the connectivity of vertices correctly. The process is not as straightforward as in 2D and requires strict rules for ordering the vertices. The implementation details can be found in the `uniformrefine3` function in the `ifem` package.

Exercise 5.2. *Prove the unisolvence of the three-dimensional Lagrange element and prove that the function space defined is globally continuous.*

Once we have understood the process for 2D and 3D, we can extend it to any d-dimensional simplex. For a d-dimensional simplex, we add the midpoint of each edge, and then connect these midpoints to create 2^d smaller simplexes. After $k-1$ times division, the number of vertices in these smaller simplexes equals the dimension of the polynomial space $\mathbb{P}_k(\mathbb{R}^d)$ used to define the finite element. Proving the unisolvence of the DoFs and showing that the global finite element space is continuous in H^1 is an essential part of the finite element method.

We will explore the unisolvence and continuity properties of Lagrange finite elements in the next lecture.

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