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A NEW DIV-DIV-CONFORMING SYMMETRIC TENSOR FINITE ELEMENT SPACE WITH APPLICATIONS TO THE BIHARMONIC EQUATION

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Abstract. A new $H(\operatorname{div}\operatorname{div})$-conforming finite element is presented, which avoids the need for supersmoothness by redistributing the degrees of freedom to edges and faces. This leads to a hybridizable mixed method with superconvergence for the biharmonic equation. Moreover, new finite element divdiv complexes are established. Finally, new weak Galerkin and C^0 discontinuous Galerkin methods for the biharmonic equation are derived.

1. Introduction

In recent years, there has been a series of developments in constructing $H(\operatorname{div}\operatorname{div})$-conforming finite elements [10, 14, 15, 26–28]. However, all these elements possess vertex degree of freedom (DoF), which makes them non-hybridizable. In this paper, we present a novel $H(\operatorname{div}\operatorname{div})$-conforming finite element that is hybridizable, enabling its efficient use in the numerical solutions of the biharmonic equation.

Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a Lipschitz domain. With the space $\mathbb{S}$ of symmetric tensors, the Sobolev space

$$H(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{S}) : \operatorname{div}\operatorname{div}\boldsymbol{\tau} \in L^2(\Omega)\}$$

with the inner div applied row-wisely to $\boldsymbol{\tau}$ resulting in a column vector for which the outer div operator is applied. The $H(\operatorname{div}\operatorname{div})$-conforming finite elements constructed in [10, 14, 15, 26–28] include the following DoFs:

$$\boldsymbol{\tau}(\mathtt{v}), \quad \mathtt{v} \in \Delta_0(T), \boldsymbol{\tau} \in \mathbb{S}, \tag{1.1}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f, \quad q \in \mathbb{P}_{k-r-1}(f), f \in \Delta_r(T), r = 1, \ldots, d-1, \text{ and } i, j = 1, \ldots, d-r, i \leq j. \tag{1.2}$$

Here, $\Delta_r(T)$ denotes the set of r-dimensional faces of the simplex T. Furthermore, $\boldsymbol{n}_i$ denotes the ith normal vector to the face f, and $(\cdot, \cdot)_f$ denotes the L^2-inner product over the face f. The new element will be constructed by redistributing the vertex and normal plane DoFs (1.1)–(1.2).

We provide a brief explanation of the redistribution process by examining DoFs of vertex $\mathtt{v}_0$. Face-normal vectors $\{\boldsymbol{n}_{F_i}, i = 1, \ldots, d\}$ form a basis of the ambient Euclidean space $\mathbb{R}^d$, $d \geq 2$, where F_i denotes the $(d-1)$-dimensional face containing

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$\mathtt{v}_0$ and opposite to $\mathtt{v}_i$ for $i = 1, \ldots, d$. We may then determine DoF $\boldsymbol{\tau}(\mathtt{v}_0) \in \mathbb{S}$ by considering the symmetric matrix $(\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}(\mathtt{v}_0)\boldsymbol{n}_{F_j})_{i,j=1,\ldots,d}$. We redistribute the diagonal entry $\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}(\mathtt{v}_0)\boldsymbol{n}_{F_i}$ to face F_i for $i = 1, \ldots, d$, while the off-diagonal entries $\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}(\mathtt{v}_0)\boldsymbol{n}_{F_j}$ with $1 \leq i < j \leq d$ to the $(d-2)$-dimensional face $e_{ij} = F_i \cap F_j$. This process can be extended to DoFs (1.2) as well by setting $\boldsymbol{n}_i = \boldsymbol{n}_{F_i}$.

In three dimensions, where $d = 3$, the faces F_i correspond to two-dimensional faces (i.e., "faces") and the e_{ij} correspond to one-dimensional faces (i.e., "edges"). We refer to this entire process as the redistribution of vertex DoFs to faces and edges. See Fig. 1 for an illustration of the redistribution.

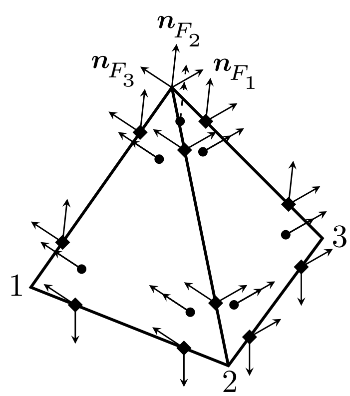


FIGURE 1. Redistribution of vertex degrees of freedom to faces and edges. $\boldsymbol{\tau}(\mathtt{v}_0) \in \mathbb{S}$ is a symmetric tensor containing 6 components. Three diagonal entries $\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}(\mathtt{v}_0)\boldsymbol{n}_{F_i}$ will be distributed to faces F_i for $i = 1, 2, 3$ and three off-diagonal entries $\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}(\mathtt{v}_0)\boldsymbol{n}_{F_j}$ to the edges $e_{ij} = F_i \cap F_j$ with $1 \leq i < j \leq 3$.

Upon redistribution, we use the geometric decomposition of the Lagrange element to merge facewise DoFs into normal-normal components as shown below:

$$(\boldsymbol{n}_F^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_F, q)_F, \quad q \in \mathbb{P}_k(F), F \in \Delta_{d-1}(T),$$

and merge the off-diagonal DoFs as shown below:

$$(\boldsymbol{n}_{F_1}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_2}, q)_e, \quad q \in \mathbb{P}_k(e), e \in \Delta_{d-2}(T), \tag{1.3}$$

where F_1 and F_2 are the two faces of the element T that share the edge e.

To ensure the $H(\operatorname{div}\operatorname{div})$-conformity, we modify DoF (1.3) on $\boldsymbol{n}_{F_1}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_2}$ to an edge jump term given by

$$\operatorname{tr}_e(\boldsymbol{\tau}) = \operatorname{tr}_e^T(\boldsymbol{\tau}) = \boldsymbol{n}_{F_1,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_1,\partial T} + \boldsymbol{n}_{F_2,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_2,\partial T},$$

where $\boldsymbol{n}_{F,e}$ denotes the normal direction of e on F induced by the orientation of F, and $\boldsymbol{n}_{F_i,\partial T}$ is the outward normal direction of face F_i with respect to ∂T. Here T represents a simplex and $\intercal$ is the transpose operator.

We provide DoFs in (2.3) and prove the unisolvence to the shape function space $\mathbb{P}_k(T;\mathbb{S})$ for $k \geq 3$. Afterwards, we define the global space $\Sigma_k^{\operatorname{div}\operatorname{div}-}$:

$$\Sigma_k^{\operatorname{div}\operatorname{div}-} := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_T \in \mathbb{P}_k(T;\mathbb{S}) \text{ for each } T \in \mathcal{T}_h,$$
$$\text{DoFs on } \operatorname{tr}_1(\boldsymbol{\tau}) \text{ and } \operatorname{tr}_2(\boldsymbol{\tau}) \text{ are single-valued}\},$$

where the traces $\mathrm{tr}_1(\boldsymbol{\tau}) = \boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n}$ and $\mathrm{tr}_2(\boldsymbol{\tau}) = \boldsymbol{n}_{\partial T}^\intercal \operatorname{div} \boldsymbol{\tau} + \operatorname{div}_F(\boldsymbol{\tau} \boldsymbol{n}_{\partial T})$ are continuous for $\boldsymbol{\tau} \in \Sigma_k^{\operatorname{div}\operatorname{div}-}$. However, the edge jump $\sum_{T\in\omega_e} \mathrm{tr}_e(\boldsymbol{\tau})|_e$ may not vanish which prevents $\Sigma_k^{\operatorname{div}\operatorname{div}-}$ being $H(\operatorname{div}\operatorname{div})$-conforming, where $\omega_e = \{T \in \mathcal{T}_h : e \subset T\}$ is the set of all simplices containing e. To obtain an $H(\operatorname{div}\operatorname{div})$-conforming subspace, we further define the subspace $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$ as the subspace of $\Sigma_k^{\operatorname{div}\operatorname{div}-}$ satisfying the constraint:

$$\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} := \{\boldsymbol{\tau} \in \Sigma_k^{\operatorname{div}\operatorname{div}-} : \sum_{T\in\omega_e} \mathrm{tr}_e(\boldsymbol{\tau})|_e = 0 \text{ for all } e \in \mathring{\mathcal{E}}_h\}.$$

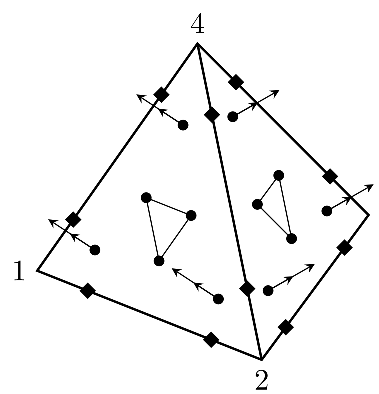
A similar constraint can be found in [18] when considering hybridization of edge elements. The space $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$ is $H(\operatorname{div}\operatorname{div})$-conforming and compared with other existing elements, the imposed continuity is minimal [23, Proposition 3.6] and no supersmoothness imposed in lower dimensional subsimplices. In particular, no vertex DoFs are needed.

The requirement $k \geq 3$ can be relaxed to $k \geq 2$ by enriching the shape function space

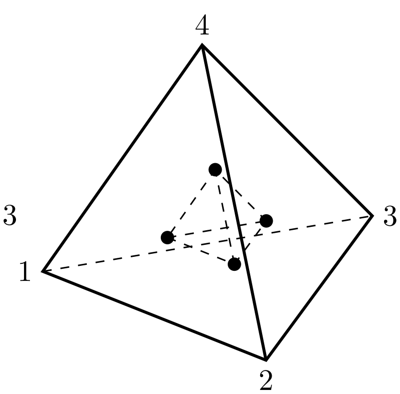
$$\Sigma_{k^+}(T;\mathbb{S}) := \mathbb{P}_k(T;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^\intercal \mathbb{H}_{k-1}(T),$$

which is in the spirit of the Raviart-Thomas (RT) element for $H(\operatorname{div})$-conforming vector finite element [2, 35]. A Raviart-Thomas type $H(\operatorname{div}\operatorname{div})$-conforming finite element space $\Sigma_{k^+}^{\operatorname{div}\operatorname{div}}$ for symmetric tensors can be constructed for $k \geq 2$.

Motivated by the construction in [22] in 2D, we further construct a lower order space $H(\operatorname{div}\operatorname{div})$-conforming finite element $\Sigma_{1^{++}}$ by enriching $\mathbb{P}_1(T;\mathbb{S})$ by some quadratic and cubic polynomials. The 3D version is illustrated in Fig. 2.



(a) The lowest degree $H(\operatorname{div}\operatorname{div})$-conforming element $\Sigma_{1^{++}}(T;\mathbb{S})$ (with 36 DoFs) for $\boldsymbol{\sigma}$



(b) Discontinuous $\mathbb{P}_1(T)$ element (with 12 DoFs) for $\boldsymbol{u}$

FIGURE 2. The lowest degree pair $\Sigma_{1^{++}}(T;\mathbb{S}) - \mathbb{P}_1(T)$ in three dimensions

The symmetric tensor finite element with only normal-normal continuity for $k \geq 0$ is shown in [24, 25, 30, 33, 34]. For the discretization of the biharmonic equation in two dimensions, referred to as the Hellan-Herrmann-Johnson (HHJ) mixed method [1, 9, 19, 24, 25, 30], the normal-normal continuous finite element for symmetric tensors is employed. Notably, there is currently no existing HHJ method for dimensions greater than two. The normal-normal continuous finite element for symmetric tensors is also adopted in [33, 34] to discretize the linear elasticity, known as the tangential-displacement normal-normal-stress (TDNNS) method. We also refer to [16] for an $H(\operatorname{rot}\operatorname{rot})$-conforming finite element for symmetric tensors on the Clough-Tocher split in two dimensions.

The $H(\operatorname{div}\operatorname{div})$-conforming finite element constructed in this paper is applicable for discretizing the biharmonic equation for all dimensions $d \geq 2$ and offers optimal convergence for symmetric tensors, along with a fourth-order higher superconvergence for the post-processed deflection. Through a hybridization technique, the implementation of the mixed method developed in this paper can be treated as a generalization of hybridized HHJ methods from 2D to arbitrary dimensions.

The $H(\operatorname{div}\operatorname{div})$-conforming space $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$ might be somewhat challenging to implement in practical applications. This complexity arises from the stringent continuity requirements placed on $\operatorname{tr}_1(\boldsymbol{\tau})$ and $\operatorname{tr}_2(\boldsymbol{\tau})$, as well as the patch constraint imposed on edge jumps. To mitigate these challenges, we employ a hybridization technique [1, 21] that effectively relaxes these continuity conditions. We utilize the discontinuous stress space $\Sigma_k^{-1} = V_k^{-1}(\mathcal{T}_h; \mathbb{S})$, and broken space

$$\mathring{M}_{k-2,k-1,k,k}^{-1} = V_{k-2}^{-1}(\mathcal{T}_h) \times V_{k-1}^{-1}(\mathring{\mathcal{F}}_h) \times V_k^{-1}(\mathring{\mathcal{F}}_h) \times V_k^{-1}(\mathring{\mathcal{E}}_h),$$

where V_r^{-1} denotes the discontinuous polynomial space of degree r with respect to some finite set, $\mathcal{T}_h$ is a triangulation, $\mathring{\mathcal{F}}_h$ is the set of interior $(d-1)$-dimensional faces, and $\mathring{\mathcal{E}}_h$ the set of interior $(d-2)$-dimensional faces. Spaces on $\mathring{\mathcal{F}}_h$ and $\mathring{\mathcal{E}}_h$ can be thought of as Lagrange multipliers for the required continuity. For example, $V_{k-1}^{-1}(\mathring{\mathcal{F}}_h)$ is for $\operatorname{tr}_2(\boldsymbol{\sigma})$ which is one degree lower than that of $\boldsymbol{\sigma}$ as $\operatorname{tr}_2(\boldsymbol{\sigma})$ consists of first-order derivatives of $\boldsymbol{\sigma}$.

Define the weak $(\operatorname{div}\operatorname{div})_w$ operator

$$(\operatorname{div}\operatorname{div})_w \boldsymbol{\sigma} := ((\operatorname{div}\operatorname{div})_T \boldsymbol{\sigma}, -h_F^{-1}[\operatorname{tr}_2(\boldsymbol{\sigma})]|_F, h_F^{-3}[\boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n}]|_F, h_e^{-2}[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e).$$

A hybridized mixed finite element method for the biharmonic equation is: find $\boldsymbol{\sigma}_h \in \Sigma_k^{-1}$ and $u_h \in \mathring{M}_{k-2,k-1,k,k}^{-1}$ s.t.

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + ((\operatorname{div}\operatorname{div})_w \boldsymbol{\tau}, u_h)_{0,h} = 0 \qquad \forall \boldsymbol{\tau} \in \Sigma_k^{-1}, \tag{1.4a}$$

$$((\operatorname{div}\operatorname{div})_w \boldsymbol{\sigma}_h, v)_{0,h} = -(f, v_0) \qquad \forall v \in \mathring{M}_{k-2,k-1,k,k}^{-1}, \tag{1.4b}$$

with appropriate modification of (f, v_0) for $k = 0, 1, 2$. We will establish the following discrete inf-sup condition,

$$\inf_{v \in \mathring{M}_{k-2,k-1,k,k}^{-1}} \sup_{\boldsymbol{\tau} \in \Sigma_k^{-1}} \frac{((\operatorname{div}\operatorname{div})_w \boldsymbol{\tau}, v)_{0,h}}{\|\boldsymbol{\tau}\|_{\operatorname{div}\operatorname{div}_w} \|v\|_{0,h}} = \alpha > 0, \quad k \geq 0,$$

from which the well-posedness of (1.4) follows. When $k = 0$, (1.4) is equivalent to using the Morley-Wang-Xu element [37] for the biharmonic equation. In other words, (1.4) generalise the popular quadratic Morley element to higher order and to higher dimensions.

Optimal convergence rates will be established for the solution $(\boldsymbol{\sigma}_h, u_h)$ to (1.4):

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |Q_M u - u_h|_{2,h} + \|Q_M u - u_h\|_{0,h} \lesssim h^{k+1} |u|_{k+3}.$$

Post-processing techniques can be used to obtain u_h^* with $k \geq 3$ satisfying

$$\|\nabla_h^2(u-u_h^*)\|_0 \lesssim h^{k+1}|u|_{k+3}, \quad \|u-u_h^*\|_0 \lesssim h^{\min\{2k-2,k+3\}}\|u\|_{k+3}.$$

Hybridization (1.4) can be also generalized to the Raviart-Thomas type $\Sigma_{k^+}^{-1} - \mathring{M}_{k-1,k-1,k,k}^{-1}$ for $k \geq 2$ and $\Sigma_{1^{++}}^{-1} - \mathring{M}_{1,1,1,1}^{-1}$ for $k=1$.

We define the weak Hessian operator ∇_w^2 as the adjoint of $(\operatorname{div}\operatorname{div})_w$ with respect to a mesh-dependent inner product $(\cdot,\cdot)_{0,h}$. Using the operator ∇_w^2, we can interpret the hybridization (1.4) as a weak Galerkin method for the biharmonic equation, which does not require any additional stabilization:

$$(\nabla_w^2 u_h, \nabla_w^2 v) = (f, v_0) \quad \forall v \in \mathring{M}_{k-2,k-1,k,k}^{-1}, \tag{1.5}$$

with appropriate modification of computing (f, v_0) for low order cases. Restricting (1.5) to different subspaces of $\mathring{M}_{k-2,k-1,k,k}^{-1}$ will derive new discrete methods:

- Embedding the H^2-non-conforming virtual element on simplices in [11] into the broken space $\mathring{M}_{k-2,k-1,k,k}^{-1}$, we acquire a stabilization-free non-conforming virtual element method for the biharmonic equation.
- Embedding the continuous Lagrange element $\mathring{V}_k$ into $\mathring{M}_{k-2,k-1,k,k}^{-1}$, we obtain a parameter-free C^0 discontinuous Galerkin (DG) method for the biharmonic equation, which generalizes the 2D scheme in [29] to arbitrary dimension $d \geq 2$.

In three dimensions, we construct the finite element $\operatorname{div}\operatorname{div}$ complex, for $k \geq 3$,

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-2}^{-1}(\mathcal{T}_h) \to 0, \tag{1.6}$$

where $\mathbf{RT} := \{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\}$, and V_{k+2}^H is the vectorial Hermite element space [17]. Since no supersmooth DoFs for space $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$, we construct $H(\operatorname{sym}\operatorname{curl};\mathbb{T})$-conforming finite element space $\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}}$ simpler than those in [13, 15, 26, 27]. Lower order finite element $\operatorname{div}\operatorname{div}$ complexes for $k = 1, 2$ in three dimensions are also constructed. The first half of the complex (1.6) can be replaced by

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^L \xrightarrow{\operatorname{dev}\operatorname{grad}} \overline{\Sigma}_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}}, \tag{1.7}$$

and the second half by $\xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k^+}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-1}^{-1}(\mathcal{T}_h) \to 0$, which leads to several variants of (1.6); see Section 5 for details.

With the weak $\operatorname{div}\operatorname{div}_w$ operator, for $k \geq 1$, we can construct the distributional finite element divdiv complex

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_k^{-1} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{k-2,k-1,k,k}^{-1} \to 0.$$

The normal-normal continuous finite element Σ_k^{nn} can be treated as a subspace of Σ_k^{-1} and the corresponding distributional divdiv complex becomes, for $k \geq 1$,

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_k^{\mathrm{nn}} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{k-2,k-1,\cdot,k}^{-1} \to 0,$$

which can be treated as a generalization of 2D distributive divdiv complex involving HHJ elements developed in [9] to 3D. Again the first half can be replaced by (1.7) for $k \geq 0$ and more variants, including $k = 0$ case, can be found in Section 5.2.

The rest of this paper is organized as follows. Hybridizable $H(\operatorname{div}\operatorname{div})$-conforming finite elements in arbitrary dimension are constructed in Section 2. A mixed finite element method together with error analysis, post-processing, and duality argument are presented in Section 3. Then in Section 4, the hybridization and its equivalence to other methods are presented for the mixed finite element method of the biharmonic equation. Several new finite element divdiv complexes in three dimensions are devised in Section 5.

2. $H(\operatorname{div}\operatorname{div})$-conforming finite elements

In this section, we discuss $H(\operatorname{div}\operatorname{div})$-conforming finite elements. We review existing finite elements that enforce conformity by ensuring continuity on the normal plane of lower-dimensional subsimplices, which is known as supersmoothness. By using a redistribution technique, we obtain a new element without such supersmoothness. Additionally, we construct a Raviart-Thomas type element using enriched polynomial spaces.

2.1. **Notation.** Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded polytope. Given a bounded domain $D \subset \mathbb{R}^d$ and a non-negative integer k, let $H^k(D)$ be the usual Sobolev space of functions over D, whose norm and semi-norm are denoted by $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ respectively. Define $H_0^k(D)$ as the closure of $C_0^\infty(D)$ with respect to the norm $\|\cdot\|_{k,D}$. Let $(\cdot,\cdot)_D$ be the standard inner product on $L^2(D)$. If D is Ω, we abbreviate $\|\cdot\|_{k,D}$, $|\cdot|_{k,D}$ and $(\cdot,\cdot)_D$ by $\|\cdot\|_k$, $|\cdot|_k$ and $(\cdot,\cdot)$, respectively. Denote by h_D the diameter of D.

For a d-dimensional simplex T, we let $\Delta(T)$ denote all the subsimplices of T, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension ℓ, for $0 \leq \ell \leq d$.

For $f \in \Delta_\ell(T)$ with $0 \leq \ell \leq d$, let $\boldsymbol{n}_{f,1}, \ldots, \boldsymbol{n}_{f,d-\ell}$ be linearly independent unit normal vectors, and $\boldsymbol{t}_{f,1}, \ldots, \boldsymbol{t}_{f,\ell}$ be its orthonormal tangential vectors. We abbreviate $\boldsymbol{n}_{F,1}$ as $\boldsymbol{n}_F$ or $\boldsymbol{n}$ when $\ell = d-1$. We also abbreviate $\boldsymbol{n}_{f,i}$ and $\boldsymbol{t}_{f,i}$ as $\boldsymbol{n}_i$ and $\boldsymbol{t}_i$ respectively if not causing any confusion. For a $(d-1)$-dimensional face $F \in \partial T$ and a $(d-2)$-dimensional face $e \in \partial F$, $\boldsymbol{n}_{F,e}$ denotes the normal direction of e on F induced by the orientation of F. When $d = 2$, e is a vertex and F is an edge. Then $\boldsymbol{n}_{F,e} = \boldsymbol{t}_F$ if e is the end point of F for the orientation given by $\boldsymbol{t}_F$ and $\boldsymbol{n}_{F,e} = -\boldsymbol{t}_F$ otherwise. We use $\boldsymbol{n}_{\partial T}$ to denote the unit outward normal vector of ∂T which is a piecewise constant vector function.

Given a face $F \in \Delta_{d-1}(T)$, and a vector $\boldsymbol{v} \in \mathbb{R}^d$, define

$$\Pi_F \boldsymbol{v} := (\boldsymbol{n}_F \times \boldsymbol{v}) \times \boldsymbol{n}_F = (\boldsymbol{I} - \boldsymbol{n}_F \boldsymbol{n}_F^{\intercal})\boldsymbol{v}$$

as the projection of $\boldsymbol{v}$ onto the face F. For a scalar function v, define the surface gradient

$$\nabla_F v := \Pi_F \nabla v = \nabla v - \frac{\partial v}{\partial n_F}\boldsymbol{n}_F = \sum_{i=1}^{d-1} \frac{\partial v}{\partial t_{F,i}} \boldsymbol{t}_{F,i},$$

namely the projection of ∇v to the face F, which is independent of the choice of the normal vectors. Denote by $\operatorname{div}_F \boldsymbol{v} := \nabla_F \cdot (\Pi_F \boldsymbol{v})$ the corresponding surface divergence.

Denote by $\mathcal{T}_h$ a conforming triangulation of Ω with each element being a simplex, where $h := \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{F}_h$, $\mathring{\mathcal{F}}_h$, $\mathcal{E}_h$ and $\mathring{\mathcal{E}}_h$ be the set of all $(d-1)$-dimensional faces, interior $(d-1)$-dimensional faces, $(d-2)$-dimensional faces and interior $(d-2)$-dimensional faces, respectively. Set $\mathcal{F}_h^\partial := \mathcal{F}_h \backslash \mathring{\mathcal{F}}_h$ and $\mathcal{E}_h^\partial := \mathcal{E}_h \backslash \mathring{\mathcal{E}}_h$. For $e \in \mathcal{E}_h$, denote by $\omega_e := \{T \in \mathcal{T}_h : e \subset T\}$ as the set of all simplices containing e. We use ∇_h, ∇_h^2 and $(\operatorname{div}\operatorname{div})_h$ to represent the element-wise gradient, Hessian and div div with respect to $\mathcal{T}_h$. Consider two adjacent simplices T_1 and T_2 sharing an interior face F. Define the average and the jump of a function w on F as

$$\{w\} := \frac{1}{2}\big((w|_{T_1})|_F + (w|_{T_2})|_F\big), \quad [w] := (w|_{T_1})|_F \boldsymbol{n}_F \cdot \boldsymbol{n}_{\partial T_1} + (w|_{T_2})|_F \boldsymbol{n}_F \cdot \boldsymbol{n}_{\partial T_2}.$$

On a face F lying on the boundary $\partial\Omega$, the above terms become

$$\{w\} := w|_F, \quad [w] := w|_F.$$

For a bounded domain $D \subset \mathbb{R}^d$ and a non-negative integer k, let $\mathbb{P}_k(D)$ stand for the set of all polynomials over D with the total degree no more than k. When $k < 0$, set $\mathbb{P}_k(D) := \{0\}$. Let $Q_{k,D}$ be the L^2-orthogonal projector onto $\mathbb{P}_k(D)$, and Q_k its element-wise version with respect to $\mathcal{T}_h$. Let $\mathbb{H}_k(D) := \mathbb{P}_k(D) \backslash \mathbb{P}_{k-1}(D)$ be the space of homogeneous polynomials of degree k. In the binomial coefficient notation $\binom{n}{k}$, if $n \geq 0, k < 0$, we set $\binom{n}{k} := 0$.

Let $V_k^{-1}(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} \mathbb{P}_k(T)$ for $k \geq 0$ and abbreviate as V_k^{-1} when the dependence of $\mathcal{T}_h$ is not emphasized.

Set $\mathbb{M} := \mathbb{R}^{d\times d}$. Denote by $\mathbb{S}$, $\mathbb{K}$ and $\mathbb{T}$ the subspace of symmetric matrices, skew-symmetric matrices and traceless matrices of $\mathbb{M}$, respectively. For a space $B(D)$ defined on D, let $B(D;\mathbb{X}) := B(D) \otimes \mathbb{X}$ be its vector or tensor version for $\mathbb{X}$ being $\mathbb{R}^d$, $\mathbb{M}$, $\mathbb{S}$, $\mathbb{K}$ and $\mathbb{T}$.

Throughout this paper, we use "$\lesssim \cdots$" to mean that "$\leq C \cdots$", where letter C is a generic positive constant independent of h, which may stand for different values at its different occurrences. The notation $A \eqsim B$ means $B \lesssim A \lesssim B$.

2.2. **Trace and continuity.** We consider the continuity of a piecewise smooth tensor function to be in the Sobolev space

$$H(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : \operatorname{div}\operatorname{div}\boldsymbol{\tau} \in L^2(\Omega)\},$$

which plays a central role in our later constructions. We start from the Green's identity established in [14, 15] for the operator divdiv.

The trace $\operatorname{tr}^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}$, as a distribution, is defined as the difference

$$\langle \operatorname{tr}^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}, \operatorname{tr}^{\nabla^2} v\rangle_{\partial T} := (\operatorname{div}\operatorname{div}\boldsymbol{\sigma}, v)_T - (\boldsymbol{\sigma}, \nabla^2 v)_T.$$

We decompose $\operatorname{tr}^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}$ and $\operatorname{tr}^{\nabla^2} v$ into two face-wise trace operators and one edge trace operator.

Lemma 2.1 (Lemma 5.2 in [14]). *We have for any $\boldsymbol{\sigma} \in \mathcal{C}^2(T;\mathbb{S})$ and $v \in H^2(T)$ that*

$$\begin{aligned}(\operatorname{div}\operatorname{div}\boldsymbol{\sigma}, v)_T &= (\boldsymbol{\sigma}, \nabla^2 v)_T \\ &- \sum_{F\in\partial T}[(\operatorname{tr}_1(\boldsymbol{\sigma}), \operatorname{tr}_1(v))_F - (\operatorname{tr}_2(\boldsymbol{\sigma}), \operatorname{tr}_2(v))_F] - \sum_{e\in\Delta_{d-2}(T)}(\operatorname{tr}_e(\boldsymbol{\sigma}), \operatorname{tr}_e(v))_e,\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
\operatorname{tr}_1(\boldsymbol{\sigma}) &= \boldsymbol{n}_{\partial T}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T}, & \operatorname{tr}_1(v) &= \partial_n v \,|_{\partial T},\\
\operatorname{tr}_2(\boldsymbol{\sigma}) &= \boldsymbol{n}_{\partial T}^{\intercal}\operatorname{div}\boldsymbol{\sigma} + \operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}_{\partial T}), & \operatorname{tr}_2(v) &= v\,|_{\partial T},\\
\operatorname{tr}_e(\boldsymbol{\sigma}) &= \sum_{F\in\partial T, e\in\partial F}\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T}, & \operatorname{tr}_e(v) &= v\,|_{\Delta_{d-2}(T)}\,.
\end{aligned}$$

When summing over all elements and assuming the test function v is smooth enough, e.g. $v \in C^2(\Omega)$, we can merge the terms on the interior faces and edges. For an interior face $F \in \mathring{\mathcal{F}}_h$, denote by T_1, T_2 two elements containing F. Introduce the jumps

$$[\operatorname{tr}_1(\boldsymbol{\sigma})]_F := \boldsymbol{n}_{\partial T_1}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_1}\mid_F - \boldsymbol{n}_{\partial T_2}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_2}\mid_F,$$
$$[\operatorname{tr}_2(\boldsymbol{\sigma})]_F := (\boldsymbol{n}_{\partial T_1}^{\intercal}\operatorname{div}\boldsymbol{\sigma} + \operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_1}))\mid_F + (\boldsymbol{n}_{\partial T_2}^{\intercal}\operatorname{div}\boldsymbol{\sigma} + \operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_2}))\mid_F,$$
$$[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e := \sum_{T\in\omega_e}\sum_{F\in\partial T, e\in\partial F}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T})|_e.$$

We recall the results from [23] using our notation.

Lemma 2.2 (Proposition 3.6 in [23]). *Let $\boldsymbol{\sigma} \in L^2(\Omega;\mathbb{S})$ and $\boldsymbol{\sigma}|_T \in H^2(T;\mathbb{S})$ for each $T \in \mathcal{T}_h$. Then $\boldsymbol{\sigma} \in H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ if and only if*

1. $[\operatorname{tr}_1(\boldsymbol{\sigma})]_F = 0$ *for all* $F \in \mathring{\mathcal{F}}_h$;
2. $[\operatorname{tr}_2(\boldsymbol{\sigma})]_F = 0$ *for all* $F \in \mathring{\mathcal{F}}_h$;
3. $[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e = 0$ *for all* $e \in \mathring{\mathcal{E}}_h$.

Enforcing the jump condition $[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e = 0$ in $H(\operatorname{div}\operatorname{div})$-conforming finite element constructions is a challenging task as the constraint is imposed in the patch of e. The continuity of $\boldsymbol{\sigma}$ projected onto the normal plane $\mathcal{N}_e$ of $e \in \mathring{\mathcal{E}}_h$ is sufficient but by no means necessary. More specifically, as a $(d-2)$-dimensional subsimplex, the dimension of the normal plane $\mathcal{N}_e$ is two. To enforce the continuity condition, we choose two orthonormal directions $\boldsymbol{n}_1, \boldsymbol{n}_2$ normal to e for each edge $e \in \mathcal{E}_h$. It is important to note that $\mathcal{N}_e$ depends solely on e and not on the elements containing it. We denote the space of 2×2 symmetric matrices on $\mathcal{N}_e$ by $\mathbb{S}(\mathcal{N}_e)$, and define $Q_{\mathcal{N}_e}(\boldsymbol{\sigma}) := (\boldsymbol{n}_i^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_j)_{i,j=1,2}$ as the projection of $\boldsymbol{\sigma} \in \mathbb{S}$ onto $\mathbb{S}(\mathcal{N}_e)$.

Lemma 2.3. *Let $\boldsymbol{\sigma} \in L^2(\Omega;\mathbb{S})$ and $\boldsymbol{\sigma}|_T \in H^2(T;\mathbb{S})$ for each $T \in \mathcal{T}_h$. If $Q_{\mathcal{N}_e}(\boldsymbol{\sigma})$ is continuous on e, then $[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e = 0$ for all $e \in \mathring{\mathcal{E}}_h$.*

Proof. For each F containing $e \in \mathring{\mathcal{E}}_h$, F is also interior and thus there exist exactly two elements T_1, T_2 in the edge patch ω_e s.t. $F \in \partial T_i, i = 1, 2$. The normal vector $\boldsymbol{n}_{F,e}$ is induced by the orientation of F which is independent of the elements but $\boldsymbol{n}_{\partial T}$ is the outward normal direction depending on the element T containing F, and $\boldsymbol{n}_{\partial T_1}\mid_F = -\boldsymbol{n}_{\partial T_2}\mid_F$. Therefore $(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_1} + \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_{\partial T_2})|_e = 0$ and consequently $[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e = 0$. $\square$

2.3. $H(\operatorname{div}\operatorname{div})$**-conforming finite elements.** Several $H(\operatorname{div}\operatorname{div})$-conforming finite elements have been constructed in a series of recent works [10, 14, 15, 26–28]. In the following, we recall the version presented in [14, Theorem 5.10] with a slight change in the notation: r in (2.2b) represents the dimension of the subsimplex while in [14, Theorem 5.10], it is the co-dimension.

Recall that, for a simplex T and an integer $k \geq 0$, the first kind Nedéléc element [31] is

$$\mathrm{ND}_k(T) = \mathbb{P}_k(T;\mathbb{R}^d) \oplus \mathbb{H}_k(T;\mathbb{K})\boldsymbol{x} = \operatorname{grad}\mathbb{P}_{k+1}(T) \oplus \mathbb{P}_k(T;\mathbb{K})\boldsymbol{x}.$$

Let $\mathbf{RM} := \mathrm{ND}_0(T)$ be the kernel of the operator $\operatorname{def} := \operatorname{sym}\operatorname{grad}$. We have $\mathbf{RM} \subseteq \mathrm{ND}_{k-3}(T)$ when $k \geq 3$.

For $k \geq 3$, the shape function space is $\Sigma_k(T;\mathbb{S}) := \mathbb{P}_k(T;\mathbb{S})$ and degrees of freedom (DoFs) are given by

$$\boldsymbol{\tau}(\mathtt{v}), \quad \mathtt{v} \in \Delta_0(T), \tag{2.2a}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f, \quad q \in \mathbb{P}_{k-r-1}(f), f \in \Delta_r(T), r = 1,\dots,d-1, \text{ and } i,j = 1,\dots,d-r, i \leq j, \tag{2.2b}$$

$$(\operatorname{tr}_2(\boldsymbol{\tau}), q)_F, \quad q \in \mathbb{P}_{k-1}(F), F \in \partial T, \tag{2.2c}$$

$$(\Pi_F \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \partial T, \tag{2.2d}$$

$$(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathrm{ND}_{k-3}(T) \backslash \mathbf{RM}, \tag{2.2e}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \ker(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(T;\mathbb{S}), \tag{2.2f}$$

We can view DoFs in (2.2a) as a special case of those in (2.2b) if we treat $\mathbb{R}^d$ as the normal plane of the vertex $\mathtt{v}$. DoFs (2.2a)–(2.2b) will determine the trace $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$ and also imply the continuity of $\boldsymbol{\tau}$ on the normal plane of edges. Notice that DoF (2.2b) only exists for subsimplex with dimension $r \leq k-1$. DoF (2.2c) is to impose the continuity of $\operatorname{tr}_2(\boldsymbol{\tau}) = \boldsymbol{n}_F^{\intercal} \operatorname{div} \boldsymbol{\tau} + \operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}_F)$, which is modified from the DoF of $\boldsymbol{n}^{\intercal} \operatorname{div} \boldsymbol{\tau}$ on F. To have the surjection $\boldsymbol{n}_F^{\intercal} \operatorname{div} \mathbb{P}_k(T;\mathbb{S}) = \mathbb{P}_{k-1}(F)$, the degree $k \geq 3$ is required; see [14, Lemma 5.3]. Moreover, $k \geq 3$ is also required so that $\mathrm{ND}_{k-3}(T) \backslash \mathbf{RM}$ in DoF (2.2e) is meaningful. The space $\mathrm{ND}_{k-3}(T) \backslash \mathbf{RM}$ can be any subspace $X \subset \mathrm{ND}_{k-3}(T)$ satisfying $\mathrm{ND}_{k-3}(T) = \mathbf{RM} \oplus X$. Since the kernel of the operator def is $\mathbf{RM}$, in (2.2e), we can also write $\mathrm{ND}_{k-3}(T)$ only.

For $k = 0, 1, 2$, one can check by direct calculation that the number of DoFs is more than the dimension of the shape function space. See also Remark 2.10.

Lemma 2.4 (Theorem 5.10 in [14])**.** *For $k \geq 3$, the DoFs* (2.2) *are unisolvent for the space $\mathbb{P}_k(T;\mathbb{S})$.*

Remark 2.5. In [14, Theorem 5.10], the requirement $k \geq \max\{d, 3\}$ is presented. The condition $k \geq d$ is to ensure DoF (2.2b) exists on $(d-1)$-dimensional faces so that the inf-sup condition holds. Based on the key decomposition in [14, Fig. 5.1] and the characterization of each component established in Lemma 4.5 for $\operatorname{tr}^{\operatorname{div}}(\mathbb{P}_k(T;\mathbb{S}))$ with $k \geq 1$, Lemma 4.11 for $E_0'(\mathbb{S})$ with $k \geq 2$, Lemma 5.3 for $\operatorname{tr}^{\operatorname{div}} \operatorname{div} F_r(\mathbb{S})$ with $k \geq 3$, and Lemma 5.4 for $F_0'(\mathbb{S})$ with $k \geq 3$, the unisolvence holds with condition $k \geq 3$ only.

The finite element space $\Sigma_k^{\operatorname{div}\operatorname{div}}$ is defined as follows

$$\Sigma_k^{\operatorname{div}\operatorname{div}} := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_T \in \mathbb{P}_k(T;\mathbb{S}) \text{ for each } T \in \mathcal{T}_h, \text{ DoFs (2.2a), (2.2b), and (2.2c) are single-valued}\}.$$

The single-valued DoFs in (2.2a) and (2.2b) imply the continuity of the $Q_{\mathcal{N}_f}(\boldsymbol{\tau})$ function for all lower-dimensional subsimplices f of T. In particular, the edge jump vanishes, i.e., $[\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0$ as proven in Lemma 2.3. The continuity of $\operatorname{tr}_1(\boldsymbol{\tau})$ and $\operatorname{tr}_2(\boldsymbol{\tau})$ are imposed by DoFs (2.2a)–(2.2c). Therefore, we can conclude that $\Sigma_k^{\operatorname{div}\operatorname{div}} \subset H(\operatorname{div}\operatorname{div}, \Omega;\mathbb{S})$ in view of Lemma 2.2.

DoF (2.2d) for the tangential-normal component $\Pi_F \boldsymbol{\tau}\boldsymbol{n}$ is considered as a local DoF, i.e., it is not single-valued across simplices. If DoF (2.2d) is also single-valued, then the function is also in $H(\operatorname{div}, \Omega;\mathbb{S})$ and the corresponding element, which is firstly introduced by Hu, Ma, and Zhang [28], is $H(\operatorname{div}\operatorname{div};\mathbb{S}) \cap H(\operatorname{div};\mathbb{S})$-conforming.

When $k \geq \max\{d,3\}$, we have the discrete divdiv stability [14, Lemma 5.12]. Namely $\operatorname{div}\operatorname{div}: \Sigma_k^{\operatorname{div}\operatorname{div}} \to V_{k-2}^{-1}(\mathcal{T}_h)$ is surjective and the following inf-sup condition holds with a constant α independent of h

$$\inf_{p_h \in V_{k-2}^{-1}(\mathcal{T}_h)} \sup_{\boldsymbol{\tau}_h \in \Sigma_k^{\operatorname{div}\operatorname{div}}} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div}\operatorname{div}} \|p_h\|_0} = \alpha > 0, \quad k \geq \max\{d,3\}.$$

Although the element is well-defined for $k \geq 3$, the constraint $k \geq d$ is required for the inf-sup condition. When $k \geq d$, DoF (2.2b) includes the moment $\int_F \boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n} \,\mathrm{d}S$ for $F \in \partial T$, which is required by the fact that the range space $\operatorname{div}\operatorname{div}\Sigma_k^{\operatorname{div}\operatorname{div}}$ should include all piecewise linear functions.

Implementing the $H(\operatorname{div}\operatorname{div})$-conforming element defined by DoF (2.2) can be challenging due to the high degree $k \geq \max\{d,3\}$ and the relatively complex degrees of freedom. In two dimensions, an $H(\operatorname{div}\operatorname{div}) \cap H(\operatorname{div})$-conforming element has been successfully implemented and applied to discretize the biharmonic equation using the basis for Hu-Zhang $H(\operatorname{div};\mathbb{S})$-element, as described in [28].

We will present a new $H(\operatorname{div}\operatorname{div})$-conforming finite element with minimal smoothness. For $k \geq 3$, the shape function space is still $\mathbb{P}_k(T;\mathbb{S})$ and the following DoFs (2.3) are proposed:

$$(\operatorname{tr}_e(\boldsymbol{\tau}), q)_e, \quad q \in \mathbb{P}_k(e), e \in \Delta_{d-2}(T), \tag{2.3a}$$

$$(\boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_k(F), F \in \partial T, \tag{2.3b}$$

$$(\operatorname{tr}_2(\boldsymbol{\tau}), q)_F, \quad q \in \mathbb{P}_{k-1}(F), F \in \partial T, \tag{2.3c}$$

$$(\Pi_F \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \partial T, \tag{2.3d}$$

$$(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathrm{ND}_{k-3}(T), \tag{2.3e}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \ker(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(T;\mathbb{S}). \tag{2.3f}$$

Comparing with DoFs (2.2), the difference is that DoFs (2.2a)–(2.2b) are redistributed to edges and faces to form DoFs (2.3a)–(2.3b).

We now briefly explain the redistribution process. Without loss of generality, consider vertex $\mathtt{v}_0$. Choose $\{\boldsymbol{n}_{F_i}, i = 1, \dots, d\}$ as a basis of $\mathbb{R}^d$, where F_i is the $(d-1)$-dimensional face containing $\mathtt{v}_0$ for $i = 1, \dots, d$. DoF $\boldsymbol{\tau}(\mathtt{v}_0) \in \mathbb{S}$ is determined by the symmetric matrix $(\boldsymbol{n}_{F_i}^\intercal \boldsymbol{\tau}(\mathtt{v}_0) \boldsymbol{n}_{F_j})_{i,j=1,\dots,d}$. We redistribute the diagonal entry $\boldsymbol{n}_{F_i} \boldsymbol{\tau}(\mathtt{v}_0) \boldsymbol{n}_{F_i}$ to face F_i, for $i = 1, \dots, d$, and the off-diagonal $\boldsymbol{n}_{F_i} \boldsymbol{\tau}(\mathtt{v}_0) \boldsymbol{n}_{F_j}, 1 \leq i < j \leq d$, to edge $e_{ij} = F_i \cap F_j$. Such redistribution can be generalized to DoF (2.2b). For a lower dimensional subsimplex $f \in \Delta_r(T)$, $r = 1, \dots, d-1$, use $\{\boldsymbol{n}_{F_i}, f \in \Delta_r(F_i), i = 1, \dots, d-r\}$ as the basis of the normal plane $\mathcal{N}_f$ of f. We can redistribute the diagonal $\boldsymbol{n}_{F_i}^\intercal \boldsymbol{\tau} \boldsymbol{n}_{F_i}|_f$ to face F_i and off-diagonal $\boldsymbol{n}_{F_i}^\intercal \boldsymbol{\tau} \boldsymbol{n}_{F_j}|_f$ to edge $e_{ij} = F_i \cap F_j$.

After the redistribution, we merge DoFs. A function $u \in \mathbb{P}_k(T)$ can be determined by

$$(u, q)_T, \quad q \in \mathbb{P}_k(T). \tag{2.4}$$

Recall that the geometric decomposition of the Lagrange element in [3, (2.6)] is

$$\mathbb{P}_k(T) = \bigoplus_{r=0}^{d} \bigoplus_{f \in \Delta_r(T)} b_f \mathbb{P}_{k-(r+1)}(f), \tag{2.5}$$

where $b_f \in \mathbb{P}_{r+1}(f), b_f|_{\partial f} = 0$, is the $\mathbb{P}_{r+1}$-polynomial bubble function on f. Based on (2.5), DoF (2.4) can be decomposed into

$$(u, q)_f, \quad q \in \mathbb{P}_{k-r-1}(f), f \in \Delta_r(T), r = 0, 1, \dots, d. \tag{2.6}$$

Vice versa, DoFs in (2.6) can be merged into (2.4).

After redistribution, we merge DoFs facewisely and edgewisely. For example, on a face F, we will have DoFs

$$(\boldsymbol{n}_F^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_F, q)_f, \quad q \in \mathbb{P}_{k-r-1}(f), f \in \Delta_r(F), r = 0, 1, \ldots, d-1. \tag{2.7}$$

By the decomposition of the Lagrange element (2.5), we can merge (2.7) to DoF (2.3b). Similarly on an edge e shared by F_1 and F_2, we merge DoFs for $\boldsymbol{n}_{F_1}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_2}$ to

$$(\boldsymbol{n}_{F_1}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_2}, q)_e, \quad q \in \mathbb{P}_k(e), e \in \Delta_{d-2}(T). \tag{2.8}$$

To switch from DoF (2.8) to edge jump DoF (2.3a), we require Lemma 2.6.

Lemma 2.6. *For a $(d-2)$-dimensional face $e \in \Delta_{d-2}(T)$, let F_1 and F_2 be the two $(d-1)$-dimensional faces in $\Delta_{d-1}(T)$ sharing e, and $\boldsymbol{n}_{F_i} = \boldsymbol{n}_{F_i,\partial T}$ for $i = 1, 2$. Then*

$$\{\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1}, \boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2}, \operatorname{sym}(\boldsymbol{n}_{F_1,e} \otimes \boldsymbol{n}_{F_1}) + \operatorname{sym}(\boldsymbol{n}_{F_2,e} \otimes \boldsymbol{n}_{F_2})\}$$

and

$$\{\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1}, \boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2}, \operatorname{sym}(\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_2})\}$$

are bases of the symmetric matrix space $\mathbb{S}(\mathcal{N}_e)$ on the normal plane of e.

Proof. Clearly,

$$\mathbb{S}(\mathcal{N}_e) = \operatorname{span}\{\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1}, \boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2}, \operatorname{sym}(\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_2})\},$$

and $\operatorname{sym}(\boldsymbol{n}_{F_1,e} \otimes \boldsymbol{n}_{F_1}) + \operatorname{sym}(\boldsymbol{n}_{F_2,e} \otimes \boldsymbol{n}_{F_2}) \in \mathbb{S}(\mathcal{N}_e)$.

Now we prove that $\boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1}$, $\boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2}$ and $\operatorname{sym}(\boldsymbol{n}_{F_1,e} \otimes \boldsymbol{n}_{F_1}) + \operatorname{sym}(\boldsymbol{n}_{F_2,e} \otimes \boldsymbol{n}_{F_2})$ are linearly independent. Assume constants c_1, c_2 and c_3 satisfy

$$c_1 \boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1} + c_2 \boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2} + c_3\big(\operatorname{sym}(\boldsymbol{n}_{F_1,e} \otimes \boldsymbol{n}_{F_1}) + \operatorname{sym}(\boldsymbol{n}_{F_2,e} \otimes \boldsymbol{n}_{F_2})\big) = 0.$$

Let us show that $c_1 = c_2 = c_3 = 0$. Multiplying $\operatorname{sym}(\boldsymbol{n}_{F_1,e} \otimes \boldsymbol{n}_{F_2,e})$ on both sides of the last equation, we get

$$\frac{1}{2} c_3 (\boldsymbol{n}_{F_1} \cdot \boldsymbol{n}_{F_2,e} + \boldsymbol{n}_{F_2} \cdot \boldsymbol{n}_{F_1,e}) = 0.$$

Noting that both $\boldsymbol{n}_{F_1} \cdot \boldsymbol{n}_{F_2,e}$ and $\boldsymbol{n}_{F_2} \cdot \boldsymbol{n}_{F_1,e}$ are positive, we get $c_3 = 0$. And this implies

$$c_1 \boldsymbol{n}_{F_1} \otimes \boldsymbol{n}_{F_1} + c_2 \boldsymbol{n}_{F_2} \otimes \boldsymbol{n}_{F_2} = 0.$$

Thus, $c_1 = c_2 = 0$. ☐

We are in the position to prove the unisolvence. Recall that in the binomial coefficient notation $\binom{n}{k}$, if $n \geq 0, k < 0$, we set $\binom{n}{k} := 0$.

Lemma 2.7. *For $k \geq 3$, the DoFs (2.3) are unisolvent for the space $\mathbb{P}_k(T; \mathbb{S})$.*

Proof. For a d-simplex T, the number of subsimplexes of dimension r is $\binom{d+1}{r+1}$. The dimension of $\mathbb{P}_{k-r-1}(f)$ with $\dim f = r$ is $\binom{k-r-1+r}{k-r-1}$ which also holds for $r \geq k$ as $\dim \mathbb{P}_{k-r-1}(f) = 0$. The normal plane $\mathcal{N}_f$ of f, will have dimension $d - r$ and the symmetric tensor on $\mathcal{N}_f$ will have dimension $\binom{d-r+1}{2}$ which can be split into off-diagonals and diagonal, i.e., $\binom{d-r+1}{2} = \binom{d-r}{2} + d - r$. The number of DoFs (2.2a)–(2.2b) is

$$\sum_{r=0}^{d-1} \binom{d+1}{r+1}\binom{k-1}{k-r-1}\binom{d-r+1}{2}$$

$$= \frac{1}{2}d(d+1)\sum_{r=0}^{d-2}\binom{d-1}{r+1}\binom{k-1}{k-r-1} + (d+1)\sum_{r=0}^{d-1}\binom{d}{r+1}\binom{k-1}{k-r-1} \tag{2.9}$$

$$= \frac{1}{2}d(d+1)\binom{k+d-2}{k} + (d+1)\binom{k+d-1}{k}, \tag{2.10}$$

which equals the number of DoFs (2.3a)–(2.3b). Hence the number of DoFs (2.3) matches the number of DoFs (2.2) which is the dimension of the space $\mathbb{P}_k(T;\mathbb{S})$ by Lemma 2.4. In the derivation above, (2.9) corresponds to the redistribution of DoFs to edges and faces, and (2.10) is the merge of DoFs for the Lagrange element on edges and faces.

Let $\boldsymbol{\tau} \in \mathbb{P}_k(T;\mathbb{S})$ and suppose that all DoFs given by (2.3) vanish. Using Lemma 2.6, we know that the vanishing DoFs (2.3a)–(2.3b) imply that DoF (2.8) also vanishes. Moreover, the vanishing of (2.8) and (2.3b) implies that DoFs (2.2a)–(2.2b) are also zero. Therefore, by the unisolvence property stated in Lemma 2.4, we conclude that $\mathbb{P}_k(T;\mathbb{S})$ is unisolvent. ∎

Define the global space

$$\begin{aligned}\Sigma_k^{\operatorname{div}\operatorname{div}-} := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : &\boldsymbol{\tau}|_T \in \mathbb{P}_k(T;\mathbb{S}) \text{ for each } T \in \mathcal{T}_h,\\ &\text{DoFs (2.3b) and (2.3c) are single-valued}\}.\end{aligned}$$

By construction, for $\boldsymbol{\tau} \in \Sigma_k^{\operatorname{div}\operatorname{div}-}$, both $\operatorname{tr}_1(\boldsymbol{\tau})$ and $\operatorname{tr}_2(\boldsymbol{\tau})$ will be continuous. But the edge jumps $[\operatorname{tr}_e(\boldsymbol{\tau})]|_e$ may not vanish which prevents $\Sigma_k^{\operatorname{div}\operatorname{div}-}$ being $H(\operatorname{div}\operatorname{div})$-conforming in view of Lemma 2.2. The edge jump condition $[\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0$ is imposed patch-wisely on ω_e. Inside each element, $\operatorname{tr}_e(\boldsymbol{\tau})$ may not be zero and for different elements the edge jumps are in general different. Therefore (2.3a) is not single-valued when defining $\Sigma_k^{\operatorname{div}\operatorname{div}-}$.

Define the subspace

$$\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} := \{\boldsymbol{\tau} \in \Sigma_k^{\operatorname{div}\operatorname{div}-} : [\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0 \text{ for all } e \in \mathring{\mathcal{E}}_h\}.$$

That is we add constraints on the DoFs of the element-wise edge traces: $\operatorname{tr}_e^{T_1}(\boldsymbol{\tau}) + \operatorname{tr}_e^{T_2}(\boldsymbol{\tau}) + \cdots + \operatorname{tr}_e^{T_{|\omega_e|}}(\boldsymbol{\tau})|_e = 0$ to get an $H(\operatorname{div}\operatorname{div})$-conforming subspace.

Let $I_h^{\operatorname{div}\operatorname{div}} : H^2(\Omega;\mathbb{S}) \to \Sigma_k^{\operatorname{div}\operatorname{div}-}$ be the canonical interpolation operator based on the DoFs (2.3). Namely $N(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}) = N(\boldsymbol{\tau})$ for all DoFs N in (2.3). To save notation, we will abbreviate $I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}$ as $\boldsymbol{\tau}_I$. Noting that

$$[\operatorname{tr}_e(\boldsymbol{\tau}_I)]|_e = Q_{k,e}([\operatorname{tr}_e(\boldsymbol{\tau})]|_e) = 0 \quad \forall e \in \mathring{\mathcal{E}}_h, \boldsymbol{\tau} \in H^2(\Omega;\mathbb{S}),$$

so indeed $\boldsymbol{\tau}_I \in \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$.

Lemma 2.8. $I_h^{\operatorname{div}\operatorname{div}}$ *is a Fortin operator in the sense that: for* $\boldsymbol{\tau} \in H^2(\Omega;\mathbb{S})$,

$$\operatorname{div}\operatorname{div}(\boldsymbol{\tau}_I) = Q_{k-2}(\operatorname{div}\operatorname{div}\boldsymbol{\tau}). \tag{2.11}$$

Proof. It can be proved by using the Green's identity (2.1) and the definition of $I_h^{\operatorname{div}\operatorname{div}}$. ∎

Using the Fortin operator, we arrive at the following inf-sup condition.

Lemma 2.9. *We have the inf-sup condition*

$$\inf_{p_h\in V_{k-2}^{-1}} \sup_{\boldsymbol{\tau}_h\in\Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}}} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{div\,div}}\|p_h\|_0} = \alpha > 0, \quad \textit{for } k\geq 3. \tag{2.12}$$

Proof. For $p_h \in V_{k-2}^{-1}$, there exists a function $\boldsymbol{\tau}\in H^2(\Omega;\mathbb{S})$ [4,32] such that

$$\|\boldsymbol{\tau}\|_2 \lesssim \|p_h\|_0, \quad \operatorname{div}\operatorname{div}\boldsymbol{\tau} = p_h.$$

Let $\boldsymbol{\tau}_h = \boldsymbol{\tau}_I \in \boldsymbol{\Sigma}_{k,\mathrm{new}}^{\mathrm{div\,div}}$. By (2.11),

$$\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h = Q_{k-2}(\operatorname{div}\operatorname{div}\boldsymbol{\tau}) = p_h.$$

Apply the scaling argument to get

$$\|\boldsymbol{\tau}_h\|_{\mathrm{div\,div}} \lesssim \|\boldsymbol{\tau}\|_2 \lesssim \|p_h\|_0.$$

Finally, we finish the proof of (2.12). $\square$

Comparing with the existing $H(\operatorname{div}\operatorname{div})$-conforming elements constructed in [10, 14, 15, 26–28], we do not enforce the normal plane continuity on lower dimensional subsimplexes and thus no requirement $k\geq d$ for the inf-sup condition. However, the condition $k\geq 3$ is still needed to ensure $\mathbf{RM} = \ker(\operatorname{def}) \subseteq \mathrm{ND}_{k-3}(T)$ in DoF (2.3e). Remark 2.10 shows $\mathbb{P}_k(T;\mathbb{S})$, $k\leq 2$, is not feasible.

Remark 2.10. For a linear polynomial $v\in\mathbb{P}_1(T)$, by identity (2.1) and the fact $\nabla^2 v = 0$, we have for $\boldsymbol{\tau}\in\mathbb{P}_k(T;\mathbb{S})$ that

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_T = \sum_{F\in\partial T}[(\operatorname{tr}_2(\boldsymbol{\tau}), v)_F - (\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F] - \sum_{F\in\partial T}\sum_{e\in\partial F}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e. \tag{2.13}$$

When $k\leq 2$, for $\boldsymbol{\tau}\in\mathbb{P}_k(T;\mathbb{S})$, $\operatorname{div}\operatorname{div}\boldsymbol{\tau}\in\mathbb{P}_0(T)$. We can choose a non-zero function $v\in\mathbb{P}_1(T)\cap L_0^2(T)$ such that $(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_T = 0$, hence it follows

$$\sum_{F\in\partial T}[(\operatorname{tr}_2(\boldsymbol{\tau}), v)_F - (\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F] - \sum_{F\in\partial T}\sum_{e\in\partial F}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e = 0.$$

This means the DoFs (2.3a)–(2.3c) for traces are not linearly independent when $k\leq 2$. The range of the $\operatorname{div}\operatorname{div}$ operator should contain $\mathbb{P}_1(T)$ piecewisely.

2.4. **Raviart-Thomas type elements.** We enrich the range of the $\operatorname{div}\operatorname{div}$ operator with the addition of high order inner moments. Take the space of shape functions as

$$\Sigma_{k^+}(T;\mathbb{S}) := \mathbb{P}_k(T;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(T), \quad k\geq 2.$$

The additional component $\boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(T)$ expands the range of the $\operatorname{div}\operatorname{div}$ operator to $\mathbb{P}_{k-1}(T)$ as $\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(T)) = \mathbb{H}_{k-1}(T)$, which is one degree higher than the range $\operatorname{div}\operatorname{div}\mathbb{P}_k(T;\mathbb{S}) = \mathbb{P}_{k-2}(T)$.

For $k\geq 3$, the degrees of freedom are nearly identical to those given in (2.3), with the exception of enriching the DoF in (2.3e) to

$$(\boldsymbol{\tau}, \operatorname{def}\boldsymbol{q})_T \quad \text{for} \quad \boldsymbol{q}\in\mathbb{P}_{k-2}(T;\mathbb{R}^d). \tag{2.14}$$

The degree of freedom $(\boldsymbol{\tau}, \operatorname{def}\boldsymbol{q})_T$ is increased from $\boldsymbol{q}\in\mathrm{ND}_{k-3}(T) = \operatorname{grad}\mathbb{P}_{k-2}(T)\oplus\mathbb{P}_{k-3}(T;\mathbb{K})\boldsymbol{x}$ in (2.3e) to $\mathbb{P}_{k-2}(T;\mathbb{R}^d) = \operatorname{grad}\mathbb{P}_{k-1}(T)\oplus\mathbb{P}_{k-3}(T;\mathbb{K})\boldsymbol{x}$. All boundary DoFs (2.3a)–(2.3d) remain the same as $(\boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(T))\boldsymbol{n}|_F\in\mathbb{P}_k(F;\mathbb{R}^d)$.

For $k=2$, $\ker(\operatorname{def}) = \mathrm{ND}_0(K) \not\subseteq \mathbb{P}_0(T;\mathbb{R}^d)$ in DoF (2.14). We propose the following DoFs for $\Sigma_{2^+}(T;\mathbb{S})$ which is a generalization of the $H(\operatorname{div}\operatorname{div})$-conforming finite element constructed in [15] by the redistribution process:

$$(\operatorname{tr}_e(\boldsymbol{\tau}), q)_e, \quad q \in \mathbb{P}_2(e), e \in \Delta_{d-2}(T), \tag{2.15a}$$

$$(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_2(F), F \in \partial T, \tag{2.15b}$$

$$(\operatorname{tr}_2(\boldsymbol{\tau}), q)_F, \quad q \in \mathbb{P}_1(F), F \in \partial T, \tag{2.15c}$$

$$(\Pi_f \boldsymbol{\tau}\boldsymbol{n}_{F_r}, \boldsymbol{q})_f, \quad \boldsymbol{q} \in \mathbb{B}_2^{\operatorname{div}}(f), f = f_{0:r-2} \in \Delta_{r-2}(F_r), r = d, \ldots, 3, \tag{2.15d}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \ker(\boldsymbol{x}^{\intercal}\cdot\boldsymbol{x}) \cap \mathbb{P}_1(T;\mathbb{S}), \tag{2.15e}$$

where $f_{0:r} = \operatorname{Convex}(\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_r)$ is the r-dimensional simplex spanned by the vertices $\{\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_r\}$. A proof of the unisolvence can be found in Appendix A (Theorem A.5).

Define the global spaces, for $k \geq 2$,

$$\begin{aligned}\Sigma_{k^+}^{\operatorname{div}\operatorname{div}} :=& \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_T \in \Sigma_{k^+}(T;\mathbb{S}) \text{ for each } T \in \mathcal{T}_h,\\ & \text{DoFs (2.3b) and (2.3c) are single-valued}, [\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0 \text{ for all } e \in \mathring{\mathcal{E}}_h\}.\end{aligned}$$

We have $\Sigma_{k^+}^{\operatorname{div}\operatorname{div}} \subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$.

Similar to the proof of (2.12) by using the canonical interpolation operator $I_h^{\operatorname{div}\operatorname{div}}$, we have the inf-sup condition

$$\inf_{p_h \in V_{k-1}^{-1}} \sup_{\boldsymbol{\tau}_h \in \Sigma_{k^+}^{\operatorname{div}\operatorname{div}}} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div}\operatorname{div}}\|p_h\|_0} = \alpha > 0, \quad \text{for } k \geq 2. \tag{2.16}$$

2.5. **A lower order $H(\operatorname{div}\operatorname{div})$-conforming finite element.** For $k=1$, we enrich the $\mathbb{P}_1(T;\mathbb{S})$ space by adding some quadratic and cubic polynomials. Take the shape function space as

$$\Sigma_{1^{++}}(T;\mathbb{S}) = \mathbb{P}_1(T;\mathbb{S}) \oplus \operatorname{sym}(\boldsymbol{x} \otimes \mathbb{H}_1(T;\mathbb{R}^d)) \oplus \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_1(T). \tag{2.17}$$

The range $\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_1(T)) = \mathbb{H}_1(T)$ and $\operatorname{div}\operatorname{div}\operatorname{sym}(\boldsymbol{x}\otimes\mathbb{H}_1(T;\mathbb{R}^d)) = \mathbb{P}_0(T)$. Consequently $\operatorname{div}\operatorname{div}\Sigma_{1^{++}}(T;\mathbb{S}) = \mathbb{P}_1(T)$.

When $\boldsymbol{\tau} \in \Sigma_{1^{++}}(T;\mathbb{S})$, we can see that $\operatorname{tr}_e(\boldsymbol{\tau}) \in \mathbb{P}_1(e)$ for $e \in \Delta_{d-2}(T)$, and $(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})|_F, \operatorname{tr}_2(\boldsymbol{\tau})|_F \in \mathbb{P}_1(F)$ for $F \in \partial T$. Hence, we propose the following DoFs:

$$(\operatorname{tr}_e(\boldsymbol{\tau}), q)_e, \quad q \in \mathbb{P}_1(e), e \in \Delta_{d-2}(T), \tag{2.18a}$$

$$(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_1(F), F \in \partial T, \tag{2.18b}$$

$$(\operatorname{tr}_2(\boldsymbol{\tau}), q)_F, \quad q \in \mathbb{P}_1(F), F \in \partial T. \tag{2.18c}$$

Lemma 2.11. *The DoFs* (2.18) *are unisolvent for the space* $\Sigma_{1^{++}}(T;\mathbb{S})$.

Proof. DoFs (2.18a)–(2.18b) are redistribution of vertex DoFs for $\mathbb{P}_1(T;\mathbb{S})$. The enrichment in (2.17) has dimension d^2+d while the number of DoF (2.18c) is $(d+1)d$. Therefore the number of DoF (2.18) is equal to $\dim\Sigma_{1^{++}}(T;\mathbb{S}) = \frac{1}{2}d(d+1)(d+3)$.

Assume $\boldsymbol{\tau} \in \Sigma_{1^{++}}(T;\mathbb{S})$, and all the DoFs (2.18) vanish. Then

$$\operatorname{tr}_1(\boldsymbol{\tau}) = 0, \quad \operatorname{tr}_2(\boldsymbol{\tau}) = 0, \quad Q_{\mathcal{N}_e}(\boldsymbol{\tau}) = 0 \text{ for } e \in \Delta_{d-2}(T). \tag{2.19}$$

Apply the integration by parts to get $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$. Consequently $\boldsymbol{\tau} \in \mathbb{P}_1(T;\mathbb{S}) + \operatorname{sym}(\boldsymbol{x}\otimes\mathbb{P}_1(T;\mathbb{R}^d))$.

Let $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \operatorname{sym}(\boldsymbol{x}\otimes\boldsymbol{q})$ with $\boldsymbol{\tau}_1 \in \mathbb{P}_1(T;\mathbb{S})$ and $\boldsymbol{q} \in \mathbb{H}_1(T;\mathbb{R}^d)$. Then $\operatorname{div}\boldsymbol{q} = 0$ follows from $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$. Since $\operatorname{tr}_2(\boldsymbol{\tau}_1)$ is piecewise constant, the fact $\operatorname{tr}_2(\boldsymbol{\tau}) = 0$ in

(2.19) means $\operatorname{tr}_2(\operatorname{sym}(\boldsymbol{x}\otimes\boldsymbol{q}))|_F \in \mathbb{P}_0(F)$ for face $F \in \partial T$. By $\operatorname{div}(\boldsymbol{x}\boldsymbol{q}^{\intercal}) = \boldsymbol{q}+\boldsymbol{x}\operatorname{div}\boldsymbol{q}$, $\operatorname{div}(\boldsymbol{q}\boldsymbol{x}^{\intercal}) = (d+1)\boldsymbol{q}$, and $\operatorname{div}_F(\boldsymbol{x}\boldsymbol{q}\cdot\boldsymbol{n}) = d\boldsymbol{q}\cdot\boldsymbol{n}$, we get

$$\operatorname{tr}_2(\operatorname{sym}(\boldsymbol{x}\otimes\boldsymbol{q}))|_F = (d+1)\boldsymbol{q}\cdot\boldsymbol{n} + \frac{1}{2}\boldsymbol{x}\cdot\boldsymbol{n}(\operatorname{div}\boldsymbol{q} + \operatorname{div}_F\boldsymbol{q}) \in \mathbb{P}_0(F).$$

This indicates $(\boldsymbol{q}\cdot\boldsymbol{n})|_F \in \mathbb{P}_0(F)$, which means $\boldsymbol{q} \in \mathbf{RT}$. By $\boldsymbol{q} \in \mathbb{H}_1(T;\mathbb{R}^d)\cap\ker(\operatorname{div})$, $\boldsymbol{q} = 0$. Now $\boldsymbol{\tau} \in \mathbb{P}_1(T;\mathbb{S})$. The third identity in (2.19) implies $\boldsymbol{\tau}$ vanishes on all the vertices of T, therefore $\boldsymbol{\tau} = 0$. ☐

Define the global space

$$\begin{aligned}\Sigma_{1++}^{\operatorname{div}\operatorname{div}} :=\{&\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_T \in \Sigma_{1++}(T;\mathbb{S}) \text{ for each } T \in \mathcal{T}_h,\\ &\text{DoFs (2.18b) and (2.18c) are single-valued}, [\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0 \text{ for all } e \in \mathring{\mathcal{E}}_h\}.\end{aligned}$$

We have $\Sigma_{1++}^{\operatorname{div}\operatorname{div}} \subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. Again using the canonical interpolation operator $I_h^{\operatorname{div}\operatorname{div}}$, it holds the inf-sup condition

$$\inf_{p_h\in V_1^{-1}} \sup_{\boldsymbol{\tau}_h\in\Sigma_{1++}^{\operatorname{div}\operatorname{div}}} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{\operatorname{div}\operatorname{div}}\|p_h\|_0} = \alpha > 0. \tag{2.20}$$

In two dimensions, i.e., $d = 2$, the finite element space $\Sigma_{1++}^{\operatorname{div}\operatorname{div}}$ has been constructed in [22]. Our construction of $\Sigma_{1++}^{\operatorname{div}\operatorname{div}}$ for general $d \geq 2$ is motivated by their work.

3. A mixed method for the biharmonic equation

This section will discuss a mixed finite element method for solving the biharmonic equation. Optimal convergence rates are obtained. Post-processing techniques will be introduced to further improve the accuracy of the solution.

3.1. **Mixed methods for the biharmonic equation.** Let $f \in L^2(\Omega)$ be given. Consider the biharmonic equation

$$\begin{cases}\Delta^2 u = f \quad \text{in } \Omega,\\ u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0.\end{cases} \tag{3.1}$$

The mixed formulation is: find $\boldsymbol{\sigma} \in H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}), u \in L^2(\Omega)$ s.t.

$$(\boldsymbol{\sigma},\boldsymbol{\tau}) + (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, u) = 0 \qquad \forall\boldsymbol{\tau} \in H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}), \tag{3.2a}$$

$$(\operatorname{div}\operatorname{div}\boldsymbol{\sigma}, v) = -(f, v) \qquad \forall v \in L^2(\Omega). \tag{3.2b}$$

Notice that the Dirichlet boundary condition $u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0$ is built naturally into the weak formulation.

We will use either the pair $\Sigma_{k,\text{new}}^{\operatorname{div}\operatorname{div}} - V_{k-2}^{-1}$ or $\Sigma_{k^+}^{\operatorname{div}\operatorname{div}} - V_{k-1}^{-1}$, and unify the notation as

$$\Sigma_{k,r}^{\operatorname{div}\operatorname{div}} - V_r^{-1} := \begin{cases}\Sigma_{k,\text{new}}^{\operatorname{div}\operatorname{div}} - V_{k-2}^{-1}, & r = k-2, k \geq 3,\\ \Sigma_{k^+}^{\operatorname{div}\operatorname{div}} - V_{k-1}^{-1}, & r = k-1, k \geq 2,\\ \Sigma_{1++}^{\operatorname{div}\operatorname{div}} - V_1^{-1}, & r = k = 1.\end{cases}$$

A mixed finite element method for biharmonic equation (3.1) is to find $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}} \times V_r^{-1}$ with $r \geq 1$, s.t.

$$(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, u_h) = 0 \qquad \forall\boldsymbol{\tau} \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}}, \tag{3.3a}$$

$$(\operatorname{div}\operatorname{div}\boldsymbol{\sigma}_h, v) = -(f, v) \qquad \forall v \in V_r^{-1}. \tag{3.3b}$$

The mixed method (3.3) is well-posed due to the discrete inf-sup conditions (2.12), (2.16) and (2.20). By the standard procedure, we have the following error estimates.

Lemma 3.1. *Let $u \in H_0^2(\Omega)$ be the solution of biharmonic equation (3.1) and $\boldsymbol{\sigma} = -\nabla^2 u$. Let $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}} \times V_r^{-1}$ be the solution of the mixed method (3.3) for $r \geq 1$. Assume $u \in H^{k+3}(\Omega)$ and $f \in H^{r+1}(\Omega)$. We have*

$$\|\operatorname{div}\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 \lesssim h^{r+1}\|f\|_{r+1},$$

$$\|Q_r u - u_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim h^{k+1}\|\boldsymbol{\sigma}\|_{k+1}, \tag{3.4}$$

$$\|u - u_h\|_0 \lesssim h^{r+1}\|u\|_{r+1}. \tag{3.5}$$

Proof. By (3.3b),

$$\|\operatorname{div}\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_0 = \|f - Q_r f\|_0 \lesssim h^{r+1}\|f\|_{r+1}.$$

From (3.2) and (3.3), we have the error equation

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, Q_r u - u_h) = 0 \quad \forall \boldsymbol{\tau} \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}}. \tag{3.6}$$

Taking $\boldsymbol{\tau} = \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I$ and noticing $\operatorname{div}\operatorname{div}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I) = 0$, we obtain the partial orthogonality $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_I) = 0$ and thus

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_0 \lesssim h^{k+1}\|\boldsymbol{\sigma}\|_{k+1}.$$

By the inf-sup condition, we can find $\boldsymbol{\tau} \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}}$ s.t. $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = Q_r u - u_h$ and obtain the estimate for $\|Q_r u - u_h\|_0$ by the Cauchy-Schwarz inequality.

Estimate (3.5) can be obtained by the triangle inequality and standard error estimate of the L^2 projection $\|u - Q_r u\|_0$. □

Observing that when the parameter r satisfies $r = k-1$ or $r = k-2$, the error estimate (3.4) exhibits one or two orders of convergence higher than that of (3.5). It is expected that a refined interior approximation of higher accuracy than u_h can be obtained via post-processing techniques.

3.2. **Post-processing.** Following the post-processing in [10] rather than those in [19, 36], we will construct a new superconvergent approximation to deflection u by using the optimal estimate of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0$ and the superconvergent estimate of $\|Q_r u - u_h\|_0$ in (3.4).

Define a new approximation $u_h^* \in V_{k+2}^{-1}$ to u elementwisely as a solution of the following problem: for any $T \in \mathcal{T}_h$,

$$(\nabla^2 u_h^*, \nabla^2 v)_T = -(\boldsymbol{\sigma}_h, \nabla^2 v)_T \quad \forall v \in \mathbb{P}_{k+2}(T), \tag{3.7a}$$

$$(u_h^*, v)_T = (u_h, v)_T \qquad \forall v \in \mathbb{P}_1(T). \tag{3.7b}$$

Theorem 3.2. *Let $u \in H_0^2(\Omega)$ be the solution of biharmonic equation (3.1) and $\boldsymbol{\sigma} = -\nabla^2 u$. Let $u_h^* \in V_{k+2}^{-1}$ be the solution of (3.7) for $r \geq 1$. Assume $u \in H^{k+3}(\Omega)$. We have*

$$\|u - u_h^*\|_0 + \|\nabla_h^2(u - u_h^*)\|_0 \lesssim h^{k+1}|u|_{k+3}.$$

Proof. For simplicity, let $z \in V_{k+2}^{-1}(\mathcal{T}_h)$ be defined by $z|_T = (I - Q_{1,T})(Q_{k+2,T}u - u_h^*)$. Since $Q_{1,T}z = 0$, we have

$$\|z\|_{0,T} \eqsim h_T|z|_{1,T} \eqsim h_T^2|z|_{2,T}. \tag{3.8}$$

Take $v = z|_T$ in (3.7a) to obtain

$$(\nabla^2(u - u_h^*), \nabla^2 z)_T = -(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla^2 z)_T.$$

Noting the definition of z, we have

$$|z|_{2,T}^2 = (\nabla^2(Q_{k+2,T}u - u_h^*), \nabla^2 z)_T = (\nabla^2(Q_{k+2,T}u - u), \nabla^2 z)_T - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla^2 z)_T,$$

which implies

$$|Q_{k+2,T}u - u_h^*|_{2,T} = |z|_{2,T} \lesssim |u - Q_{k+2,T}u|_{2,T} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}. \tag{3.9}$$

Hence $|u - u_h^*|_{2,h} \lesssim h^{k+1}|u|_{k+3}$ follows from the triangle inequality, the estimate of $Q_{k+2,T}$, and error estimate (3.4).

On the other side, by (3.7b), we have

$$\|Q_{1,T}(Q_{k+2,T}u - u_h^*)\|_{0,T} = \|Q_{1,T}(Q_{r,T}u - u_h)\|_{0,T} \leq \|Q_{r,T}u - u_h\|_{0,T},$$

which together with (3.8) yields

$$\begin{aligned}\|Q_{k+2,T}u - u_h^*\|_{0,T} &\leq \|Q_{1,T}(Q_{k+2,T}u - u_h^*)\|_{0,T} + \|z\|_{0,T}\\ &\lesssim \|Q_{r,T}u - u_h\|_{0,T} + h_T^2|z|_{2,T}.\end{aligned}$$

By the triangle inequality and (3.9),

$$\begin{aligned}\|u - u_h^*\|_{0,T} \lesssim\ & \|u - Q_{k+2,T}u\|_{0,T} + \|Q_{r,T}u - u_h\|_{0,T}\\ &+ h_T^2(|u - Q_{k+2,T}u|_{2,T} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}).\end{aligned} \tag{3.10}$$

Hence, $\|u - u_h^*\|_0 \lesssim h^{k+1}|u|_{k+3}$ follows from the estimate of $Q_{k+2,T}$ and (3.4). □

3.3. **Duality argument.** To further enhance the convergence rate of $\|Q_r u - u_h\|_0$ and achieve a superconvergent L^2-error estimate for the post-processed approximation, we employ a duality argument. Consider the biharmonic equation

$$\begin{cases}\Delta^2 \widetilde{u} = Q_r u - u_h & \text{in } \Omega,\\ \widetilde{u}|_{\partial\Omega} = \partial_n \widetilde{u}|_{\partial\Omega} = 0.\end{cases}$$

Let $\widetilde{\boldsymbol{\sigma}} := -\nabla^2 \widetilde{u}$. We assume that $\widetilde{u} \in H^4(\Omega) \cap H_0^2(\Omega)$ and the bound

$$\|\widetilde{\boldsymbol{\sigma}}\|_2 + \|\widetilde{u}\|_4 \lesssim \|Q_r u - u_h\|_0. \tag{3.11}$$

In two dimensions, when Ω is a bounded polygonal domain with all the inner angles smaller than 126.383696°, the regularity estimate (3.11) holds [6, Theorem 2].

Theorem 3.3. *Let $u \in H_0^2(\Omega)$ be the solution of biharmonic equation* (3.1) *and $\boldsymbol{\sigma} = -\nabla^2 u$. Let $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{k,r}^{\operatorname{div}\operatorname{div}} \times V_r^{-1}$ be the solution of the mixed method* (3.3) *for $r \geq 1$. Let u_h^* be obtained by the post-processing* (3.7) *using $\boldsymbol{\sigma}_h$ and u_h. Assume $u \in H^{k+3}(\Omega)$, $f \in H^{r+1}(\Omega)$ and the regularity estimate* (3.11) *holds. We have*

$$\|Q_r u - u_h\|_0 + \|u - u_h^*\|_0 \lesssim h^{k+3}\|u\|_{k+3} + h^{\min\{2r+2, r+5\}}\|f\|_{r+1}.$$

Proof. Set $v = Q_r u - u_h$ for simplicity. By (2.11), (3.6) and integration by parts,

$$\begin{aligned}\|Q_r u - u_h\|_0^2 &= -(\operatorname{div}\operatorname{div}\widetilde{\boldsymbol{\sigma}}, v) = -(\operatorname{div}\operatorname{div}\widetilde{\boldsymbol{\sigma}}_I, v) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}_I)\\ &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}_I - \widetilde{\boldsymbol{\sigma}}) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla^2 \widetilde{u})\\ &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}_I - \widetilde{\boldsymbol{\sigma}}) - (\operatorname{div}\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \widetilde{u})\\ &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}_I - \widetilde{\boldsymbol{\sigma}}) + (f - Q_r f, \widetilde{u} - Q_r \widetilde{u}).\end{aligned}$$

Apply the Cauchy-Schwarz inequality and interpolation error estimate to get

$$\|Q_r u - u_h\|_0^2 \lesssim h^2\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0|\widetilde{\boldsymbol{\sigma}}|_2 + h^{\min\{r+1,4\}}\|f - Q_r f\|_0\|\widetilde{u}\|_4.$$

Thus the bound on $\|Q_r u - u_h\|_0$ follows from the regularity estimate (3.11), and the bound on $\|u - u_h^*\|_0$ follows from (3.10) and (3.4). □

4. Hybridization

This section will discuss the hybridization of the mixed finite element method (3.3). Spaces of Lagrange multipliers are introduced to relax the continuity of $\operatorname{tr}_1(\boldsymbol{\tau})$, $\operatorname{tr}_2(\boldsymbol{\tau})$, and the patch constraint imposed on edge jumps. Weak divdiv stability will be proved. Equivalent weak Galerkin and non-conforming virtual element methods formulation will also be provided, as well as a C^0 discontinuous Galerkin (CDG) method.

4.1. **Broken spaces and weak differential operators.** For $k \geq 0$, define

$$\Sigma_{k,r}^{-1} := \prod_{T\in\mathcal{T}_h} \Sigma_{k,r}(T;\mathbb{S})$$

with

$$\Sigma_{k,r}(T;\mathbb{S}) := \begin{cases} \mathbb{P}_k(T;\mathbb{S}), & r = k-2, \\ \mathbb{P}_k(T;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(T), & r = k-1. \end{cases}$$

We also write $\Sigma_k^{-1} = \Sigma_{k,k-2}^{-1}$ and $\Sigma_{k^+}^{-1} = \Sigma_{k,k-1}^{-1}$ for $k \geq 0$ when r is not emphasized. The case $\Sigma_{1^{++}}^{-1}$ is defined by the enriched local space (2.17) and is not included in this notation system. Define the discontinuous polynomial spaces

$$V_r^{-1}(\mathcal{F}_h) := \prod_{F\in\mathcal{F}_h} \mathbb{P}_r(F), \quad V_r^{-1}(\mathcal{E}_h) := \prod_{e\in\mathcal{E}_h} \mathbb{P}_r(e).$$

Spaces for the scalar function are: for $r = k-2$ or $k-1$

$$M_{r,k-1,k,k}^{-1} = V_r^{-1}(\mathcal{T}_h) \times V_{k-1}^{-1}(\mathcal{F}_h) \times V_k^{-1}(\mathcal{F}_h) \times V_k^{-1}(\mathcal{E}_h),$$
$$\mathring{M}_{r,k-1,k,k}^{-1} = V_r^{-1}(\mathcal{T}_h) \times V_{k-1}^{-1}(\mathring{\mathcal{F}}_h) \times V_k^{-1}(\mathring{\mathcal{F}}_h) \times V_k^{-1}(\mathring{\mathcal{E}}_h).$$

When the index is less than zero, we use $\cdot$ to de-emphasize it. For example, when $k = 1$, $r = k-2 = -1$, the space is denoted by $M_{\cdot,0,1,1}^{-1}$; for $k = 0$, it is $M_{\cdot,\cdot,0,0}^{-1}$. Spaces on $\mathring{\mathcal{F}}_h$ and $\mathring{\mathcal{E}}_h$ can be thought of as Lagrange multiplier for the required continuity. For example, $V_{k-1}^{-1}(\mathring{\mathcal{F}}_h)$ is for $\operatorname{tr}_2(\boldsymbol{\sigma})$ which is one degree lower than that of $\boldsymbol{\sigma}$ as $\operatorname{tr}_2(\boldsymbol{\sigma})$ consists of first-order derivatives of $\boldsymbol{\sigma}$. Space $\mathring{M}_{r,k-1,k,k}^{-1}$ can be treated as a subspace of $M_{r,k-1,k,k}^{-1}$ by zero extension to boundary faces and edges. A function $v \in M_{r,k-1,k,k}^{-1}$ can be written as $v = (v_0, v_b, v_n, v_e)$, where v_0 represents function value in the interior, v_b on faces, v_e on edges, and v_n for the normal derivative on faces.

Introduce the inner products $(\cdot,\cdot)_{0,h}$ with weight:

$$((u_0,u_b,u_n,u_e),(v_0,v_b,v_n,v_e))_{0,h} = \sum_{T\in\mathcal{T}_h}(u_0,v_0)_T + \sum_{F\in\mathcal{F}_h} h_F(u_b,v_b)_F$$
$$+ \sum_{F\in\mathcal{F}_h} h_F^3(u_n,v_n)_F + \sum_{e\in\mathcal{E}_h} h_e^2(u_e,v_e)_e.$$

The induced norm is denoted by $\|\cdot\|_{0,h}$. Different scalings are introduced such that all terms have the same scaling as the L^2-inner product (u_0,v_0).

We will use either the pair $\Sigma_{k,k-2}^{-1} - \mathring{M}_{k-2,k-1,k,k}^{-1}$ or $\Sigma_{k,k-1}^{-1} - \mathring{M}_{k-1,k-1,k,k}^{-1}$, and unify the notation as $\Sigma_{k,r}^{-1} - \mathring{M}_{r,k-1,k,k}^{-1}$.

Define weak divdiv operator $(\operatorname{div}\operatorname{div})_w : \Sigma_{k,r}^{-1} \to \mathring{M}_{r,k-1,k,k}^{-1}$ as

$$(\operatorname{div}\operatorname{div})_w\boldsymbol{\sigma} := ((\operatorname{div}\operatorname{div})_T\boldsymbol{\sigma}, -h_F^{-1}[\operatorname{tr}_2(\boldsymbol{\sigma})]|_F, h_F^{-3}[\boldsymbol{n}^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}]|_F, h_e^{-2}[\operatorname{tr}_e(\boldsymbol{\sigma})]|_e),$$

and extend to $\overline{(\operatorname{div}\operatorname{div})}_w : \Sigma_{k,r}^{-1} \to M_{r,k-1,k,k}^{-1}$ by including boundary faces and edges. The negative power scaling is introduced to match the scaling of the second order derivative $(\operatorname{div}\operatorname{div})_T\boldsymbol{\sigma}$. When $\boldsymbol{\sigma} \in \Sigma_{k,r}^{-1} \cap H(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$, $(\operatorname{div}\operatorname{div})_w\boldsymbol{\sigma} = (\operatorname{div}\operatorname{div})\boldsymbol{\sigma}$ but $\overline{(\operatorname{div}\operatorname{div})}_w\boldsymbol{\sigma} \neq \operatorname{div}\operatorname{div}\boldsymbol{\sigma}$ pointwisely as terms on the boundary faces and edges are included. However, $\overline{(\operatorname{div}\operatorname{div})}_w\boldsymbol{\sigma} = (\operatorname{div}\operatorname{div})_w\boldsymbol{\sigma} = \operatorname{div}\operatorname{div}\boldsymbol{\sigma}$ in the distribution sense as the test function vanishes on the boundary.

For $v = (v_0, v_b, v_n, v_e) \in M_{r,k-1,k,k}^{-1}$, $k \geq 0$, define weak Hessian $\nabla_w^2 v \in \Sigma_{k,r}^{-1}$ s.t. for all $\boldsymbol{\sigma} \in \Sigma_{k,r}(T; \mathbb{S})$ and $T \in \mathcal{T}_h$:

$$\begin{aligned}(\nabla_w^2 v, \boldsymbol{\sigma})_T :=& (v_0, \operatorname{div}\operatorname{div}_h \boldsymbol{\sigma})_T \\ & - (v_b, \operatorname{tr}_2(\boldsymbol{\sigma}))_{\partial T} + (v_n \boldsymbol{n}_F \cdot \boldsymbol{n}, \boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n})_{\partial T} + \sum_{e \in \Delta_{d-2}(T)} (v_e, \operatorname{tr}_e(\boldsymbol{\sigma}))_e.\end{aligned} \tag{4.1}$$

Using integration by parts, we also have an equivalent formula on $\nabla_w^2 v$

$$\begin{aligned}(\nabla_w^2 v, \boldsymbol{\sigma})_T =& (\nabla_h^2 v_0, \boldsymbol{\sigma})_T + (v_0 - v_b, \operatorname{tr}_2(\boldsymbol{\sigma}))_{\partial T} - (\partial_n v_0 - v_n \boldsymbol{n}_F \cdot \boldsymbol{n}, \boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n})_{\partial T} \\ & + \sum_{e \in \Delta_{d-2}(T)} (v_e - v_0, \operatorname{tr}_e(\boldsymbol{\sigma}))_e.\end{aligned} \tag{4.2}$$

For piecewise smooth $v \in H^2(\Omega)$, define $Q_M v \in M_{r,k-1,k,k}^{-1}$ by local L^2-projection

$$Q_M v = (Q_{r,T} v, Q_{k-1,F} v, Q_{k,F} \partial_{n_F} v, Q_{k,e} v)_{T \in \mathcal{T}_h, F \in \mathcal{F}_h, e \in \mathcal{E}_h},$$

then by definition

$$\nabla_w^2 Q_M v = Q_\Sigma \nabla^2 v, \tag{4.3}$$

where Q_Σ is the L^2-projection to the space $\Sigma_{k,r}^{-1}$.

By definition, we have the following formulae on the integration by parts.

Lemma 4.1. *We have the integration by parts*

$$\begin{aligned}((\operatorname{div}\operatorname{div})_w \boldsymbol{\sigma}, v)_{0,h} &= (\boldsymbol{\sigma}, \nabla_w^2 v), \quad \boldsymbol{\sigma} \in \Sigma_{k,r}^{-1}, v \in \mathring{M}_{r,k-1,k,k}^{-1}, \\ (\overline{(\operatorname{div}\operatorname{div})}_w \boldsymbol{\sigma}, v)_{0,h} &= (\boldsymbol{\sigma}, \nabla_w^2 v), \quad \boldsymbol{\sigma} \in \Sigma_{k,r}^{-1}, v \in M_{r,k-1,k,k}^{-1}.\end{aligned}$$

As a consequence, for $\boldsymbol{\sigma} \in \Sigma_{k,r}^{-1}, v \in C_0^\infty(\Omega)$, we have

$$((\operatorname{div}\operatorname{div})_w \boldsymbol{\sigma}, Q_M v)_{0,h} = (\boldsymbol{\sigma}, \nabla^2 v) = \langle \operatorname{div}\operatorname{div}\boldsymbol{\sigma}, v\rangle,$$

where the last $\langle \cdot, \cdot \rangle$ is the duality pair. Namely $(\operatorname{div}\operatorname{div})_w$ can be viewed as a discretization of $\operatorname{div}\operatorname{div}$ operator in the distributional sense.

4.2. **Weak divdiv stability.** Introduce the norm square

$$\|\boldsymbol{\tau}\|_{\operatorname{div}\operatorname{div}_w}^2 := \|\boldsymbol{\tau}\|_0^2 + \|(\operatorname{div}\operatorname{div})_w \boldsymbol{\tau}\|_{0,h}^2,$$

and

$$\|\boldsymbol{\tau}\|_{\overline{\operatorname{div}\operatorname{div}_w}}^2 := \|\boldsymbol{\tau}\|_0^2 + \|\overline{(\operatorname{div}\operatorname{div})}_w \boldsymbol{\tau}\|_{0,h}^2.$$

Theorem 4.2. *We have the inf-sup condition: there exist constants* α *and* $\bar{\alpha}$ *independent of* h *s.t.*

$$\inf_{v\in \mathring{M}^{-1}_{r,k-1,k,k}} \sup_{\boldsymbol{\tau}\in\Sigma^{-1}_{k,r}} \frac{((\operatorname{div}\operatorname{div})_w\boldsymbol{\tau}, v)_{0,h}}{\|\boldsymbol{\tau}\|_{\operatorname{div}\operatorname{div}_w}\|v\|_{0,h}} = \alpha > 0, \quad \text{for } k\geq 0, \tag{4.4}$$

$$\inf_{v\in M^{-1}_{r,k-1,k,k}/\mathbb{P}_1} \sup_{\boldsymbol{\tau}\in\Sigma^{-1}_{k,r}} \frac{(\overline{(\operatorname{div}\operatorname{div})}_w\boldsymbol{\tau}, v)_{0,h}}{\|\boldsymbol{\tau}\|_{\overline{\operatorname{div}\operatorname{div}}_w}\|v\|_{0,h}} = \bar{\alpha} > 0, \quad \text{for } k\geq 0. \tag{4.5}$$

Proof. The proof of (4.4) and (4.5) is similar. We will prove (4.4) for $r = k-2$ which also works for $r = k-1$ with appropriate change of DoFs to define $\Sigma^{\operatorname{div}\operatorname{div}}_{k^+}$ rather than $\Sigma^{\operatorname{div}\operatorname{div}}_{k,\mathrm{new}}$.

Step 1. We first consider the case $r\geq 1$ for which an $H(\operatorname{div}\operatorname{div})$-conforming finite element either $\Sigma^{\operatorname{div}\operatorname{div}}_{k}, k\geq 3$, or $\Sigma^{\operatorname{div}\operatorname{div}}_{k^+}, k\geq 2$, has been constructed.

For $e\in\mathring{\mathcal{E}}_h$, let $|\omega_e|$ be the number of elements in ω_e. First consider a tensor $\boldsymbol{\tau}_b\in\Sigma^{-1}_k$ with DoFs

$$\operatorname{tr}_2(\boldsymbol{\tau}_b)|_F = -\frac{1}{2}h_F v_b, \quad \boldsymbol{n}^{\intercal}\boldsymbol{\tau}_b\boldsymbol{n}|_F = \frac{1}{2}h_F^3 v_n(\boldsymbol{n}_F\cdot\boldsymbol{n}) \quad \text{on } F\in\partial T,$$
$$(\boldsymbol{n}^{\intercal}_{F_1,e}\boldsymbol{\tau}_b\boldsymbol{n}_{F_1} + \boldsymbol{n}^{\intercal}_{F_2,e}\boldsymbol{\tau}_b\boldsymbol{n}_{F_2})|_e = \frac{1}{|\omega_e|}h_e^2 v_e \quad \text{on } e\in\Delta_{d-2}(T)$$

for each $T\in\mathcal{T}_h$, and others in (2.3) vanish. Consequently,

$$(\operatorname{div}\operatorname{div})_w\boldsymbol{\tau}_b = ((\operatorname{div}\operatorname{div})_T\boldsymbol{\tau}_b, v_b, v_n, v_e)_{T\in\mathcal{T}_h}, \text{ and } \|\boldsymbol{\tau}_b\|_{\operatorname{div}\operatorname{div}_w}\lesssim\|v\|_{0,h}.$$

Then by the inf-sup condition (2.12), we can find $\boldsymbol{\tau}_0\in\Sigma^{\operatorname{div}\operatorname{div}}_{k,\mathrm{new}}$ s.t. $\operatorname{div}\operatorname{div}\boldsymbol{\tau}_0 = v_0 - (\operatorname{div}\operatorname{div})_h\boldsymbol{\tau}_b$, and $\|\boldsymbol{\tau}_0\|_{\operatorname{div}\operatorname{div}}\lesssim\|v_0\|_0 + \|(\operatorname{div}\operatorname{div})_h\boldsymbol{\tau}_b\|_0\lesssim\|v\|_{0,h}$.

Set $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_b$. We have $(\operatorname{div}\operatorname{div})_w\boldsymbol{\tau} = v$ and $\|\boldsymbol{\tau}\|_{\operatorname{div}\operatorname{div}_w}\lesssim\|v\|_{0,h}$, which verifies the inf-sup condition (4.4).

The pair $\Sigma^{-1}_{1^{++}} - \mathring{M}^{-1}_{1,1,1,1}$ can be proved similarly as an $H(\operatorname{div}\operatorname{div})$-conforming finite element space $\Sigma^{\operatorname{div}\operatorname{div}}_{1^{++}}$ can be constructed. However, for Σ^{-1}_k, $k = 1,2$, and $\Sigma^{-1}_{1^+}$, no finite elements have been constructed and will be treated differently.

Step 2. Consider $k = 2$. Given $v\in\mathring{M}^{-1}_{0,1,2,2}\subset\mathring{M}^{-1}_{1,1,2,2}$, by the established inf-sup condition for $\Sigma^{-1}_{2^+} - \mathring{M}^{-1}_{1,1,2,2}$, we can find $\boldsymbol{\tau}\in\Sigma^{-1}_{2^+}$ s.t. $\operatorname{div}\operatorname{div}_w\boldsymbol{\tau} = v$. We claim $\boldsymbol{\tau}\in\Sigma^{-1}_2$ as $\operatorname{div}\operatorname{div}_T\boldsymbol{\tau}\in\mathbb{P}_0(T)$ and the range of the enrichment $\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_1(T)) = \mathbb{H}_1(T)$. This finishes the weak divdiv stability for $\operatorname{div}\operatorname{div}_w:\Sigma^{-1}_2\to\mathring{M}^{-1}_{0,1,2,2}$.

Step 3. Consider $k = 1$. Given $v\in\mathring{M}^{-1}_{\cdot,0,1,1}\subset\mathring{M}^{-1}_{1,1,1,1}$, by the established inf-sup condition for $\Sigma^{-1}_{1^{++}} - \mathring{M}^{-1}_{1,1,1,1}$, we can find $\boldsymbol{\tau}\in\Sigma^{-1}_{1^{++}}$ s.t. $\operatorname{div}\operatorname{div}_w\boldsymbol{\tau} = v$. As $v_0 = 0$, we conclude $\operatorname{div}\operatorname{div}_h\boldsymbol{\tau} = 0$. Consequently $\boldsymbol{\tau}\in\mathbb{P}_1(T;\mathbb{S}) + \operatorname{sym}(\boldsymbol{x}\otimes\mathbb{P}_1(T;\mathbb{R}^d))$. By the proof of Lemma 2.11, we can derive $\boldsymbol{\tau}\in\mathbb{P}_1(T;\mathbb{S})$ from the fact $\operatorname{tr}_2(\boldsymbol{\tau})\in\mathbb{P}_0(F)$. Namely we obtain the stability for the pair $\Sigma^{-1}_1 - \mathring{M}^{-1}_{\cdot,0,1,1}$. Then by adding $\boldsymbol{x}^{\intercal}\boldsymbol{x}\mathbb{P}_0(T)$ element-wise, we obtain the stability for $\Sigma^{-1}_{1^+} - \mathring{M}^{-1}_{0,0,1,1}$. This finishes all $k = 1$ cases.

Step 4. Consider $k = 0$. We shall use the non-conforming finite element space as the bridge. The space $\mathring{M}^{-1}_{\cdot,\cdot,0,0}$ can be identified as the Morley-Wang-Xu (MWX) element $\mathring{V}^{\mathrm{MWX}}_2$ [37] through the bijection $Q_M:\mathring{V}^{\mathrm{MWX}}_2\to\mathring{M}^{-1}_{\cdot,\cdot,0,0}$. Similar as (4.3), it holds $\nabla_w^2 Q_M \chi = Q_\Sigma \nabla_h^2 \chi$ for $\chi \in \mathring{V}_2^{\mathrm{MWX}}$. Given $v \in \mathring{M}^{-1}_{\cdot,\cdot,0,0}$, let $w_h \in \mathring{V}_2^{\mathrm{MWX}}$ satisfy

$$(\nabla_h^2 w_h, \nabla_h^2 \chi) = (v, Q_M \chi)_{0,h}, \quad \chi \in \mathring{V}_2^{\mathrm{MWX}}.$$

Take $\boldsymbol{\tau} = \nabla_h^2 w_h \in \Sigma_0^{-1}$, then $\operatorname{div}\operatorname{div}_w \boldsymbol{\tau} = v$, and

$$\|\boldsymbol{\tau}\|_0^2 = (\boldsymbol{\tau}, \nabla_w^2 Q_M w_h) = (v, Q_M w_h)_{0,h} \le \|v\|_{0,h} \|Q_M w_h\|_{0,h}.$$

By the norm equivalence $\|Q_M w_h\|_{0,h} \eqsim \|w_h\|_0$ of the MWX element and the Poincaré inequality $\|w_h\|_0 \lesssim \|\nabla_h^2 w_h\|_0$ [37, Lemma 8], we have $\|\boldsymbol{\tau}\|_0 \lesssim \|v\|_{0,h}$, which means $\|\boldsymbol{\tau}\|_{\operatorname{div}\operatorname{div}_w} \lesssim \|v\|_{0,h}$. Thus the inf-sup condition (4.4) holds for $k = 0$. $\square$

As the adjoint of the $\operatorname{div}\operatorname{div}_w$, ∇_w^2 is injective. We obtain another version of the inf-sup condition.

Corollary 4.3. *We have*

$$\inf_{v \in \mathring{M}^{-1}_{r,k-1,k,k}} \sup_{\boldsymbol{\tau} \in \Sigma^{-1}_{k,r}} \frac{(\operatorname{div}\operatorname{div}_w \boldsymbol{\tau}, v)_{0,h}}{\|\boldsymbol{\tau}\|_0 \|\nabla_w^2 v\|_0} = 1, \quad \text{for } k \ge 0. \tag{4.6}$$

Proof. We can take $\boldsymbol{\tau} = \nabla_w^2 v$ to finish the proof as $\nabla_w^2 : \mathring{M}^{-1}_{r,k-1,k,k} \to \Sigma^{-1}_{k,r}$ is injective and $\|\nabla_w^2 \cdot\|_0$ is a norm on $\mathring{M}^{-1}_{r,k-1,k,k}$. $\square$

4.3. **Hybridized discretization of the biharmonic equation.** A hybridization of the mixed finite element discretization (3.3) of the biharmonic equation is: Find $\boldsymbol{\sigma}_h \in \Sigma^{-1}_{k,r}$ and $u_h \in \mathring{M}^{-1}_{r,k-1,k,k}$ s.t.

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div}\operatorname{div}_w \boldsymbol{\tau}, u_h)_{0,h} = 0 \qquad \forall \boldsymbol{\tau} \in \Sigma^{-1}_{k,r}, \tag{4.7a}$$

$$(\operatorname{div}\operatorname{div}_w \boldsymbol{\sigma}_h, v)_{0,h} = -(f, v_0) \qquad \forall v \in \mathring{M}^{-1}_{r,k-1,k,k}, \tag{4.7b}$$

with appropriate modification of (f, v_0) for the case $r \le 0$ which will be discussed later.

More generally, for a given function $f_h = (f_0, f_b, f_n, f_e) \in \mathring{M}^{-1}_{r,k-1,k,k}$, we consider the mixed variational problem

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div}\operatorname{div}_w \boldsymbol{\tau}, u_h)_{0,h} = 0 \qquad \forall \boldsymbol{\tau} \in \Sigma^{-1}_{k,r}, \tag{4.8a}$$

$$(\operatorname{div}\operatorname{div}_w \boldsymbol{\sigma}_h, v)_{0,h} = (f_h, v)_{0,h} \qquad \forall v \in \mathring{M}^{-1}_{r,k-1,k,k}. \tag{4.8b}$$

The biharmonic equation is a special case with $f_h = (-Q_r f, 0, 0, 0)$.

Lemma 4.4. *The hybridized mixed finite element method* (4.8) *has a unique solution $\boldsymbol{\sigma}_h \in \Sigma^{-1}_{k,r}$ and $u_h = ((u_h)_0, (u_h)_b, (u_h)_n, (u_h)_e) \in \mathring{M}^{-1}_{r,k-1,k,k}$ for $k \ge 0$, and*

$$\|\boldsymbol{\sigma}_h\|_{\operatorname{div}\operatorname{div}_w} + \|u_h\|_{0,h} \lesssim \|f_h\|_{0,h}. \tag{4.9}$$

Moreover, when $r \ge 1$, the solution $(\boldsymbol{\sigma}_h, (u_h)_0) \in \Sigma^{\operatorname{div}\operatorname{div}}_{k,r} \times V_r^{-1}$ to (4.7) *is the solution of the mixed finite element method* (3.3).

Proof. The discrete method (4.8) is well-posed thanks to the weak divdiv stability (4.4). The stability (4.9) is from the Babuska-Brezzi theory.

For the biharmonic equation (4.7), $f_h = (-Q_r f, 0, 0, 0)$. Therefore $\boldsymbol{\sigma}_h \in \Sigma^{\operatorname{div}\operatorname{div}}_{k,r}$ and $\operatorname{div}\operatorname{div} \boldsymbol{\sigma}_h = Q_r f$. By restricting $\boldsymbol{\tau} \in \Sigma^{\operatorname{div}\operatorname{div}}_{k,r}$ in (4.7a), we conclude $(\boldsymbol{\sigma}_h, (u_h)_0)$ is the solution to (3.3). $\square$

Notice that the mixed formulation (3.3) is only presented for $r \geq 1, k \geq 2$, where $H(\operatorname{div}\operatorname{div})$-conforming finite elements are constructed. While the hybridized version is well-posed for all $k \geq 0$.

Using the stability result (4.9), we can prove the following discrete Poincaré inequality.

Lemma 4.5. *On the space* $\mathring{M}^{-1}_{r,k-1,k,k}$*, we have*

$$\|u\|_{0,h} \lesssim \|\nabla_w^2 u\|_0, \quad u \in \mathring{M}^{-1}_{r,k-1,k,k} \quad \text{for } k \geq 0. \tag{4.10}$$

Proof. For $f_h = u$ in (4.8), we can find $\boldsymbol{\sigma} \in \Sigma^{-1}_{k,r}$ s.t. $\operatorname{div}\operatorname{div}_w \boldsymbol{\sigma} = u$ and $\|\boldsymbol{\sigma}\|_0 \lesssim \|u\|_{0,h}$. Set $v = u$ in (4.8b), we obtain

$$\|u\|_{0,h}^2 = (\operatorname{div}\operatorname{div}_w \boldsymbol{\sigma}, u)_{0,h} = (\boldsymbol{\sigma}, \nabla_w^2 u) \leq \|\boldsymbol{\sigma}\|_0 \|\nabla_w^2 u\|_0 \lesssim \|u\|_{0,h}\|\nabla_w^2 u\|_0,$$

which implies the desired inequality. □

We now present error analysis of scheme (4.7) for $r \geq 1$ which is equivalent to the mixed finite element method (3.3). Thus we focus on the error estimate of u_h.

Theorem 4.6. *Let* $u \in H_0^2(\Omega)$ *be the solution of biharmonic equation* (3.1) *and* $\boldsymbol{\sigma} = -\nabla^2 u$*. Let* $\boldsymbol{\sigma}_h \in \Sigma^{-1}_{k,r}, u_h \in \mathring{M}^{-1}_{r,k-1,k,k}$ *be the solution of the discrete method* (4.7) *for* $r \geq 1$ *and* $k \geq 2$*. Assume* $u \in H^{k+3}(\Omega)$*. We have*

$$\|\nabla_w^2(Q_M u - u_h)\|_0 + \|Q_M u - u_h\|_{0,h} \lesssim h^{k+1}|u|_{k+3}.$$

Proof. In (4.7), as $\boldsymbol{\sigma}_h$ is discontinuous, we can eliminate $\boldsymbol{\sigma}_h$ elementwisely and use the weak Hessian to obtain an equivalent formulation: find $u_h \in \mathring{M}^{-1}_{r,k-1,k,k}$, s.t.

$$(\nabla_w^2 u_h, \nabla_w^2 v) = (f, v_0) \quad \forall v \in \mathring{M}^{-1}_{r,k-1,k,k}.$$

For $r \geq 1$, we have the canonical interpolation $\boldsymbol{\sigma}_I \in \Sigma_{k,r}$ satisfying

$$(\boldsymbol{\sigma}_I, \nabla_w^2 v) = (\operatorname{div}\operatorname{div}_w \boldsymbol{\sigma}_I, v)_{0,h} = (\operatorname{div}\operatorname{div} \boldsymbol{\sigma}_I, v_0) = (Q_r \operatorname{div}\operatorname{div} \boldsymbol{\sigma}, v_0) = -(f, v_0).$$

On the other hand, we have the property $\nabla_w^2 Q_M u = Q_\Sigma \nabla^2 u = -Q_\Sigma \boldsymbol{\sigma}$.

Let $v = Q_M u - u_h$. We then have

$$\|\nabla_w^2(Q_M u - u_h)\|_0^2 = -(Q_\Sigma \boldsymbol{\sigma}, \nabla_w^2 v) - (f, v_0) = (\boldsymbol{\sigma}_I - Q_\Sigma \boldsymbol{\sigma}, \nabla_w^2 v).$$

Then the error estimate on $\|\nabla_w^2(Q_M u - u_h)\|_0$ follows from Cauch-Schwarz inequality, triangle inequality, and the estimate of $\|\boldsymbol{\sigma} - Q_\Sigma \boldsymbol{\sigma}\|$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|$.

Estimate on $\|Q_M u - u_h\|_{0,h}$ is a consequence of the Poincaré inequality (4.10). □

Note that Theorem 4.6 covers only the case $r \geq 1, k \geq 2$. We now give corrections to low order cases: $k = 0, 1, 2$ and $r \leq 0$.

For $k = 1, 2$, we define $v^{\mathrm{CR}} \in \mathbb{P}_1(T)$ by $Q_{0,F} v^{\mathrm{CR}} = Q_{0,F} v_b$ for $F \in \partial T$. The load term (f, v_0) is replaced by (f, v^{CR}) for $k = 1$ and by $(f, v^{\mathrm{CR}} + v_0 - Q_0 v^{\mathrm{CR}})$ for $k = 2, r = 0$.

For $k = 0$, $v = (v_n, v_e) \in M^{-1}_{\cdot,\cdot,0,0}$, we define $v_0 = Q_M^{-1} v \in \mathring{V}_2^{\mathrm{MWX}}$ and $v_b = Q_M^{-1} v$ on ∂T. With this v_b, we can define v^{CR}. From this point of view, (4.7) generalizes the well-known $\mathbb{P}_2$ Morley element to higher order and arbitrary dimensions.

We can write $(f, v_0 + (I - Q_r)v^{\mathrm{CR}})$ for all $k \geq 0$ cases. We will present the error analysis after we identify (4.7) with the non-conforming virtual element methods (VEM).

4.4. **Equivalence to other methods.** In (4.7), as $\boldsymbol{\sigma}_h$ is discontinuous, we can eliminate $\boldsymbol{\sigma}_h$ elementwisely and use the weak Hessian to obtain a weak Galerkin formulation: find $u_h \in \mathring{M}^{-1}_{r,k-1,k,k}$, s.t.

$$(\nabla_w^2 u_h, \nabla_w^2 v) = (f, v_0 + (I - Q_r)v^{\mathrm{CR}}) \quad \forall v \in \mathring{M}^{-1}_{r,k-1,k,k}, \quad k \geq 0. \tag{4.11}$$

The discrete method (4.11) is well-posed, since $\|\nabla_w^2(\cdot)\|$ constitutes a norm on the space $\mathring{M}^{-1}_{r,k-1,k,k}$ by (4.6). Indeed (4.11) is equivalent to (4.7). Moreover, the weak divdiv stability, which is equivalent to the coercivity of the bilinear form $(\nabla_w^2\cdot, \nabla_w^2\cdot)$, obviates the need for any additional stabilization. This not only simplifies the implementation, but also facilitates the error analysis. Some weak Galerkin methods without extrinsic stabilization for the biharmonic equation are designed recently on polytopal meshes in [39, 40].

For a simplex T, recall the local space of the H^2-non-conforming virtual element introduced in [11] for $r = k-2$ or $k-1$

$$V_{k+2}^{\mathrm{VEM}}(T) := \big\{u \in H^2(T) : \mathrm{tr}_1(\nabla^2 u)|_F \in \mathbb{P}_k(F), \mathrm{tr}_2(\nabla^2 u)|_F \in \mathbb{P}_{k-1}(F),$$
$$\mathrm{tr}_e(\nabla^2 u) \in \mathbb{P}_k(e) \forall F \in \partial T, e \in \Delta_{d-2}(T), \Delta^2 u \in \mathbb{P}_r(T)\big\}.$$

Define the global virtual element space

$$\mathring{V}_{k+2}^{\mathrm{VEM}} := \big\{u \in L^2(\Omega) : u|_T \in V_{k+2}^{\mathrm{VEM}}(T) \text{ for } T \in \mathcal{T}_h, Q_{k-1,F}u, Q_{k,F}(\partial_{n_F} u), Q_{k,e}u$$
$$\text{are single-valued for } F \in \mathring{\mathcal{F}}_h, e \in \mathring{\mathcal{E}}_h, \text{ and vanish on boundary } \partial\Omega\big\}.$$

The well-posedness of VEM space using DoFs $(Q_{r,T}u, Q_{k-1,F}u, Q_{k,F}(\partial_{n_F}u), Q_{k,e}u)$ can be found in [11]. In general a function $v \in \mathring{V}_{k+2}^{\mathrm{VEM}}$ is non-polynomial, with the only exception of $k = 0$, and thus its point-wise value may not be known. Instead several projections to polynomial spaces using DoFs will be used.

Given a function $(v_0, v_b, v_n, v_e) \in M^{-1}_{r,k-1,k,k}$, we can define an H^2 non-conforming virtual element function $v \in \mathring{V}_{k+2}^{\mathrm{VEM}}$ by $Q_M v = (v_0, v_b, v_n, v_e)$. That is $Q_M : \mathring{V}_{k+2}^{\mathrm{VEM}} \to \mathring{M}^{-1}_{r,k-1,k,k}$ is a bijection. Similar as (4.3), it holds

$$\nabla_w^2 Q_M v = Q_\Sigma \nabla_h^2 v \qquad \forall v \in \mathring{V}_{k+2}^{\mathrm{VEM}}. \tag{4.12}$$

We have a unified construction $v^{\mathrm{CR}} = I^{\mathrm{CR}} v$ where I^{CR} is the interpolation operator to the non-conforming linear element space. The face integral $\int_F v$ is a DoF when $k \geq 1$ and when $k = 0$, $\int_F v$ is computable as v is a quadratic polynomial.

Then (4.11) becomes: find $u_h \in \mathring{V}_{k+2}^{\mathrm{VEM}}$, for $k \geq 0$, s.t.

$$(Q_\Sigma \nabla_h^2 u_h, Q_\Sigma \nabla_h^2 v) = (f, v^{\mathrm{CR}} + Q_r(v - v^{\mathrm{CR}})) \quad \forall v \in \mathring{V}_{k+2}^{\mathrm{VEM}}. \tag{4.13}$$

So we obtain a stabilization free non-conforming VEM for the biharmonic equation on triangular meshes due to the weak divdiv stability.

We will use the following norm equivalence, whose proof can be found in Appendix B, and the error analysis of VEM to provide another convergence analysis of (4.7)

$$\|Q_\Sigma \nabla_h^2 v\|_0 \eqsim \|\nabla_h^2 v\|_0, \quad v \in \mathring{V}_{k+2}^{\mathrm{VEM}}, k \geq 0. \tag{4.14}$$

Theorem 4.7. *Let $u \in H_0^2(\Omega)$ be the solution of biharmonic equation* (3.1) *and $\boldsymbol{\sigma} = -\nabla^2 u$. Let $\boldsymbol{\sigma}_h \in \Sigma^{-1}_{k,r}, u_h \in \mathring{M}^{-1}_{r,k-1,k,k}$ be the solution of the discrete method*

(4.7) *for* $k \geq 0$. *Assume* $u \in H^{k+3}(\Omega)$. *We have*

$$\|\nabla_w^2(Q_M u - u_h)\|_0 + \|Q_M u - u_h\|_{0,h} \lesssim h^{k+1}(|u|_{k+3} + \delta_{k0} h \|f\|_0).$$

Proof. Due to the equivalence between (4.13) and (4.7), it is equivalent to prove

$$\|\nabla_w^2 Q_M(u - u_h)\|_0 + \|Q_M(u - u_h)\|_{0,h} \lesssim h^{k+1}(|u|_{k+3} + \delta_{k0} h\|f\|_0),$$

where $u_h \in \mathring{V}_{k+2}^{\rm VEM}$ is the solution of the virtual element method (4.13).

We outline the proof and refer to [11] for details. Notice that there is an index shift in the notation. Results in [11] are applied to $\mathring{V}_{k+2}^{\rm VEM}$ with degree $k+2$ for $k \geq 0$.

Let $I_h u$ be the nodal interpolation of u based on the DoFs of $V_{k+2}^{\rm VEM}(T)$ [11, (2.6)-(2.9)]. Then $Q_M u = Q_M(I_h u)$ and thus $Q_\Sigma \nabla_h^2(I_h u) = \nabla_w^2 Q_M(I_h u) = \nabla_w^2 Q_M u = Q_\Sigma \nabla^2 u$. Set $v = I_h u - u_h$. We have the error equation

$$\begin{aligned}\|Q_\Sigma \nabla_h^2(I_h u - u_h)\|_0^2 =& ((Q_\Sigma - I)\nabla^2 u, \nabla_h^2 v) + (\nabla^2 u, \nabla_h^2 v) - (f, v)\\ &+ (f, (I - Q_r)(v - v^{\rm CR})).\end{aligned}$$

The first term is bounded by

$$((Q_\Sigma - I)\nabla^2 u, \nabla_h^2 v) \leq \|(Q_\Sigma - I)\nabla^2 u\|_0 \|\nabla_h^2 v\|_0 \lesssim h^{k+1}|u|_{k+3}\|\nabla_h^2 v\|_0.$$

The second term is the consistence error [11, Lemma 5.5 and 5.6]

$$(\nabla^2 u, \nabla_h^2 v_h) - (f, v_h) \lesssim h^{k+1}(|u|_{k+3} + \delta_{k0} h\|f\|_0)\|\nabla_h^2 v_h\|_0.$$

The third term is a perturbation and can be bounded by

$$\begin{aligned}(f, (I - Q_r)(v - v^{\rm CR})) &= ((I - Q_r)f, v - v^{\rm CR})\\ &\lesssim h^{k+1}\big((1 - \delta_{k0})|f|_{k-1} + \delta_{k0} h\|f\|_0\big)\|\nabla_h^2 v\|_0.\end{aligned}$$

Putting together, we have

$$\begin{aligned}\|Q_\Sigma \nabla_h^2(I_h u - u_h)\|_0^2 &\lesssim h^{k+1}(|u|_{k+3} + \delta_{k0} h\|f\|_0)\|\nabla_h^2 v\|_0\\ &\lesssim h^{k+1}(|u|_{k+3} + \delta_{k0} h\|f\|_0)\|Q_\Sigma \nabla_h^2 v\|_0,\end{aligned}$$

where in the last step, we have used the norm equivalence (4.14). Canceling one $\|Q_\Sigma \nabla_h^2 v\|_0$ to get the desired error estimate. □

In view of $\boldsymbol{\sigma}_h = -Q_\Sigma \nabla_h^2 u_h$, the post-processing u_h^* defined by (3.7) is indeed the local H^2 projection of u_h to the polynomial space, i.e.,

$$(\nabla^2 u_h^*, \nabla^2 v)_T = (\nabla^2 u_h, \nabla^2 v)_T, \quad v \in \mathbb{P}_{k+2}(T), T \in \mathcal{T}_h.$$

When some partial continuity is imposed on Σ_k^{-1}, we can simplify the pair space. For example, consider the normal-normal continuous element $\Sigma_{k,r}^{\rm nn}$ by asking DoFs on $\boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n}$ are unique, then there is no need of Lagrange multiplier for u_n. We have the surjectivity

$$\Sigma_k^{\rm nn} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{k-2,k-1,\cdot,k}^{-1} \to 0 \quad \text{ for } k \geq 0.$$

Given a function $(u_0, u_b, u_e) \in \mathring{M}_{k-2,k-1,\cdot,k}^{-1}$, for $k \geq 1$, using (u_0, u_b), we can define a weak gradient $\nabla_w(u_0, u_b) \in \mathbb{P}_{k-1}(T; \mathbb{R}^d)$ by

$$(\nabla_w(u_0, u_b), \boldsymbol{q})_T = -(u_0, \operatorname{div} \boldsymbol{q})_T + (u_b, \boldsymbol{n}^\intercal \boldsymbol{q})_{\partial T}, \quad \boldsymbol{q} \in \mathbb{P}_{k-1}(T; \mathbb{R}^d),$$

and a surface weak gradient $\nabla_{w,F}(u_b, u_e) \in \mathbb{P}_k(F; \mathbb{R}^{d-1})$ using (u_b, u_e) by

$$(\nabla_{w,F}(u_b, u_e), \boldsymbol{q})_F = -(u_b, \operatorname{div}_F \boldsymbol{q})_F + (u_e, \boldsymbol{n}_{F,e}^\intercal \boldsymbol{q})_{\partial F}, \quad \boldsymbol{q} \in \mathbb{P}_k(F; \mathbb{R}^{d-1}),$$

where $\mathbb{P}_{k-1}(F;\mathbb{R}^{d-1})$ is the polynomial vector tangential to F. For $k=0$, we only have u_e on edges and can define u_b as the non-conforming linear element on F based on u_e on ∂F. After that, using the average of u_b, to define the non-conforming linear element inside T.

With this notation, we have a simpler formulation of $\operatorname{div}\operatorname{div}_w$

$$\begin{aligned}(\operatorname{div}\operatorname{div}_w \boldsymbol{\tau}, (v_0, v_b, v_e))_{0,h} = &- \sum_{T\in\mathcal{T}_h} (\operatorname{div}_h \boldsymbol{\tau}, \nabla_w(v_0, v_b))_T \\ &+ \sum_{F\in\mathring{\mathcal{F}}_h} ([\Pi_F \boldsymbol{\tau}\boldsymbol{n}], \nabla_{w,F}(v_b, v_e))_F.\end{aligned} \tag{4.15}$$

In computation, (4.15) provides an alternative discretization without relatively complicated trace $\operatorname{tr}_2(\boldsymbol{\tau})$ and $\operatorname{tr}_e(\boldsymbol{\tau})$.

In two dimensions, the space $\mathring{M}^{-1}_{k-2,k-1,\cdot,k}$ can be identified as the Lagrange element $\mathring{V}^L_{k+1}$. The weak gradient operators become the gradient operators and (4.15) is the bilinear form used in the HHJ formulation. Therefore restricting to the pair $\Sigma_k^{\mathrm{nn}} - \mathring{M}^{-1}_{k-2,k-1,\cdot,k}$, we generalize HHJ to high dimensions whose hybridization is exactly (4.7) with appropriate correction on (f, v_0) for low order cases.

4.5. **A C^0 DG method for the biharmonic equation.** A C^0 discontinuous Galerkin (CDG) method for biharmonic equation can be developed by embedding the Lagrange element space $\mathring{V}_k(\mathcal{T}_h)$ into the broken space $\mathring{M}^{-1}_{r,k-1,k,k}$. This approach enables us to preserve the optimal order of convergence while reducing the size of the linear algebraic system.

We start with the embedding, for $k \geq 2$,

$$\begin{aligned}E^{\mathrm{CDG}} &: \mathring{V}_k(\mathcal{T}_h) \to \mathring{M}^{-1}_{r,k-1,k,k},\\ E^{\mathrm{CDG}} u &:= (Q_{r,T} u, Q_{k-1,F} u, \{\partial_{n_F} u\}|_F, u|_e)_{T\in\mathcal{T}_h, F\in\mathring{\mathcal{F}}_h, e\in\mathring{\mathcal{E}}_h}.\end{aligned}$$

For the boundary face $F \in \partial\mathcal{F}_h$ and $F\subset\partial T$, modify the jump and the average as

$$[u] = 2u|_T, \quad \{u\} = \frac{1}{2} u|_T. \tag{4.16}$$

By (4.1), for any $\boldsymbol{\tau} \in \Sigma^{-1}_{k,r}(T;\mathbb{S})$, the weak Hessian $\nabla_w^2 E^{\mathrm{CDG}} u$ is

$$\begin{aligned}(\nabla_w^2 E^{\mathrm{CDG}} u, \boldsymbol{\tau})_T =& (u, (\operatorname{div}\operatorname{div})_T \boldsymbol{\tau})_T + (\{\partial_{n_F} u\}\boldsymbol{n}_F\cdot\boldsymbol{n}, \boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})_{\partial T\cap\mathring{\mathcal{F}}_h}\\ &- (u, \operatorname{tr}_2(\boldsymbol{\tau}))_{\partial T} + \sum_{e\in\Delta_{d-2}(T)} (u, [\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}]|_T)_e\\ =& (\nabla_h^2 u, \boldsymbol{\tau})_T - \frac{1}{2}([\partial_n u], \boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})_{\partial T},\end{aligned} \tag{4.17}$$

where we use the fact $\partial_n u - \{\partial_{n_F} u\}\boldsymbol{n}_F\cdot\boldsymbol{n} = \frac{1}{2}[\partial_n u]$.

Let $(0,0,u_n,0) \in \mathring{M}^{-1}_{r,k-1,k,k}$ be given. By the definition of the weak Hessian, we have

$$(\nabla_w^2 u_n, \boldsymbol{\tau})_T := (\nabla_w^2(0,0,u_n,0), \boldsymbol{\tau})_T = (u_n \boldsymbol{n}_F\cdot\boldsymbol{n}, \boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})_{\partial T},$$

where u_n is defined on faces only, while $\nabla_w^2 u_n$ is element-wise polynomial. This quantity is sometimes referred to as the "lifting" of a boundary trace in the literature [5, 8, 38].

To save notation, define $\nabla_w^2 u := \nabla_w^2 E^{\rm CDG} u$ for $u \in \mathring{V}_k$. We can write (4.17) as

$$\nabla_w^2 u = \nabla_h^2 u - \frac{1}{2}\nabla_w^2 [\partial_n u]\boldsymbol{n}_F \cdot \boldsymbol{n}_{\partial T}, \tag{4.18}$$

where $[\partial_n u] \in V_{k-1}^{-1}(\mathcal{F}_h)$ and $\boldsymbol{n}_F \cdot \boldsymbol{n}_{\partial T} = \pm 1$ accounting for the consistency of orientation of face F.

Restricting the bilinear form $(\nabla_w^2 \cdot, \nabla_w^2 \cdot)$ to the subspace $E^{\rm CDG}\mathring{V}_k(\mathcal{T}_h)$, we obtain a C^0 DG formulation.

Lemma 4.8. *For $u, v \in \mathring{V}_k(\mathcal{T}_h)$, for $k \geq 2$, we have*

$$(\nabla_w^2 u, \nabla_w^2 v) = a^{\rm CDG}(u, v),$$

where

$$a^{\rm CDG}(u,v) = \sum_{T\in\mathcal{T}_h} (\nabla_h^2 u, \nabla_h^2 v)_T - \sum_{F\in\mathcal{F}_h} [(\{\partial_{nn} u\}, [\partial_n v])_F + ([\partial_n u], \{\partial_{nn} v\})_F] + \frac{1}{4}(\nabla_w^2[\partial_n u], \nabla_w^2[\partial_n v]).$$

Proof. It is a straightforward substitution of (4.18) into $(\nabla_w^2 u, \nabla_w^2 v)$. The cross term

$$\frac{1}{2}\sum_{T\in\mathcal{T}_h} (\nabla_h^2 u, \nabla_w^2[\partial_n v]\boldsymbol{n}_F \cdot \boldsymbol{n}_{\partial T}) = \frac{1}{2}\sum_{T\in\mathcal{T}_h} (\partial_{nn} u, [\partial_n v])_{\partial T} = \sum_{F\in\mathcal{F}_h} (\{\partial_{nn} u\}, [\partial_n v])_F,$$

where the scaling 2 or 1/2 in (4.16) is introduced for the unity of notation for interior and boundary faces. □

We obtain a C^0 DG method for the biharmonic equation: Find $u_h \in \mathring{V}_k(\mathcal{T}_h)$ s.t.

$$a^{\rm CDG}(u_h, v) = (f, Q_r v) \quad \forall v \in \mathring{V}_k(\mathcal{T}_h). \tag{4.19}$$

The boundary condition $u|_{\partial\Omega} = 0$ is built into the space $\mathring{V}_k(\mathcal{T}_h)$ while $\partial_n u|_{\partial\Omega} = 0$ is weakly imposed in DG sense.

It is worth noting that the widely-used interior penalty C^0 DG (IPCDG) method for the biharmonic equation [7, 20] requires a stabilization term in the form $\gamma(h_F^{-1}[\partial_n u], [\partial_n v])_{\mathcal{F}_h}$, where γ is chosen to be sufficiently large. In contrast, the CDG method (4.19) employs the bilinear form of the weak Hessian of jumps, i.e., $(\nabla_w^2[\partial_n u], \nabla_w^2[\partial_n v])$, as a parameter-free stabilization technique. It coincides with the approach proposed in [29, (2.9)] for the two-dimensional case.

The error analysis can be carried out following the approach in 2D [29]. To save the space, we only present the result below.

Theorem 4.9. *Let $u \in H_0^2(\Omega)$ be the solution of biharmonic equation* (3.1)*. Let $u_h \in \mathring{V}_k(\mathcal{T}_h)$ be the solution of the discrete method* (4.19) *for $k \geq 2$. Assume $u \in H^{k+1}(\Omega)$. We have*

$$\|\nabla^2 u - \nabla_w^2 u_h\|_0 \lesssim h^{k-1}(|u|_{k+1} + |f|_{\max\{k-3,0\}}).$$

The resulting linear algebraic system from the C^0 DG discretization is significantly reduced compared to the hybridized version. Despite the use of this simpler element, the method retains the optimal order of convergence. Hence, the C^0 DG method provides an attractive alternative to the hybridized approach. On the other hand, the hybridized mixed finite element method (1.4) can be post-processed to improve the convergence rate.

5. Finite element divdiv complexes in three dimensions

In this section we will first present finite element divdiv complexes involving conforming finite element spaces. Then we construct the distributional finite element divdiv complexes using the weak divdiv operator.

5.1. **Conforming finite element divdiv complexes.**

The three-dimensional divdiv complex is [4,32]

$$\mathbf{RT} \xrightarrow{\subset} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{dev}\operatorname{grad}} H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T}) \xrightarrow{\operatorname{sym}\operatorname{curl}} H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\operatorname{div}} L^2(\Omega) \to 0,$$

where $\mathbf{RT} = \{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\}$, $H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$ is the space of traceless tensor $\boldsymbol{\sigma} \in L^2(\Omega;\mathbb{T})$ such that $\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma} \in L^2(\Omega;\mathbb{S})$ with the row-wise curl operator.

5.1.1. *Finite element complexes starting from Hermite element.* We start from the vectorial Hermite element space in three dimensions [17]

$$\begin{aligned} V_{k+2}^H := \{\boldsymbol{v}_h \in H^1(\Omega;\mathbb{R}^3) : & \boldsymbol{v}_h|_T \in \mathbb{P}_{k+2}(T;\mathbb{R}^3) \text{ for each } T \in \mathcal{T}_h, \\ & \nabla \boldsymbol{v}_h(\delta) \text{ is single-valued at each vertex } \delta \text{ of } \mathcal{T}_h\}. \end{aligned}$$

Since no supersmooth DoFs in (2.3), we can use DoFs for $H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$-conforming finite elements simpler than those in [13, 15, 26]. Take the space of shape functions as $\mathbb{P}_{k+1}(T;\mathbb{T})$. The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta), \quad \delta \in \Delta_0(T), \boldsymbol{\tau} \in \mathbb{T}, \tag{5.1a}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e, \quad q \in \mathbb{P}_{k-1}(e), e \in \Delta_1(T), i = 1, 2, \tag{5.1b}$$

$$(\boldsymbol{n} \times \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \times \boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathbb{B}_{k+1}^{tt}(F;\mathbb{S}), F \in \partial T, \tag{5.1c}$$

$$(\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathbb{B}_{k+1}^{\operatorname{div}_F}(F), F \in \partial T, \tag{5.1d}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, T;\mathbb{T}), \tag{5.1e}$$

where

$$\begin{aligned} \mathbb{B}_{k+1}^{tt}(F;\mathbb{S}) &:= \{\boldsymbol{\tau} \in \mathbb{P}_{k+1}(F;\mathbb{S}) : \boldsymbol{\tau}(\mathtt{v}) = 0 \text{ for } \mathtt{v} \in \Delta_0(F), \boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}|_{\partial F} = 0\}, \\ \mathbb{B}_{k+1}^{\operatorname{div}_F}(F) &:= \{\boldsymbol{v} \in \mathbb{P}_{k+1}(F;\mathbb{R}^2) : \boldsymbol{v} \cdot \boldsymbol{n}_{F,e}|_{\partial F} = 0\}, \\ \mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, T;\mathbb{T}) &:= \{\boldsymbol{\tau} \in \mathbb{P}_{k+1}(T;\mathbb{T}) : (\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n})|_{\partial T} = 0, \\ &\qquad (\boldsymbol{n} \times \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \times \boldsymbol{n})|_{\partial T} = 0\}. \end{aligned}$$

Characterization of $\mathbb{B}_{k+1}^{\operatorname{div}_F}(F)$ can be found in [12, Lemma 4.2] and $\mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, T;\mathbb{T})$ in [15, Lemma 5.7]. In particular, we know $\dim \mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, T;\mathbb{T}) = \frac{1}{3}(4k^3 + 6k^2 - 10k)$ and $\dim \mathbb{B}_{k+1}^{\operatorname{div}_F}(F) = 2\dim \mathbb{P}_{k+1}(F) - 3 \times (k+2) = k^2 + 2k$. The bubble space $\dim \mathbb{B}_{k+1}^{tt}(F;\mathbb{S}) = 3\dim \mathbb{P}_{k+1}(F) - 3 \times 3 - 3 \times k = \frac{3}{2}(k^2 + 3k)$.

Lemma 5.1. *The DoFs* (5.1) *are unisolvent for the space* $\mathbb{P}_{k+1}(T;\mathbb{T})$ *for* $k \geq 0$.

Proof. The number of DoFs (5.1) is

$$4 \times 8 + 6 \times 2k + 4 \times \left(\frac{3}{2}(k^2 + 3k) + (k^2 + 2k)\right) + \frac{1}{3}(4k^3 + 6k^2 - 10k) = 8\binom{k+4}{3},$$

which equals $\dim \mathbb{P}_{k+1}(T;\mathbb{T})$.

Assume $\boldsymbol{\tau} \in \mathbb{P}_{k+1}(T;\mathbb{T})$ and all the DoFs (5.1) vanish. Clearly $(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t})|_e = 0$ follows from the vanishing DoFs (5.1a)–(5.1b) for $e \in \Delta_1(T)$ and $i = 1, 2$. Notice that for $e \in \Delta_1(T)$ being an edge of face $F \in \partial T$, we have

$$\boldsymbol{n}_{F,e}^{\intercal}\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_F)\boldsymbol{n}_{F,e} = \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, \quad \boldsymbol{n}_F^{\intercal}(\boldsymbol{\tau}\times\boldsymbol{n}_F)\boldsymbol{n}_{F,e} = \boldsymbol{n}_F^{\intercal}\boldsymbol{\tau}\boldsymbol{t}. \tag{5.2}$$

Hence $(\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n})|_F \in \mathbb{B}_{k+1}^{tt}(F;\mathbb{S})$ and $(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})|_F \in \mathbb{B}_{k+1}^{\operatorname{div}_F}(F)$ for $F \in \partial T$. Then we get from the vanishing DoFs (5.1c)–(5.1d) that $\boldsymbol{\tau} \in \mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, T;\mathbb{T})$, which together with the vanishing DoF (5.1e) yields $\boldsymbol{\tau} = 0$. ☐

The finite element space $\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}}$ is defined as follows

$$\begin{aligned}\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{T}) :&\boldsymbol{\tau}|_T \in \mathbb{P}_{k+1}(T;\mathbb{T}) \text{ for each } T \in \mathcal{T}_h,\\ &\text{all the DoFs (5.1) are single-valued}\}.\end{aligned}$$

DoFs (5.1a)–(5.1b) on $e \in \Delta_1(T)$ determine $(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t})|_e$. By (5.2), $(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t})|_e$ and (5.1c) determine $(\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n})|_F$, and $(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t})|_e$ and (5.1d) determine $(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})|_F$. Therefore, $\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \subset H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$ by the characterization of traces of functions in $H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$ given in [15].

Theorem 5.2. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The finite element* div div *complex*

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-2}^{-1} \to 0, \ \textit{for } k \geq 3, \tag{5.3}$$

is exact. Similarly, the finite element div div *complex*

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k^+}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-1}^{-1} \to 0, \quad \textit{for } k \geq 2,$$

involving Raviart-Thomas type space $\Sigma_{k^+}^{\operatorname{div}\operatorname{div}}$ is exact.

Proof. The proof of two complexes is similar. So we focus on (5.3).

Clearly (5.3) is complex. We have proved the div div operator is surjective. For $\boldsymbol{\tau} \in \ker(\operatorname{sym}\operatorname{curl}) \cap \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}}$, there exists a $v \in H^1(\Omega)$ s.t. $\operatorname{dev}\operatorname{grad} v = \boldsymbol{\tau}$. As $\boldsymbol{\tau}$ is piecewise polynomial, so is v. And the continuity of $\boldsymbol{\tau}$ at vertices implies v is C^1 at vertices. Therefore we verified $\boldsymbol{\tau} \in \ker(\operatorname{sym}\operatorname{curl}) \cap \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} = \operatorname{dev}\operatorname{grad} V_{k+2}^H$.

It remains to verify $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl}\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}}$ by dimension count. It is easy to show the constraints $[\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0$ for all $e \in \mathring{\mathcal{E}}_h$ are linearly independent. Therefore

$$\begin{aligned}\dim \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} &= \dim \Sigma_k^{\operatorname{div}\operatorname{div}-} - (k+1)|\mathring{\mathcal{E}}_h|\\ &= (k+1)(k^2+k+2)|\mathcal{T}_h| + (k+1)^2|\mathcal{F}_h| - (k+1)|\mathring{\mathcal{E}}_h|.\end{aligned}$$

Hence we have

$$\begin{aligned}&\dim(\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} \cap \ker(\operatorname{div}\operatorname{div})) = \dim \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} - \dim V_{k-2}^{-1}(\mathcal{T}_h)\\ =&\frac{1}{6}(k+1)(5k^2+7k+12)|\mathcal{T}_h| + (k+1)^2|\mathcal{F}_h| - (k+1)|\mathring{\mathcal{E}}_h|.\end{aligned}$$

While

$$\begin{aligned}&\dim \operatorname{sym}\operatorname{curl}\Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} = \dim \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} - \dim V_{k+2}^H + \dim \mathbf{RT}\\ =&\frac{1}{6}(k-1)k(5k+17)|\mathcal{T}_h| + (k^2+5k)|\mathcal{F}_h| - (k-3)|\mathcal{E}_h| - 4|\mathcal{V}_h| + 4.\end{aligned}$$

Then

$$\begin{aligned}&\dim(\Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div})) - \dim \operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\mathrm{sym\,curl}}\\ &= (6k+2)|\mathcal{T}_h| - (3k-1)|\mathcal{F}_h| - (k+1)|\mathring{\mathcal{E}}_h| + (k-3)|\mathcal{E}_h| + 4|\mathcal{V}_h| - 4\\ &= k(6|\mathcal{T}_h| - 3|\mathcal{F}_h| + |\mathcal{E}_h^{\partial}|) + 2|\mathcal{T}_h| + |\mathcal{F}_h| + |\mathcal{E}_h^{\partial}| - 4|\mathcal{E}_h| + 4|\mathcal{V}_h| - 4.\end{aligned}$$

By the relation $4|\mathcal{T}_h| = 2|\mathcal{F}_h| - |\mathcal{F}_h^{\partial}|$ and $3|\mathcal{F}_h^{\partial}| = 2|\mathcal{E}_h^{\partial}|$,

$$6|\mathcal{T}_h| - 3|\mathcal{F}_h| + |\mathcal{E}_h^{\partial}| = -\frac{3}{2}|\mathcal{F}_h^{\partial}| + |\mathcal{E}_h^{\partial}| = 0.$$

This together with the Euler's formula $|\mathcal{V}_h| - |\mathcal{E}_h| + |\mathcal{F}_h| - |\mathcal{T}_h| = 1$ yields

$$2|\mathcal{T}_h| + |\mathcal{F}_h| + |\mathcal{E}_h^{\partial}| - 4|\mathcal{E}_h| + 4|\mathcal{V}_h| - 4 = -4|\mathcal{T}_h| + 4|\mathcal{F}_h| - 4|\mathcal{E}_h| + 4|\mathcal{V}_h| - 4 = 0.$$

Combining the last three identities gives

$$\dim(\Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div})) = \dim(\operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\mathrm{sym\,curl}}).$$

Therefore, $\Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\mathrm{sym\,curl}}$. □

5.1.2. *Finite element complexes starting from Lagrange element.* We present finite element divdiv complexes with the lowest smoothness in three dimensions.

We start from the vectorial Lagrange element space V_{k+2}^L. Define the $H(\operatorname{sym}\operatorname{curl}, \Omega; \mathbb{T})$-conforming space with the lowest smoothness

$$\overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} := \{\boldsymbol{\tau} \in H(\operatorname{sym}\operatorname{curl}, \Omega; \mathbb{T}) : \boldsymbol{\tau}|_T \in \mathbb{P}_{k+1}(T; \mathbb{T}) \text{ for each } T \in \mathcal{T}_h\}.$$

Although $\overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}}$ exists, it is hard to give local DoFs. Notice that $\Sigma_{k+1}^{\mathrm{sym\,curl}} \subseteq \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}}$.

Theorem 5.3. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The finite element* div div *complexes*

(5.4)
$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^L \xrightarrow{\operatorname{dev}\operatorname{grad}} \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-2}^{-1} \to 0, \ \textit{for } k \geq 3,$$

and

(5.5)
$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^L \xrightarrow{\operatorname{dev}\operatorname{grad}} \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k^+}^{\mathrm{div\,div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_{k-1}^{-1} \to 0, \ \textit{for } k \geq 2,$$

are exact.

Proof. By the similarity of two complexes, we focus on the exactness of complex (5.4).

By the exactness of complex (5.3), we have

$$\operatorname{div}\operatorname{div} \Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} = V_{k-2}^{-1}, \quad \operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\mathrm{sym\,curl}} = \Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div}).$$

Noting that $\Sigma_{k+1}^{\mathrm{sym\,curl}} \subseteq \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}}$, it follows

$$\operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\mathrm{sym\,curl}} \subseteq \operatorname{sym}\operatorname{curl} \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} \subseteq \Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div}).$$

Hence $\operatorname{sym}\operatorname{curl} \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} = \Sigma_{k,\mathrm{new}}^{\mathrm{div\,div}} \cap \ker(\operatorname{div}\operatorname{div})$.

Clearly $\operatorname{dev}\operatorname{grad} V_{k+2}^L \subseteq (\overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} \cap \ker(\operatorname{sym}\operatorname{curl}))$. On the other side, for $\boldsymbol{\tau} \in \overline{\Sigma}_{k+1}^{\mathrm{sym\,curl}} \cap \ker(\operatorname{sym}\operatorname{curl})$, there exists $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3)$ satisfying $\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl} \boldsymbol{v}$. On each tetrahedron $T \in \mathcal{T}_h$, $\operatorname{sym}\operatorname{curl}(\boldsymbol{v}|_T) \in \mathbb{P}_{k+1}(T;\mathbb{T})$, then $\boldsymbol{v}|_T \in \mathbb{P}_{k+2}(T;\mathbb{R}^3)$ and $\boldsymbol{v} \in V^L_{k+2}$. Therefore complex (5.4) is exact. $\square$

5.1.3. *Lower order finite element divdiv complexes.* The previous divdiv complexes did not cover the case $k = 0, 1$. In this subsection, we consider $k = 1$ and refer to Section 5.2 for $k = 0$ in the distributional sense. For the $H(\operatorname{sym}\operatorname{curl}, \Omega; \mathbb{T})$-conforming finite element, we take the space of shape functions as

$$\Sigma_{2^+}(T;\mathbb{T}) := \mathbb{P}_2(T;\mathbb{T}) \oplus (\boldsymbol{x} \otimes (\boldsymbol{x} \times \mathbb{H}_1(T;\mathbb{R}^3))),$$

whose dimension is $80 + 8 = 88$. Since $\operatorname{sym}\operatorname{curl}(\boldsymbol{x} \otimes \boldsymbol{v}) = \operatorname{sym}(\boldsymbol{x} \otimes \operatorname{curl}\boldsymbol{v})$ for $\boldsymbol{v} \in H^1(T;\mathbb{R}^3)$, we have $\operatorname{sym}\operatorname{curl}(\boldsymbol{x} \otimes (\boldsymbol{x} \times \mathbb{H}_1(T;\mathbb{R}^3))) = \operatorname{sym}(\boldsymbol{x} \otimes \operatorname{curl}\mathbb{H}_2(T;\mathbb{R}^3))$, which means $\dim \operatorname{sym}\operatorname{curl}(\boldsymbol{x} \otimes (\boldsymbol{x} \times \mathbb{H}_1(T;\mathbb{R}^3))) = \dim \operatorname{curl}\mathbb{H}_2(T;\mathbb{R}^3) = 8$, hence $\operatorname{sym}\operatorname{curl}$ is injective on $\boldsymbol{x} \otimes (\boldsymbol{x} \times \mathbb{H}_1(T;\mathbb{R}^3))$.

The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta), \quad \delta \in \Delta_0(T), \boldsymbol{\tau} \in \mathbb{T}, \tag{5.6a}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e, \quad q \in \mathbb{P}_0(e), e \in \Delta_1(T), i = 1, 2, \tag{5.6b}$$

$$(\boldsymbol{n} \times \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \times \boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathbb{B}^{tt}_{2^+}(F;\mathbb{S}), F \in \partial T, \tag{5.6c}$$

$$(\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathbb{B}^{\operatorname{div}_F}_2(F), F \in \partial T, \tag{5.6d}$$

where

$$\mathbb{B}^{tt}_{2^+}(F;\mathbb{S}) := \{\boldsymbol{\tau} \in \mathbb{P}_2(F;\mathbb{S}) + (\boldsymbol{x} \times \boldsymbol{n}) \otimes (\boldsymbol{x} \times \boldsymbol{n})\mathbb{P}_1(F) : \boldsymbol{\tau}(\mathrm{v}) = 0 \text{ for } \mathrm{v} \in \Delta_0(F), \boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}|_{\partial F} = 0\}.$$

By Section 3 in [10], $\dim \mathbb{B}^{tt}_{2^+}(F;\mathbb{S}) = 8$. Recall that $\dim \mathbb{B}^{\operatorname{div}_F}_2(F) = 3$.

Lemma 5.4. *The DoFs* (5.6) *are unisolvent for the space* $\Sigma_{2^+}(T;\mathbb{T})$.

Proof. The number of DoFs (5.6) is

$$4 \times 8 + 6 \times 2 + 4 \times 8 + 4 \times 3 = 88 = \dim \Sigma_{2^+}(T;\mathbb{T}).$$

Assume $\boldsymbol{\tau} \in \Sigma_{2^+}(T;\mathbb{T})$ and all the DoFs (5.6) vanish. Notice that $(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t})|_e \in \mathbb{P}_2(e)$ for $e \in \Delta_1(T)$, and $(\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n})|_F \in \mathbb{P}_2(F;\mathbb{R}^2)$ and $(\boldsymbol{n} \times \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \times \boldsymbol{n})|_F \in \mathbb{P}_2(F;\mathbb{S}) + (\boldsymbol{x} \times \boldsymbol{n}) \otimes (\boldsymbol{x} \times \boldsymbol{n})\mathbb{P}_1(F)$ for $F \in \partial T$. Hence the vanishing DoFs (5.6) imply $\boldsymbol{\tau} \in \Sigma_{2^+}(T;\mathbb{T}) \cap \mathbb{B}_3(\operatorname{sym}\operatorname{curl}, T;\mathbb{T})$. By Theorem 5.12 in [15] and $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} \in \Sigma_{1^{++}}(T;\mathbb{S})$, we get $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} = 0$. Thus, $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{q}$ with $\boldsymbol{q} \in \mathbb{P}_3(T;\mathbb{R}^3)$ satisfying $\boldsymbol{q}|_{\partial T} = 0$. Therefore, $\boldsymbol{q} = 0$ and $\boldsymbol{\tau} = 0$. $\square$

Define $H(\operatorname{sym}\operatorname{curl})$-conforming finite element spaces as follows

$$\Sigma^{\operatorname{sym}\operatorname{curl}}_{2^+} := \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{T}) : \boldsymbol{\tau}|_T \in \Sigma_{2^+}(T;\mathbb{T}) \text{ for each } T \in \mathcal{T}_h, \text{ all the DoFs (5.6) are single-valued}\},$$

$$\overline{\Sigma}^{\operatorname{sym}\operatorname{curl}}_{2^+} := \{\boldsymbol{\tau} \in H(\operatorname{sym}\operatorname{curl}, \Omega;\mathbb{T}) : \boldsymbol{\tau}|_T \in \Sigma_{2^+}(T;\mathbb{T}) \text{ for each } T \in \mathcal{T}_h\}.$$

Clearly, $\Sigma^{\operatorname{sym}\operatorname{curl}}_{2^+} \subset H(\operatorname{sym}\operatorname{curl}, \Omega;\mathbb{T})$, and $\dim \Sigma^{\operatorname{sym}\operatorname{curl}}_{2^+} = 11|\mathcal{F}_h| + 2|\mathcal{E}_h| + 8|\mathcal{V}_h|$.

Applying the argument in Theorem 5.2 and Theorem 5.3, we have the following lower order finite element divdiv complexes.

Theorem 5.5. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The finite element* $\operatorname{div}\operatorname{div}$ *complexes*

$$\mathbf{RT} \xrightarrow{\subset} V_3^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{2^+}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{1^{++}}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_1^{-1} \to 0,$$

$$\mathbf{RT} \xrightarrow{\subset} V_3^L \xrightarrow{\operatorname{dev}\operatorname{grad}} \overline{\Sigma}_{2^+}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{1^{++}}^{\operatorname{div}\operatorname{div}} \xrightarrow{\operatorname{div}\operatorname{div}} V_1^{-1} \to 0$$

are exact.

5.2. **Distributional finite element divdiv complexes.** With the weak $\operatorname{div}\operatorname{div}_w$ operator, we can construct the distributional finite element divdiv complexes. We first present finite element discretization of the distributional divdiv complex

$$\mathbf{RT} \xrightarrow{\subset} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{dev}\operatorname{grad}} H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T}) \xrightarrow{\operatorname{sym}\operatorname{curl}} L^2(\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\operatorname{div}} H^{-2}(\Omega) \to 0.$$

Theorem 5.6. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The following complex*

(5.7)
$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{k,r}^{-1} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{r,k-1,k,k}^{-1} \to 0, \text{ for } k \geq 1,$$

is exact.

Proof. The proof is similar to that for Theorem 5.2. The only difference is to verify $\dim \operatorname{sym}\operatorname{curl} \Sigma_{k+1}^{\operatorname{sym}\operatorname{curl}} = \dim \ker(\operatorname{div}\operatorname{div}_w) \cap \Sigma_k^{-1}$ by dimension count:

$$\dim \ker(\operatorname{div}\operatorname{div}_w) \cap \Sigma_k^{-1} = \dim \Sigma_k^{-1} - \dim \mathring{M}_{k-2,k-1,k,k}^{-1} = \dim \Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}} - \dim V_{k-2}^{-1}.$$

□

By dimension count and the structure of the enrichment, we have two more complexes for $k = 0, 1$.

Proposition 5.7. *For $k = 1$, the following complex*

$$\mathbf{RT} \xrightarrow{\subset} V_3^H \xrightarrow{\operatorname{dev}\operatorname{grad}} \Sigma_{2^+}^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_{1^{++}}^{-1} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{1,1,1,1}^{-1} \to 0$$

is also exact. For $k = 0$, the following complex

$$\mathbf{RT} \xrightarrow{\subset} V_2^L \xrightarrow{\operatorname{dev}\operatorname{grad}} \overline{\Sigma}_1^{\operatorname{sym}\operatorname{curl}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \Sigma_0^{-1} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{\cdot,\cdot,0,0}^{-1} \to 0$$

is exact.

We can define $\overline{\Sigma}_{k,r}^{\operatorname{div}\operatorname{div}}$, for $k = 0, 1, 2, r \leq 0$,

$$\overline{\Sigma}_{k,r}^{\operatorname{div}\operatorname{div}} = \{\boldsymbol{\tau} \in H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) : \boldsymbol{\tau}\,|_T \in \Sigma_{k,r}(T;\mathbb{S})\}.$$

Although local DoFs cannot be given for space $\overline{\Sigma}_k^{\operatorname{div}\operatorname{div}}, k = 0, 1, 2$, a discretization of the biharmonic equation can be obtained by the hybridization. For example, $\overline{\Sigma}_0^{\operatorname{div}\operatorname{div}} = \ker(\operatorname{div}\operatorname{div}_w) \cap \Sigma_0^{-1}$ is defined by applying the following constraints to Σ_0^{-1}

$$[\operatorname{tr}_e(\boldsymbol{\tau})]|_e = 0 \quad \text{for } e \in \mathring{\mathcal{E}}_h, \quad [\boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n}]|_F = 0 \quad \text{for } F \in \mathring{\mathcal{F}}_h.$$

By counting the dimension of $\overline{\Sigma}_0^{\operatorname{div}\operatorname{div}}$, these constraints are linearly independent.

Corollary 5.8. *Both conforming finite element* $\operatorname{div}\operatorname{div}$ *complexes* (5.4) *and* (5.5) *are exact for all $k = 0, 1, 2$ using space $\overline{\Sigma}_{k,r}^{\operatorname{div}\operatorname{div}}$ to replace $\Sigma_{k,\mathrm{new}}^{\operatorname{div}\operatorname{div}}$ or $\Sigma_{k^+}^{\operatorname{div}\operatorname{div}}$.*

Remark 5.9. The first half of complexes (5.7) can be replaced by

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^{L} \xrightarrow{\text{dev grad}} \overline{\Sigma}_{k+1}^{\text{sym curl}} \xrightarrow{\text{sym curl}} \cdots \quad \text{for } k \geq 0.$$

Remark 5.10. Recall that we can identity non-conforming VEM space $Q_M : \mathring{V}_{k+2}^{\text{VEM}} \to \mathring{M}_{r,k-1,k,k}^{-1}$ through Q_M. Then we can rewrite the second half of complexes (5.7) as

$$\cdots \xrightarrow{\text{sym curl}} \Sigma_{k,r}^{-1} \xrightarrow{Q_M^{-1} \operatorname{div}\operatorname{div}_w} \mathring{V}_{k+2}^{\text{VEM}} \to 0 \quad \text{for } k \geq 0.$$

When some partial continuity is imposed on Σ_k^{-1}, we can simplify the last space. For example, consider the normal-normal continuous element $\Sigma_{k,r}^{\text{nn}}$ by asking DoFs on $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$ are single valued, then there is no need of Lagrange multiplier u_n. The corresponding divdiv complexes are still exact as we only reduce the range space of $\operatorname{div}\operatorname{div}_w$; see the $\tilde{}$ operation introduced in [13]. As a result of Theorem 5.6, we will get finite element discretizations of the distributional divdiv complex

$$\mathbf{RT} \xrightarrow{\subset} H^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{dev grad}} H(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T}) \xrightarrow{\text{sym curl}} H^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\operatorname{div}} H^{-1}(\Omega) \to 0.$$

Theorem 5.11. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The following complexes*

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^{H} \xrightarrow{\text{dev grad}} \Sigma_{k+1}^{\text{sym curl}} \xrightarrow{\text{sym curl}} \Sigma_{k,r}^{\text{nn}} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{r,k-1,\cdot,k}^{-1} \to 0, \ \textit{for } k \geq 1,$$

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^{L} \xrightarrow{\text{dev grad}} \overline{\Sigma}_{k+1}^{\text{sym curl}} \xrightarrow{\text{sym curl}} \Sigma_{k,r}^{\text{nn}} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{M}_{r,k-1,\cdot,k}^{-1} \to 0, \ \textit{for } k \geq 0,$$

are exact.

In two dimensions, the space $\mathring{M}_{k-2,k-1,\cdot,k}^{-1}$ can be identified as the Lagrange element $\mathring{V}_{k+1}^{L}$. The first distributional divdiv complex constructed in [9] can be written as

$$\mathbf{RT} \xrightarrow{\subset} (V_{k+1}^{L})^2 \xrightarrow{\text{sym curl}} \Sigma_{k}^{\text{nn}} \xrightarrow{\operatorname{div}\operatorname{div}_w} \mathring{V}_{k+1}^{L} \to 0, \quad \text{for } k \geq 0.$$

Complexes in Theorem 5.11 are its generalization to 3-D.

We can further reduce the space of u to $\mathring{M}_{r,k-1,\cdot,\cdot}^{-1}$ when the normal-normal continuity and $[\operatorname{tr}_e(\cdot)] = 0$ are both imposed and denoted by $\Sigma_{k,r}^{\text{nn,e}}$ for $k \geq 1$. The space $\mathring{M}_{r,k-1,\cdot,\cdot}^{-1}$ can be identified as the H^1 non-conforming virtual element space [11, Section 2.2]

$$\mathring{V}_k^{1,\text{VEM}} := \big\{u \in L^2(\Omega) : u|_T \in V_k^{1,\text{VEM}}(T) \text{ for } T \in \mathcal{T}_h, \\ Q_{k-1,F}u \text{ is single-valued for } F \in \mathring{\mathcal{F}}_h, \text{ and vanish on boundary } \partial\Omega\big\},$$

where $V_k^{1,\text{VEM}}(T) := \big\{u \in H^1(T) : \Delta u \in \mathbb{P}_r(T), \partial_n u|_F \in \mathbb{P}_{k-1}(F) \text{ for } F \in \partial T\big\}$. The DoFs of $\mathring{V}_k^{1,\text{VEM}}$ are given by $Q_M u := \{Q_{r,T}u, Q_{k-1,F}u\}_{T\in\mathcal{T}_h, F\in\mathring{\mathcal{F}}_h}$ through which can be identified $\mathring{M}_{r,k-1,\cdot,\cdot}^{-1}$.

We then obtain a divdiv complex ending with $\mathring{V}_k^{1,\text{VEM}}$

$$\mathbf{RT} \xrightarrow{\subset} V_{k+2}^{H} \xrightarrow{\text{dev grad}} \Sigma_{k+1}^{\text{sym curl}} \xrightarrow{\text{sym curl}} \Sigma_{k,r}^{\text{nn,e}} \xrightarrow{Q_M^{-1}\operatorname{div}\operatorname{div}_w} \mathring{V}_k^{1,\text{VEM}} \to 0, \ \text{for } k \geq 1.$$

Appendix A. Unisolvence

In this appendix, we give the unisolvence of DoFs (2.15) for the space $\Sigma_{2^+}(T;\mathbb{S})$. First we recall a decomposition of a polynomial space and some barycentric calculus developed in [15].

Lemma A.1. *Let* $\mathbb{P}_3(T)\backslash\mathbb{P}_1(T) := \{q \in \mathbb{P}_3(T) : q(0) = 0, \nabla q(0) = 0\}$. *The mapping* $\boldsymbol{x}^\intercal\nabla^2\cdot\boldsymbol{x} : \mathbb{P}_3(T)\backslash\mathbb{P}_1(T) \to \mathbb{P}_3(T)\backslash\mathbb{P}_1(T)$ *is one-to-one. The mapping* $\boldsymbol{x}^\intercal\cdot\boldsymbol{x} : \mathbb{P}_1(T;\mathbb{S}) \to \mathbb{P}_3(T)\backslash\mathbb{P}_1(T)$ *is surjective.*

Proof. By direct computation $\boldsymbol{x}^\intercal(\nabla^2 q)\boldsymbol{x} = r(r-1)q$ for $q \in \mathbb{H}_r(T), r \geq 0$. $\square$

Lemma A.2. *We have the decomposition*

$$\mathbb{P}_1(T;\mathbb{S}) = \nabla^2\mathbb{P}_3(T) \oplus (\ker(\boldsymbol{x}^\intercal\cdot\boldsymbol{x}) \cap \mathbb{P}_1(T;\mathbb{S})), \tag{A.1}$$

and consequently,

$$\begin{aligned}\dim\ker(\boldsymbol{x}^\intercal\cdot\boldsymbol{x}) \cap \mathbb{P}_1(T;\mathbb{S}) &= \dim\mathbb{P}_1(T;\mathbb{S}) - \dim\mathbb{P}_3(T) + \dim\mathbb{P}_1(T)\\ &= (d+1)\binom{d+1}{2} - \binom{d+3}{3} + d + 1 = 2\binom{d+1}{3}.\end{aligned}$$

Proof. By Lemma A.1, $\nabla^2\mathbb{P}_3(T) \cap (\ker(\boldsymbol{x}^\intercal\cdot\boldsymbol{x}) \cap \mathbb{P}_1(T;\mathbb{S})) = 0$, and

$$\dim\ker(\boldsymbol{x}^\intercal\cdot\boldsymbol{x}) \cap \mathbb{P}_1(T;\mathbb{S}) = \dim\mathbb{P}_1(T;\mathbb{S}) - \dim\mathbb{P}_3(T) + \dim\mathbb{P}_1(T),$$

which ends the proof. $\square$

Define

$$\mathbb{B}_k^{\mathrm{div}}(T) := \mathbb{P}_k\left(T;\mathbb{R}^d\right) \cap H_0(\mathrm{div}, T) = \{\boldsymbol{v} \in \mathbb{P}_k(T;\mathbb{R}^d) : \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial T} = 0\}.$$

Recall the characterization of the div bubble function.

Lemma A.3 (Lemma 4.2 in [12])**.** *For an edge* $e = [\mathtt{v}_i, \mathtt{v}_j]$, *let* $b_e = \lambda_i\lambda_j$ *be the quadratic edge bubble function and* $\boldsymbol{t}_e$ *be tangential vector of* e. *Then we have*

$$\mathbb{B}_2^{\mathrm{div}}(T) = \mathrm{span}\{b_e(x)\boldsymbol{t}_e : e \in \Delta_1(T)\}. \tag{A.2}$$

As a consequence $\dim\mathbb{B}_2^{\mathrm{div}}(T) = |\Delta_1(T)| = \binom{d+1}{2}$ for a d-dimensional simplex T.

We can easily show that $b_e(x)\boldsymbol{t}_e$ is an element of $\mathbb{B}_2^{\mathrm{div}}(T)$. In order to establish (A.2), it is necessary to demonstrate that all quadratic divergence-free bubbles can be expressed in this form. See [12] for details.

Lemma A.4. *Let* $\boldsymbol{v} \in \mathbb{B}_2^{\mathrm{div}}(T)$ *satisfy* $\mathrm{div}\,\boldsymbol{v} = 0$ *and for one* $F \in \partial T$

$$(\Pi_F\boldsymbol{v}, \boldsymbol{q})_F = 0, \quad \boldsymbol{q} \in \mathbb{B}_2^{\mathrm{div}}(F). \tag{A.3}$$

Then $\boldsymbol{v} = 0$.

Proof. Without loss of generality, take $F = F_d$. Then (A.3) implies that $\boldsymbol{v}$ does not contain edge bubbles on F_d, i.e., $\boldsymbol{v} = \sum_{i=0}^{d-1} c_i b_{e_{d,i}}\boldsymbol{t}_{d,i}$ with $c_i \in \mathbb{R}$ and $\boldsymbol{t}_{d,i} = \mathtt{v}_i - \mathtt{v}_d$. By direct computation and the fact $\nabla\lambda_i\cdot\boldsymbol{t}_{d,j} = \delta_{ij}$, we have

$$\frac{|T|}{(d+2)(d+1)}c_i = (\boldsymbol{v}, \nabla\lambda_i)_T = -(\mathrm{div}\,\boldsymbol{v}, \lambda_i)_T = 0$$

for each $i = 0, 1, \ldots, d-1$. So $\boldsymbol{v} = 0$. $\square$

To facilitate the proof of unisolvence, we can select an intrinsic coordinate system. Let $\boldsymbol{t}_i := \mathtt{v}_i - \mathtt{v}_0$ for $i = 1, \ldots, d$. The set of tangential vectors $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d\}$ forms a basis of $\mathbb{R}^d$, and its dual basis is given by $\{\nabla\lambda_1, \ldots, \nabla\lambda_d\}$. We have the property that $\nabla\lambda_i \cdot \boldsymbol{t}_j = \delta_{ij}$ for $i, j = 1, \ldots, d$, where δ_{ij} is the Kronecker delta. We can then express the symmetric tensor $\boldsymbol{\tau}$ as $\boldsymbol{\tau} = \sum_{i,j=1}^d \tau_{ij}\boldsymbol{t}_i \otimes \boldsymbol{t}_j$ using coefficients τ_{ij}, which are computed as $\tau_{ij} = (\nabla\lambda_i)^\intercal \boldsymbol{\tau}(\nabla\lambda_j)$.

Since $\boldsymbol{\tau}$ is symmetric, we have that $\tau_{ij} = \tau_{ji}$ for $1 \le i, j \le d$. Therefore, we can represent $\boldsymbol{\tau}$ as a symmetric matrix function $(\tau_{ij}(x))$ in this coordinate system.

Theorem A.5. *The DoFs* (2.15) *are unisolvent for the space* $\Sigma_{2^+}(T; \mathbb{S})$.

Proof.

Step 1 (Dimension count). The number of DoF (2.15d) is

$$\binom{2}{2} + \binom{3}{2} + \ldots + \binom{d-1}{2} = \binom{d}{3},$$

and the number of DoF (2.15e) is $\dim \ker(\boldsymbol{x}^\intercal \cdot \boldsymbol{x}) \cap \mathbb{P}_1(T; \mathbb{S}) = 2\binom{d+1}{3}$. Hence the total number of DoFs (2.15) is

$$\begin{aligned}
&\binom{d+1}{2}\binom{d}{2} + (d+1)\binom{d+1}{2} + (d+1)d + \binom{d}{3} + 2\binom{d+1}{3} \\
&= \binom{d+1}{2}\binom{d+2}{2} + d,
\end{aligned}$$

which is exactly the dimension of $\Sigma_{2^+}(T; \mathbb{S})$.

Step 2 (Consequence of vanishing DoFs). Assume $\boldsymbol{\tau} \in \Sigma_{2^+}(T; \mathbb{S}) = \mathbb{P}_2(T; \mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^\intercal \mathbb{H}_1(T)$, and all the DoFs (2.15) vanish. The vanishing DoFs (2.15a)–(2.15c) imply the traces of $\boldsymbol{\tau}$ vanish

$$\operatorname{tr}_1(\boldsymbol{\tau}) = 0, \quad \operatorname{tr}_2(\boldsymbol{\tau}) = 0, \tag{A.4}$$

and

$$Q_{\mathcal{N}_f}(\boldsymbol{\tau}) = 0 \text{ for } f \in \Delta_r(T), r = 0, \ldots, d-2.$$

Then apply the integration by parts (2.1) and the fact $\operatorname{div}\operatorname{div}\boldsymbol{\tau} \in \mathbb{P}_1(T)$ to conclude $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$ and consequently

$$\boldsymbol{\tau} \in \mathbb{P}_2(T; \mathbb{S}), \quad (\boldsymbol{\tau}, \nabla^2 v)_T = 0 \quad \forall v \in H^2(T).$$

Then the vanishing DoF (2.15e) and the decomposition (A.1) imply

$$(\boldsymbol{\tau}, \boldsymbol{q})_T = 0 \quad \forall \boldsymbol{q} \in \mathbb{P}_1(T; \mathbb{S}). \tag{A.5}$$

Recall that $\boldsymbol{\tau}$ is represented as a symmetric matrix function $(\tau_{ij}(x))$ in the coordinate $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d\}$. We are going to show $\tau_{ij} = 0$ for all $1 \le i \le j \le d$. As $\boldsymbol{\tau}$ is quadratic, being orthogonal to $\mathbb{P}_1(T; \mathbb{S})$ is not enough to conclude $\boldsymbol{\tau} = 0$. More conditions will be derived from vanishing DoFs.

Step 3 (Diagonal is zero). By $\operatorname{tr}_1(\boldsymbol{\tau}) = 0$, it follows

$$\tau_{ii}|_{F_i} = |\nabla\lambda_i|^2 \boldsymbol{n}_i^\intercal \boldsymbol{\tau} \boldsymbol{n}_i|_{F_i} = 0, \quad i = 1, \ldots, d.$$

For each $i = 1, \ldots, d$, there exists $p_i \in \mathbb{P}_1(K)$ satisfying $\tau_{ii} = \lambda_i p_i$. Taking $\boldsymbol{q} = p_i \boldsymbol{n}_i \boldsymbol{n}_i^\intercal$ in (A.5) will produce

$$\tau_{ii} = 0, \quad i = 1, \ldots, d.$$

Namely the diagonal of $\boldsymbol{\tau}$ is zero. Notice that the index $i = 1, \dots d$ not including $i = 0$. Will use vanishing $\boldsymbol{n}_{F_0}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_0}|_{F_0}$ in the last step.

Step 4 (Off-diagonal: the last row/column). By $Q_{\mathcal{N}_e}(\boldsymbol{\tau}) = 0$ in (A.4), we have

$$\Pi_F(\boldsymbol{\tau}\boldsymbol{n}_F) \in \mathbb{B}_2^{\operatorname{div}}(F) \quad \text{for each } F \in \partial T.$$

As $\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_i} = 0$ in T for $i = 1, \dots, d$, it follows $\partial_{n_{F_i}}(\boldsymbol{n}_{F_i}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_i}) = 0$, and $\operatorname{tr}_2(\boldsymbol{\tau}) = 0$ becomes

$$\operatorname{div}_{F_r}(\Pi_{F_r}(\boldsymbol{\tau}\boldsymbol{n}_{F_r}))|_{F_r} = 0, \quad r = d, \dots, 1. \tag{A.6}$$

Again $r = 0$ is not included in (A.6).

Consider $r = d$ in (2.15d). As $\Pi_{F_d}(\boldsymbol{\tau}\boldsymbol{n}_{F_d}) \in \mathbb{B}_2^{\operatorname{div}}(F_d)$ and

$$(\Pi_{f_{0:d-2}}\boldsymbol{\tau}\boldsymbol{n}_{F_d}, \boldsymbol{q})_{f_{0:d-2}} = 0, \quad \boldsymbol{q} \in \mathbb{B}_2^{\operatorname{div}}(f_{0:d-2}),$$

applying Lemma A.4 to $(d-1)$-dimensional simplex F_d, we conclude $(\Pi_{F_d}\boldsymbol{\tau}\boldsymbol{n}_{F_d})|_{F_d} = 0$. Together with the vanishing normal-normal component, we have $\boldsymbol{\tau}\boldsymbol{n}_{F_d}|_{F_d} = 0$.

Then there exists $\boldsymbol{p} \in \mathbb{P}_1(T; \mathbb{R}^d)$ such that $\boldsymbol{\tau}\boldsymbol{n}_{F_d} = \lambda_d \boldsymbol{p}$. Take $\boldsymbol{q} = \operatorname{sym}(\boldsymbol{p} \otimes \boldsymbol{n}_{F_d})$ in (A.5) to conclude $\boldsymbol{\tau}\boldsymbol{n}_{F_d} = 0$ in T. That is the last column of the symmetric matrix representation of $\boldsymbol{\tau}$ is zero.

Step 5 (Off-diagonal: the r-th row/column). Assume we have proved the ℓ-th columns are zero for $\ell > r$. By symmetry and vanishing normal-normal component $\boldsymbol{n}_{F_\ell}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_r} = 0$ for $\ell \geq r$. Expand in the edge coordinate $\boldsymbol{\tau}\boldsymbol{n}_{F_r} = \sum_{i=1}^{r-1} p_i(x)\boldsymbol{t}_i$ with $p_i(x) \in \mathbb{P}_2(T)$. So

$$\Pi_{F_r}\boldsymbol{\tau}\boldsymbol{n}_{F_r}|_{F_r} = \sum_{e \in \Delta_1(f_{0:r-1})} c_e b_e(x)\boldsymbol{t}_e \in \mathbb{B}_2^{\operatorname{div}}(F_r) \text{ with } c_e \in \mathbb{R},$$

which contains only the edge bubble corresponding to edges of simplex $f_{0:r-1}$. Notice that $\Pi_{F_r}\boldsymbol{\tau}\boldsymbol{n}_{F_r}|_{f_{0:r-2}} \in \mathbb{B}_2^{\operatorname{div}}(f_{0:r-2})$. The vanishing (2.15d) on $f_{0:r-2}$ will further rule out the edge bubbles on $f_{0:r-2}$ and simplify to

$$\Pi_{F_r}\boldsymbol{\tau}\boldsymbol{n}_{F_r}|_{F_r} = \sum_{i=0}^{r-2} c_i b_{e_{r-1,i}}(x)\boldsymbol{t}_{i,r-1}.$$

Use $-(\operatorname{div}_{F_r}\Pi_{F_r}\boldsymbol{\tau}\boldsymbol{n}_{F_r}, \lambda_i)_{F_r} = \frac{|F_r|}{(d+1)d}c_i = 0$ to conclude $\Pi_{F_r}\boldsymbol{\tau}\boldsymbol{n}_{F_r}|_{F_r} = 0$. Together with the vanishing normal-normal component, we have $\boldsymbol{\tau}\boldsymbol{n}_{F_r}|_{F_r} = 0$. The rest to prove $\boldsymbol{\tau}\boldsymbol{n}_{F_r} = 0$ in T is like Step 4.

Step 6 (Entry τ_{12}). Only one entry τ_{12} is left, i.e., $\boldsymbol{\tau} = 2\tau_{12}\operatorname{sym}(\boldsymbol{t}_1\boldsymbol{t}_2^{\intercal})$. Multiplying $\boldsymbol{\tau}$ by $\nabla\lambda_0$ from both sides and restricting to F_0, we have

$$\tau_{12}|_{F_0} = \frac{1}{2}|\nabla\lambda_0|^2(\boldsymbol{n}_{F_0}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F_0})|_{F_0} = 0.$$

Again there exists $p \in \mathbb{P}_1(K)$ satisfying $\tau_{12} = \lambda_0 p$. Taking $\boldsymbol{q} = \operatorname{sym}(\boldsymbol{t}_1\boldsymbol{t}_2^{\intercal})p$ in (A.5) gives $\tau_{12} = 0$. We thus have $\boldsymbol{\tau} = 0$ and consequently prove the unisolvence. $\square$

Corollary A.6. *The DoFs*

$$(\mathrm{tr}_e(\boldsymbol{\tau}), q)_e, \quad q \in \mathbb{P}_2(e), e \in \Delta_{d-2}(T), \tag{A.7a}$$

$$(\boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_2(F), F \in \partial T, \tag{A.7b}$$

$$(\mathrm{tr}_2(\boldsymbol{\tau}), q)_F, \quad q \in \mathbb{P}_1(F)/\mathbb{R}, F \in \partial T, \tag{A.7c}$$

$$(\Pi_f \boldsymbol{\tau} \boldsymbol{n}_{F_r}, \boldsymbol{q})_f, \quad \boldsymbol{q} \in \mathbb{B}_2^{\mathrm{div}}(f), f = f_{0:r-2} \in \Delta_{r-2}(F_r), r = d, \ldots, 3, \tag{A.7d}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \ker(\boldsymbol{x}^\intercal \cdot \boldsymbol{x}) \cap \mathbb{P}_1(T; \mathbb{S}), \tag{A.7e}$$

$$(\mathrm{div}\,\mathrm{div}\, \boldsymbol{\tau}, q)_T, \quad q \in \mathbb{P}_0(T), \tag{A.7f}$$

are unisolvent for $\mathbb{P}_2(T; \mathbb{S})$.

Proof. Compared with DoFs (2.15) for $\Sigma_{2^+}(T; \mathbb{S})$, the number of DoFs (A.7) equals $\dim \mathbb{P}_2(T; \mathbb{S})$. Assume $\boldsymbol{\tau} \in \mathbb{P}_2(T; \mathbb{S})$ and all the DoFs (A.7) vanish. By the vanishing DoFs (A.7a)–(A.7c) and (A.7f), we have $\mathrm{tr}_e(\boldsymbol{\tau}) = 0$ for $e \in \Delta_{d-2}(T)$, $(\boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n})|_F = 0$ and $\mathrm{tr}_2(\boldsymbol{\tau})|_F \in \mathbb{P}_0(F)$ for $F \in \partial T$, and $\mathrm{div}\,\mathrm{div}\, \boldsymbol{\tau} = 0$. Apply (2.13) to get

$$\sum_{F \in \partial T} (\mathrm{tr}_2(\boldsymbol{\tau}), v)_F = 0, \quad v \in \mathbb{P}_1(T),$$

which implies $\mathrm{tr}_2(\boldsymbol{\tau}) = 0$. Finally, $\boldsymbol{\tau} = 0$ follows from Theorem A.5. ☐

The finite element space defined by (A.7) is not $H(\mathrm{div}\,\mathrm{div})$-conforming as $\mathrm{tr}_2(\tau)$ is not continuous. It will be used in the proof of norm equivalence in Appendix B.

Appendix B. Norm equivalence

For $u \in \mathring{M}_{r,k-1,k,k}^{-1}$ with $k \geq 0$, define a discrete H^2-norm:

$$\begin{aligned} |u|_{2,h}^2 = \sum_{T \in \mathcal{T}_h} \left(h_T^{-4} \| Q_{r,T} u^{\mathrm{CR}} - u_0 \|_{0,T}^2 + \sum_{F \in \partial T} h_T^{-3} \| Q_{k-1,F} u^{\mathrm{CR}} - u_b \|_{0,F}^2 \right) \\ + \sum_{T \in \mathcal{T}_h} \left(\sum_{F \in \partial T} h_T^{-1} \| \partial_{n_F} u^{\mathrm{CR}} - u_n \|_{0,F}^2 + \sum_{e \in \Delta_{d-2}(T)} h_T^{-2} \| Q_{k,e} u^{\mathrm{CR}} - u_e \|_{0,e}^2 \right), \end{aligned}$$

where $u^{\mathrm{CR}} = I^{\mathrm{CR}}(Q_M^{-1} u)$ with I^{CR} being the interpolation operator to the nonconforming linear element space and Q_M^{-1} is the bijection from $\mathring{M}_{r,k-1,k,k}^{-1}$ to $\mathring{V}_{k+2}^{\mathrm{VEM}}$. When $k = 0, 1, r < 0$, it is simplified to

$$|u|_{2,h}^2 = \sum_{T \in \mathcal{T}_h} \left(\sum_{F \in \partial T} h_T^{-1} \| \partial_{n_F} u^{\mathrm{CR}} - u_n \|_{0,F}^2 + \sum_{e \in \Delta_{d-2}(T)} h_T^{-2} \| Q_{k,e} u^{\mathrm{CR}} - u_e \|_{0,e}^2 \right).$$

Lemma B.1. *On the space* $\mathring{M}_{r,k-1,k,k}^{-1}$*, we have the norm equivalence*

$$\| \nabla_w^2 u \|_0 \eqsim |u|_{2,h}, \quad u \in \mathring{M}_{r,k-1,k,k}^{-1} \quad \text{for } k \geq 0. \tag{B.1}$$

Proof. By (4.1) and the Green's identity (2.1), for $\boldsymbol{\tau} \in \Sigma_{k,r}(T; \mathbb{S})$ we have

$$\begin{aligned} (\nabla_w^2 u, \boldsymbol{\tau})_T = (u_0 - Q_{r,T} u^{\mathrm{CR}}, (\mathrm{div}\,\mathrm{div})_T \boldsymbol{\tau})_T - (u_b - Q_{k-1,F} u^{\mathrm{CR}}, \mathrm{tr}_2(\boldsymbol{\tau}))_{\partial T} \\ + (u_n \boldsymbol{n}_F \cdot \boldsymbol{n} - \partial_n u^{\mathrm{CR}}, \boldsymbol{n}^\intercal \boldsymbol{\tau} \boldsymbol{n})_{\partial T} + \sum_{e \in \Delta_{d-2}(T)} (u_e - Q_{k,e} u^{\mathrm{CR}}, [\boldsymbol{n}_{F,e}^\intercal \boldsymbol{\tau} \boldsymbol{n}_{\partial T}]|_e)_e. \end{aligned} \tag{B.2}$$

Then $\|\nabla_w^2 u\|_0 \lesssim |u|_{2,h}$ follows from the Cauchy-Schwarz inequality, and the inverse trace inequality.

Next we prove $|u|_{2,h} \lesssim \|\nabla_w^2 u\|_0$.

Step 1. First consider $k = 0$. By (4.12) and the fact $Q_M^{-1} u \in \mathring{V}_2^{\mathrm{MWX}}$, $\nabla_w^2 u = \nabla_h^2 Q_M^{-1} u$. It follows from the norm equivalence and the error estimate of I^{CR} that

$$|u|_{2,h}^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{-4}\|Q_M^{-1}u - u^{\mathrm{CR}}\|_{0,T}^2 \lesssim \|\nabla_h^2 Q_M^{-1} u\|_0^2 = \|\nabla_w^2 u\|_0^2.$$

Step 2. Next consider $k = 1, 2$, and $r = k-2$. By the DoFs (A.7), we can construct $\boldsymbol{\tau} \in \Sigma_{k,r}(T;\mathbb{S})$ such that

$$\begin{aligned}
[\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{\partial T}]|_e &= h_T^{-2}(u_e - u^{\mathrm{CR}})|_e, && e \in \Delta_{d-2}(T),\\
(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})|_F &= h_T^{-1}(u_n \boldsymbol{n}_F\cdot\boldsymbol{n} - \partial_n u^{\mathrm{CR}})|_F, && F \in \Delta_{d-1}(T),\\
(I - Q_{0,F})\operatorname{tr}_2(\boldsymbol{\tau})|_F &= h_T^{-3}(u^{\mathrm{CR}} - u_b)|_F, && F \in \Delta_{d-1}(T) \text{ if } k=2,\\
\operatorname{div}\operatorname{div}_T \boldsymbol{\tau} &= h_T^{-4}(u_0 - Q_{r,T}u^{\mathrm{CR}}), && \text{if } k=2,
\end{aligned}$$

and all the other DoFs in (A.7) vanish. By the norm equivalence and the scaling argument, we have

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{0,T}^2 \lesssim{}& h_T^{-4}\|Q_{r,T}u^{\mathrm{CR}} - u_0\|_{0,T}^2 + h_T^{-3}\|Q_{k-1,F}u^{\mathrm{CR}} - u_b\|_{0,\partial T}^2\\
&+ h_T^{-1}\|u_n - \partial_{n_F}u^{\mathrm{CR}}\|_{0,\partial T}^2 + \sum_{e\in\Delta_{d-2}(T)} h_T^{-2}\|u^{\mathrm{CR}} - u_e\|_{0,e}^2.
\end{aligned}$$

Substituted into (B.2), we get

$$\begin{aligned}
h_T^{-4}\|Q_{r,T}u^{\mathrm{CR}} - u_0\|_{0,T}^2 + h_T^{-3}\|Q_{k-1,F}u^{\mathrm{CR}} - u_b\|_{0,\partial T}^2 + h_T^{-1}\|\partial_{n_F}u^{\mathrm{CR}} - u_n\|_{0,\partial T}^2&\\
+ \sum_{e\in\Delta_{d-2}(T)} h_T^{-2}\|u^{\mathrm{CR}} - u_e\|_{0,e}^2 = (\nabla_w^2 u, \boldsymbol{\tau})_T \le \|\nabla_w^2 u\|_{0,T}\|\boldsymbol{\tau}\|_{0,T}.&
\end{aligned}$$

We conclude $|u|_{2,h} \lesssim \|\nabla_w^2 u\|_0$ by combining the last two inequalities.

Step 3. Consider $k \ge 2$ and $r \ge 1$. By the DoFs (2.3) or (2.15), we can construct $\boldsymbol{\tau} \in \mathbb{P}_k(T;\mathbb{S})$ such that

$$\begin{aligned}
[\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{\partial T}]|_e &= h_T^{-2}(u_e - u_0)|_e, && e \in \Delta_{d-2}(T),\\
(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})|_F &= h_T^{-1}(u_n \boldsymbol{n}_F\cdot\boldsymbol{n} - \partial_n u_0)|_F, && F \in \Delta_{d-1}(T),\\
\operatorname{tr}_2(\boldsymbol{\tau})|_F &= h_T^{-3}(u_0 - u_b)|_F, && F \in \Delta_{d-1}(T),\\
(\boldsymbol{\tau}, \boldsymbol{q})_T &= (\nabla_h^2 u_0, \boldsymbol{q})_T, && \boldsymbol{q} \in \nabla^2\mathbb{P}_r(T),
\end{aligned} \tag{B.3}$$

and all the other DoFs in (2.3) and (2.15) vanish. By the norm equivalence and the scaling argument, we have

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{0,T}^2 \lesssim{}& \|\nabla_h^2 u_0\|_{0,T}^2 + h_T^{-3}\|Q_{k-1,F}u_0 - u_b\|_{0,\partial T}^2 + h_T^{-1}\|u_n - \partial_{n_F}u_0\|_{0,\partial T}^2\\
&+ \sum_{e\in\Delta_{d-2}(T)} h_T^{-2}\|u_0 - u_e\|_{0,e}^2.
\end{aligned}$$

By (4.2) we get

$$\begin{aligned}
\|\nabla_h^2 u_0\|_{0,T}^2 &+ h_T^{-3}\|u_0 - u_b\|_{0,\partial T}^2 + h_T^{-1}\|\partial_{n_F}u_0 - u_n\|_{0,\partial T}^2\\
&+ \sum_{e\in\Delta_{d-2}(T)} h_T^{-2}\|u_0 - u_e\|_{0,e}^2 = (\nabla_w^2 u, \boldsymbol{\tau})_T \le \|\nabla_w^2 u\|_{0,T}\|\boldsymbol{\tau}\|_{0,T}.
\end{aligned}$$

Finally, we obtain $|u|_{2,h} \lesssim \|\nabla_w^2 u\|_0$ by combining the last two inequalities. □

Lemma B.2. *We have the norm equivalence*

$$\|\nabla_w^2 Q_M v\|_0 = \|Q_\Sigma \nabla_h^2 v\|_0 \eqsim \|\nabla_h^2 v\|_0, \quad v \in \mathring{V}_{k+2}^{\mathrm{VEM}}, k \geq 0. \tag{B.4}$$

Proof. First (B.4) is obviously true for $k = 0$, since $\nabla_w^2 Q_M v = Q_\Sigma \nabla_h^2 v = \nabla_h^2 v$. Then we focus on $k \geq 1$.

By the norm equivalence (B.1), it suffices to prove

$$|Q_M v|_{2,h} \eqsim \|\nabla_h^2 v\|_0, \quad v \in \mathring{V}_{k+2}^{\mathrm{VEM}}. \tag{B.5}$$

By the definition of $|Q_M v|_{2,h}$ and v^{CR}, and the norm equivalence on $V_{k+2}^{\mathrm{VEM}}(T)$,

$$\begin{aligned}
|Q_M v|_{2,h}^2 = & \sum_{T \in \mathcal{T}_h} h_T^{-4} \|Q_{r,T}(v^{\mathrm{CR}} - v)\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} \sum_{e \in \Delta_{d-2}(T)} h_T^{-2} \|Q_{k,e}(v^{\mathrm{CR}} - v)\|_{0,e}^2 \\
& + \sum_{T \in \mathcal{T}_h} \sum_{F \in \partial T} (h_T^{-1} \|Q_{k,F} \partial_{n_F}(v^{\mathrm{CR}} - v)\|_{0,F}^2 + h_T^{-3} \|Q_{k-1,F}(v^{\mathrm{CR}} - v)\|_{0,F}^2) \\
& \eqsim \sum_{T \in \mathcal{T}_h} h_T^{-4} \|v^{\mathrm{CR}} - v\|_{0,T}^2.
\end{aligned}$$

Therefore, (B.5) follows from the inverse inequality and the interpolation estimate of the non-conforming linear element. □

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