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TERNARY AND NON-ASSOCIATIVE STRUCTURES

RICHARD KERNER

Laboratoire LPTMC, Université Pierre-et-Marie-Curie
75252 Paris Cedex 05, France
rk@ccr.jussieu.fr

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A Michel Dubois-Violette, associé et complice dans la quête d'absolu
avec amitié et admiration.

We discuss ternary algebraic structures appearing in various domains of theoretical and mathematical physics. Some of them are associative, and some are not. Their interesting and curious properties can be exploited in future applications to enlarged and generalized field theoretical models in the years to come.

Many ideas presented here have been developed and clarified in countless discussions with Michel Dubois-Violette.

Keywords: Ternary structures; non-associative algebras; Z_3-grading.

1. Introduction

The development of our vision of nature passed through many stages, including blind alleys. It was followed by the development of mathematical structures necessary for the description of growing complexity of physical theories and models. Natural numbers and fractions were generalized towards real numbers, then the complex plane has appeared as the arena for all kinds of phenomena described by second order ordinary differential equations.

The ondulatory phenomena, waves and oscillations, naturally require complex numbers. Quantum mechanical effects are principally stemming from the fact that the square of a superposition of two amplitudes is not equal to the square of the sum of their absolute values. The observables became elements of an associative, but non-commutative algebra. This enlarged the domain of algebraic structures needed to take into account the complexity of quantum phenomena.

The next step was necessary to take into account the phenomena of variable number of particles and quantum fields. This development is known as the second quantization. It involves the algebra of operators as primary object, so that all other relationships, including the space-time symmetries, can be derived from the properties of this algebra, and not vice versa. The most elegant mathematical formulation of these ideas was introduced by von Neumann.

The von Neumann algebraic approach led later to the concept of Non-Commutative Geometry developed in the last decade of XX-th century. Some of the most important contrubutors are among the participants of this Colloquium. The development of String Theory since more than thirty years now represents a radically different approach which introduces a continuum of higher order than the usual functional Hilbert space. Here we propose another alley, still quite unexplored: the non-associative and ternary algebraic structures.

The discovery of deeper layers of fundamental interactions presented above and followed by introduction of more sophisticated algebraic structures can be schematically represented as follows:

Classical Mechanics and Classical Statistical Mechanics are described by finite-dimensional real spaces and smooth functions on these manifolds (the Galilean time and space):

$$\mathbf{R}^1 \times \mathbf{R}^3, \quad \mathbf{R}^1 \times \mathbf{R}^{3N}, \quad \mathcal{C}^\infty(\mathbf{R}^1 \times \mathbf{R}^3).$$

Quantum Mechanics use the same Galilean time, but instead of the three-dimensional space it uses the complex functions on the manifold, and denumerable infinite-dimensional Hilbert space as the space of states of a given system, and finally, the algebra of hermitian operators acting on the Hilbert space:

$$\mathbf{R}^1 \times \mathbf{C}^1, \quad \mathbf{R}^1 \times \mathbf{C}^N; \quad \mathcal{C}^\infty(\mathbf{R}^1 \times \mathbf{R}^3); \quad \mathcal{H}, \quad \mathcal{A}(\mathcal{H}).$$

Quantum Field Theory performed a step forward introducing the C-star algebras, the infinite-dimensional algebras acting on the Fock space of states:

$$\mathcal{A}; \quad C^*(\mathcal{A}).$$

The so-called String Theory introduces infinities of yet higher order: functional spaces containing all possible configurations of continuously embedded manifolds.

The Non-Commutative Geometry follows an opposite direction: it tends to expunge the description of elementary particles from unnecessary degrees of freedom, leaving the strict minimum which can be described by a few points and finite-dimensional matrices.

Until now, the non-associative structures, e.g. octonions or Jordan algebras, did not seem to be of great interest to elementary particle and field-theoretical physicists. However, the situation might change if we are forced to think over again the basics of the Standard Model, in particular the Higgs–Kibble mechanism of mass generation, if the Higgs boson does not show up in the Large Hadron Collider in the next few years.

Finally, let us mention another unexploited alley: ternary algebraic structures, in which the multiplication is defined for three items instead of two, as in classical algebraic structures. They may represent another, yet unexplored stage of mathematical description of quantum fields, helping to explain why the quarks constituting fermions can be observed only in packs by three.

2. Examples of Ternary Structures

Ternary algebraic operations and cubic relations have been considered, although quite sporadically, by several authors already in the ninetieth century by Cayley [1] and Sylvester [2]. The development of Cayley's ideas, which contained a cubic generalization of matrices and their determinants, can be found in the book by Kapranov, Gelfand and Zelevinskii [3]. A discussion of the next step in generality, the so called *n-ary* algebras, can be found in [4]. Here we shall focus our attention on the *ternary* and *cubic* algebraic structures only.

We shall introduce the following distinction between these two denominations: we shall call a *ternary algebraic structure* any linear space V endowed with one or more ternary composition laws:

$$m_3 : V \otimes V \otimes V \Rightarrow V \quad \text{or} \quad m_3' : V \otimes V \otimes V \Rightarrow \mathbf{C},$$

the second law being an analogue of a scalar product in the usual (binary) case.

We shall call a *cubic structure* or an algebra generated by cubic relations, an ordinary algebra with a binary composition law:

$$m_2 : V \otimes V \Rightarrow V$$

with cubic constitutive relations for the generators: e.g. $(abc) = e^{2\pi i/3}(bca)$.

Some of ternary operations and cubic relations are so familiar that we do not pay much attention to their particular character. Let us cite as example the triple product of vectors in 3-dimensional Euclidean vector space:

$$\{a, b, c\} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

which is a tri-linear mapping from $E \otimes E \otimes E$ onto $\mathbf{R^1}$, invariant under the cyclic group Z_3.

Curiously enough, it is in the 4-dimensional Minkowskian space-time M_4 where a natural ternary composition of 4-vectors can be easily defined:

$$(X, Y, Z) \to U(X, Y, Z) \in M_4$$

with the resulting 4-vector U^μ defined via its components in a given coordinate system as follows:

$$U^\mu(X, Y, Z) = g^{\mu\sigma} \eta_{\sigma\nu\lambda\rho} X^\nu Y^\lambda Z^\rho, \quad \text{with } \mu, \nu, \ldots = 0, 1, 2, 3$$

where $g^{\mu\nu}$ is the metric tensor, and $\eta_{\mu\nu\lambda\rho}$ is the canonical volume element of M_4.

Other examples of "ternary ideas" that we should cite here are:

— The *cubic matrices* and a generalization of the determinant, called the "*hyper-determinant*", first introduced by Cayley in 1840, then found again and generalized by Kapranov, Gelfand and Zelevinskii in 1990 [3]. The simplest example of this (non-commutative and non-associative) ternary algebra is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{mkn}, \quad i, j, k \ldots = 1, 2, \ldots, N.$$

Other ternary rules can be obtained from this one by taking various linear combinations, with real or complex coefficients, of the above 3-product, e.g.

$$[a, b, c] = \{a, b, c\} + j\{b, c, a\} + j^2\{c, a, b\} \quad \text{with } j = e^{2\pi i/3}.$$

— the algebra of "*nonions*", introduced by Sylvester as a ternary analog of Hamilton's quaternions. The "nonions" are generated by two matrices:

$$\eta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}$$

and all their linearly independent powers; the constitutive relations are of cubic character:

$$\sum_{\text{perm.}(ikm)} \Gamma_i \Gamma_k \Gamma_m = \delta_{ikm} \mathbf{1}$$

where δ_{ikm} is equal to 1 when $i = k = m$ and 0 otherwise.

— a cubic analog of Laplace and d'Alembert equations, first considered by Humbert [5] in 1934: the third-order differential operator that generalized the Laplacian was

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + j^2\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + j^2\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}\right)$$
$$= \frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - 3\frac{\partial^3}{\partial x \partial y \partial z}.$$

Ternary and cubic algebras have been studied by Lawrence, Dabrowski *et al.* [6], Plyushchay and Rausch de Traubenberg [7], and other authors.

3. Examples of Ternary Relations in Physics

The quark model inspired a particular brand of ternary algebraic systems, intended to explain the non-observability of isolated quarks as a phenomenon of "algebraic confinement". One of the first such attempts has been proposed by Nambu [8] in 1973, and known under the name of "*Nambu mechanics*" since then [9].

Consider a 3-dimensional real space parameterized by Cartesian coordinates, with $\vec{r} = (x, y, z) \in \mathbf{R}^3$. Introducing two smooth functions $H(x, y, z)$ and $G(x, y, z)$, one may define the following ternary analog of the Poisson bracket and dynamical equations: for a given function $f(x, y, z)$ defined on our 3-dimensional space, its time derivative is postulated to be:

$$\frac{d\vec{r}}{dt} = (\vec{\nabla} H) \times (\vec{\nabla} G) \tag{3.1}$$

or more explicitly, because we have

$$\frac{dx}{dt} = \det\left(\frac{\partial(H,G)}{\partial(y,z)}\right), \quad \frac{dy}{dt} = \det\left(\frac{\partial(H,G)}{\partial(z,x)}\right), \quad \frac{dz}{dt} = \det\left(\frac{\partial(H,G)}{\partial(x,y)}\right),$$

we can write

$$\frac{df}{dt} = (\vec{\nabla}) \cdot (\vec{\nabla} H) \times (\vec{\nabla} G) = \det\left(\frac{\partial(f,G,H)}{\partial(x,y,z)}\right) = [f,H,G]. \tag{3.2}$$

The so defined "*ternary Poisson bracket*" satisfies obvious relations:

(i) $[A, B, C] = -[B, A, C] = [B, C, A]$;

(ii) $[A_1 A_2, B, C] = [A_1, B, C]A_2 + A_1[A_2, B, C]$;

(iii) $\vec{\nabla} \cdot \left(\frac{d\vec{r}}{dt}\right) = \vec{\nabla} \cdot (\vec{\nabla} H \times \vec{\nabla} G) = 0.$

A canonical transformation $(x, y, z) \Rightarrow (x', y', z')$ is readily defined as a smooth coordinate transformation whose determinant is equal to 1:

$$[x', y', z'] = \det\left(\frac{\partial(x',y',z')}{\partial(x,y,z)}\right) = 1,$$

so that one automatically has:

$$\frac{df}{dt} = \det\left(\frac{\partial(f,H,G)}{\partial(x,y,z)}\right) = \det\left(\frac{\partial(f,H,G)}{\partial(x',y',z')}\right). \tag{3.3}$$

It is easily seen that *linear canonical transformations* leaving this ternary Poisson bracket invariant form the group $SL(3, \mathbf{R})$.

The dynamical equations describing the *Euler top* can be cast into this new ternary mechanics scheme, if we identify the vector $\vec{r}$ with the components of the angular momentum $\vec{L} = [L_x, L_y, L_z]$, and the two "Hamiltonians" with the following functions of the above:

$$H = \frac{1}{2}[L_x^2 + L_y^2 + L_z^2], \quad G = \frac{1}{2}\left[\frac{L_x^2}{J_x} + \frac{L_y^2}{J_y} + \frac{L_z^2}{J_z}\right]. \tag{3.4}$$

Yamaleev has found an interesting link between the Nambu mechanics and ternary Z_3-graded algebras [10].

The Yang–Baxter equation provides another celebrated cubic relation imposed on the bilinear operators named $\tilde{R}$-matrices: for $\tilde{R}_{km} : V \otimes V \to V \otimes V$, one has

$$\tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23} = \tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12}, \tag{3.5}$$

where the indices refer to various choices of *two* out of *three* distinct specimens of the vector space V.

An alternative form of this relation is more widely used. Let P be the operator of permutation, $P : V_1 \otimes V_2 \to V_2 \otimes V_1$ and let us introduce another R-matrix by defining $\tilde{R} = P \circ R$. Then this relation takes on the following form:

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}. \tag{3.6}$$

Applications of this equation are innumerable indeed; they serve to solve many integrable systems, such as Toda lattices; they also give the representations of braid groups, etc.

In a given local basis of $V \otimes V, e_i \otimes e_k$, we can write, for $X = X^i e_i, Y = Y^k e_k$,

$$R(X,Y) = R^{km}_{ij} X^i Y^j e_k \otimes e_m.$$

An interesting ternary aspect of these R-matrices has been discovered by Okubo [11] in search for new solutions of Yang–Baxter equations. Introducing a supplementary real parameter θ, we can write this equation as follows:

$$R^{b'a'}_{a_1 b_1}(\theta) R^{c'a_2}_{a'c_1}(\theta') R^{c_2 b_2}_{b'c'}(\theta'') = R^{c'b'}_{b_1 c_1}(\theta'') R^{c_2 a'}_{a_1 c'}(\theta') R^{b_2 a_2}_{a'b'}(\theta), \tag{3.7}$$

with $\theta' = \theta + \theta''$.

An entire class of solutions of Yang–Baxter equation, including the ones found by de Vega and Nicolai, can be obtained in terms of *triple product systems* if the matrix R satisfies an extra symmetry condition:

$$R^{ba}_{dc}(\theta) = R^{ab}_{cd}(\theta). \tag{3.8}$$

Okubo considered the following *symplectic* and *orthogonal* triple systems, i.e. vector spaces (denoted by V) endowed simultaneously with a non-degenerate bi-linear form

$$\langle x, y \rangle : V \otimes V \to \mathbf{C}^1, \quad x, y \in V$$

and a triple product $\{x, y, z\} : V \otimes V \otimes V \to V, x, y, z \in V$ satisfying certain natural assumptions about the relationship between these two products (the details can be found in [11]).

In a chosen basis of V, $(e_1, e_2, \ldots, e_N)$, one can write

$$\langle e_i, e_k \rangle = g_{ik} = \varepsilon g_{ki}, \quad \text{and} \quad \{e_i, e_k, e_m\} = C^j_{ikm} e_j$$

where the coefficients C^j_{ikm} play the rôle of ternary structure constants.

With the help of the inverse metric tensor, g^{jk}, we can now raise the lower-case indexes, defining the contravariant basis $e^k = g^{km} e_m$. If a one-parameter family of triple products is defined, $\{e_i, e_k, e_m\}_\theta$, then we may define an R-matrix depending on the same parameter θ:

$$R^{ij}_{km} = \langle e^i, \{e^j, e_k, e_m\}_\theta \rangle$$

or equivalently,

$$\{e^b, e_c e_d\}_\theta = R^{ab}_{cd} e_a. \tag{3.9}$$

The symmetry condition $R^{ba}_{dc}(\theta) = R^{ab}_{cd}(\theta)$ can be now written as

$$\langle u, \{x, y, z\}_\theta \rangle = \langle z, \{y, x, u\}_\theta \rangle$$

and the Yang–Baxter equation becomes equivalent with an extra condition imposed on the ternary product:

$$\sum_a \{v, \{u, e_a, z\}_{\theta'}, \{e^a, x, y\}_\theta\}_{\theta"} = \sum_a \{u, \{v, e_a, x\}_{\theta'}, \{e^a, z, y\}_{\theta"}\}_\theta. \tag{3.10}$$

Using this encoded form of the Yang–Baxter equation, Okubo was able to find a series of new solutions just by finding 1-parameter families of ternary products satisfying the above constraints.

This original approach suggests another possibility of introducing ternary structures in the very fabric of traditional quantum mechanics. As we know, any bounded linear operator acting in Hilbert space $\mathcal{H}$ can be represented as

$$A : \mathcal{H} \to \mathcal{H}, \quad A = \sum_{k,m} a_{km} |e_k\rangle\langle e_m|; \tag{3.11}$$

where $|e_k\rangle$ is a basis in Hilbert space. If now

$$|x\rangle = \sum_k c_m |e_m\rangle,$$

then one has

$$A|x\rangle = \sum_{i,k,m} a_{ik} |e_i\rangle\langle e_k|c_m|e_m\rangle = \sum_{i,k,m} a_{ik} c_m \delta_{km} |e_i\rangle = \sum_{km} a_{km} c_m |e_m\rangle. \tag{3.12}$$

Each item in this sum can be considered as a result of *ternary multiplication* defined in the Hilbert space of states:

$$m(|e_i\rangle, |e_j\rangle, |e_k\rangle) = |e_i\rangle\langle e_j|e_k\rangle = \delta_{jk}|e_i\rangle = \sum_n \delta_{jk}\delta^n_i |e_n\rangle, \tag{3.13}$$

with the structure constants defined as $C^n_{ijk} = \delta_{jk}\delta^n_i$. Using this interpretational scheme, the states and the observables (operators) are no more separate entities, but can interact with each other: by superposing triplets of states, we arrive at the result which amounts to changing both the state and the observable simultaneously.

Similar constructions, often referred to as *algebraic confinement*, were considered by many authors, in particular by Lipkin quite a long time ago [12].

Consider an algebra of operators $\mathcal{O}$ acting on a Hilbert space $\mathcal{H}$ which is a free module with respect to the algebra $\mathcal{O}$, endowed with Hilbertian scalar product. Let us introduce tensor products of the algebra and the module with the following Z_3-graded matrix algebra $\mathcal{A}$ over the complex field $\mathbf{C}^1$:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2, \quad \mathcal{A} \in \mathrm{Mat}(3, \mathbf{C}).$$

The three linear subspaces of $\mathcal{A}$, of which only $\mathcal{A}_0$ forms a subalgebra, are defined as follows:

$$\mathcal{A}_0 := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \mathcal{A}_1 := \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 := \begin{pmatrix} 0 & 0 & \gamma \\ \beta & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}.$$

It is easy to check that under matrix multiplication the degrees $0, 1$ and 2 add up modulo 3: a product of two elements of degree 1 belongs to $\mathcal{A}_2$, the product of two elements of degree 2 belongs to $\mathcal{A}_1$, and the product of an element of degree 1 with an element of degree 2 belongs to $\mathcal{A}_0$, etc.

With this in mind, a generalized state vector can now belong to one of these subspaces, e.g. $|\Psi\rangle$ of degree one, its hermitian conjugate being automatically of Z_3-degree 2:

$$|\Psi\rangle := \begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix} \quad \langle\Psi| := \begin{pmatrix} 0 & 0 & \langle\psi_3| \\ \langle\psi_1| & 0 & 0 \\ 0 & \langle\psi_2| & 0 \end{pmatrix}$$

with $|\psi_k\rangle \in \mathcal{H}$. The scalar product obviously generalizes as follows:

$$\begin{aligned} \langle\Phi|\Psi\rangle &:= Tr\left[\begin{pmatrix} 0 & 0 & \langle\phi_3| \\ \langle\phi_1| & 0 & 0 \\ 0 & \langle\phi_2| & 0 \end{pmatrix}\begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix}\right] \\ &= \langle\phi_1|\psi_1\rangle + \langle\phi_2|\psi_2\rangle + \langle\phi_3|\psi_3\rangle. \end{aligned} \tag{3.14}$$

With this definition of scalar product any expectation value of an operator of degree 1 or 2 (represented by the corresponding traceless matrices) will identically vanish, e.g. for an operator of degree 1: ($\mathcal{D}_i \in \mathcal{O}$)

$$Tr\left[\begin{pmatrix} 0 & 0 & \langle\psi_3| \\ \langle\psi_1| & 0 & 0 \\ 0 & \langle\psi_2| & 0 \end{pmatrix}\begin{pmatrix} 0 & D_1 & 0 \\ 0 & 0 & D_2 \\ D_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & |\psi_1\rangle & 0 \\ 0 & 0 & |\psi_2\rangle \\ |\psi_3\rangle & 0 & 0 \end{pmatrix}\right] = 0.$$

It is clear that only the operators whose Z_3-degree is 0 may have non-vanishing expectation values, because the operators of degrees 1 and 2 are traceless. Denoting the operator of degree 1 by Q, and the operators of degree 2 by $\bar{Q}$, the only combinations that can be observed, i.e. that can lead to non-vanishing expectation values no matter what the nature of the operator and the observable it is supposed to represent, are the following products:

$$QQQ; \quad \bar{Q}\bar{Q}\bar{Q}; \quad Q\bar{Q} \quad \text{and} \quad \bar{Q}Q \tag{3.15}$$

which correspond to the observable combinations (tensor products) of the fields supposed to describe the quarks. This particular realization of "*algebraic confinement*" suggests the importance of ternary and cubic relations in algebras of observables.

4. Cubic Grassmann and Clifford Algebras

A general 3-algebra (or *ternary algebra*) is defined as internal ternary multiplication in a vector space V. Such a multiplication must be of course 3-linear, but not necessarily associative:

$$m : V \otimes V \otimes V \to V; \quad m(X, Y, Z) \in V. \tag{4.1}$$

Such a 3-product is said to be *strongly associative* if one has

$$m(X, m(S, Y, T), Z) = m(m(X, S, Y), T, Z) = m(X, S, m(Y, T, Z)). \tag{4.2}$$

Of course, any associative binary algebra can serve as starting point for introduction of a (not necessarily associative) ternary algebra, by defining its ternary product:

$$\begin{aligned} &(*) \quad (X, Y, Z) = XYZ \quad \text{(trivial)};\\ &(**) \quad \{X, Y, Z\} = XYZ + YZX + ZXY \quad \text{(symmetric)};\\ &(***) \quad [X, Y, Z] = XYZ + jYZX + j^2 ZXY \quad (j\text{-skew-symmetric}) \end{aligned} \tag{4.3}$$

where we set $j = e^{2\pi i/3}$, the primitive cubic root of unity. It is worthwhile to note that the last cubic algebra, which is a direct generalization of the Z_2-graded skew-symmetric product $[X, Y] = XY - YX$, which defines the usual Lie algebra product, contains it as a special substructure if the underlying associative algebra is unital. Indeed, if $\mathbf{1}$ is the unit of that algebra, one easily checks that substituting it in place of the second factor of the skew-symmetric ternary product, one gets:

$$\{X, \mathbf{1}, Z\} := X\mathbf{1}Z + j\mathbf{1}ZX + j^2 ZX\mathbf{1} = XZ + (j + j^2)ZX = XZ - ZX, \tag{4.4}$$

because of the identity $j + j^2 + 1 = 0$, so that the usual Lie-algebraic structure is recovered as a special case.

In general, a ternary algebra cannot be derived from an associative binary algebra. Indeed, suppose that we have, on one side, a ternary multiplication law defined by its structural constants with respect to a given basis $\{e_k\}$:

$$m(e_i, e_j, e_k) = \sum_{l=1}^{N} m^l_{ijk} e_l,$$

and on the other hand, a binary multiplication law, defined in the same basis by

$$p(e_i, e_k) = \sum_{m=1}^{N} p^m_{ik} e_m;$$

and suppose that we want to interpret the ternary multiplication as two consecutive binary multiplications:

$$m(a, b, c) = p(a, p(b, c)) = p(p(a, b), c)$$

(supposing that the binary algebra is associative). Then, after projection on the basis vectors e_k we should have

$$m^i_{jkm} = \sum_{r=1}^{N} p^r_{km} p^i_{jr}. \tag{4.5}$$

Even in the simplest case of dimension $N = 2$, we get $2^4 = 16$ equations for $2^3 = 8$ unknowns (the coefficients p^i_{jk}), which cannot be solved in general, except maybe for some very special cases.

Sitarz ([54]) proved that any *associative* n-ary algebra can be generated by a part of the $\mathcal{A}_1$, i.e. the grade 1 subspace of certain Z_{N-1}-graded associative ordinary (binary) algebra. The simplest example of this situation is given by the groups algebra of the symmetry group S_3. It contains two subspaces, which are naturally Z_2-graded. The even subspace (of degree 0) is spanned by the cyclic subgroup Z_3, while the odd subspace is spanned by three involutions, corresponding to odd permutations. As the square of each involution is the unit element, the product of three involutions gives another involution, which defines a ternary algebra (without unit element). The full ternary multiplication table contains 27 independent products.

Just as binary products can be divided into different classes reflecting their behavior under the permutation group Z_2, so all ternary products can be divided into classes according to their behavior under the actions of the permutation group S_3. These in turn are naturally separated into *symmetric cubic* and *skew-symmetric cubic* subsets.

There are four possible ternary generalizations of the *symmetric* binary product:

$$\begin{aligned}
&S_0 : x^j x^k x^m = x^{\pi(j)} x^{\pi(k)} x^{\pi(m)}, \quad \text{any permutation } \pi \in \mathcal{S}_3;\\
&S_1 : x^j x^k x^m = x^k x^m x^j \quad \text{(cyclic permutations only)};\\
&S : x^k x^m x^n + j x^m x^n x^k + j^2 x^n x^k x^m = 0;\\
&\bar{S} : \bar{x}^k \bar{x}^m \bar{x}^n + j^2 \bar{x}^m \bar{x}^n \bar{x}^k + j \bar{x}^n \bar{x}^k \bar{x}^m = 0.
\end{aligned}$$

Obviously, the spaces S and $\bar{S}$ are isomorphic, and there exist surjective homomorphisms from S and $\bar{S}$ onto S_1, and a surjective homomorphism from S_1 onto S_0. Similarly, the *skew-symmetric* cubic algebras can be defined as a direct generalization of Grassmann algebras:

$$\begin{aligned}
&\Lambda_0 : \theta^A\theta^B\theta^C + \theta^B\theta^C\theta^A + \theta^C\theta^A\theta^B + \theta^C\theta^B\theta^A + \theta^B\theta^A\theta^C + \theta^A\theta^C\theta^B = 0,\\
&\Lambda_1 : \theta^A\theta^B\theta^C + \theta^B\theta^C\theta^A + \theta^C\theta^A\theta^B = 0,\\
&\Lambda : \theta^A\theta^B\theta^C = j\theta^B\theta^C\theta^A; \quad \bar{\Lambda} : \bar{\theta}^{\bar{A}}\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}} = j^2\bar{\theta}^{\bar{B}}\bar{\theta}^{\bar{C}}\bar{\theta}^{\bar{A}}.
\end{aligned}$$

Here again, a surjective homomorphism exists from Λ_0 onto Λ_1, then two surjective homomorphisms can be defined from Λ_1 onto Λ or onto $\bar{\Lambda}$.

The natural Z_3-grading attributes degree 1 to variables θ^A and degree 2 to the variables $\bar{\theta}^{\bar{B}}$; the degrees add up modulo 3 under the associative multiplication.

Then the algebras Λ and $\bar{\Lambda}$ can be merged into a bigger one if we postulate the extra binary commutation relations between variables θ^A and $\bar{\theta}^{\bar{B}}$:

$$\theta^A \bar{\theta}^{\bar{B}} = j\bar{\theta}^{\bar{B}}\theta^A, \quad \bar{\theta}^{\bar{B}}\theta^A = j^2\theta^A\bar{\theta}^{\bar{B}}.$$

If $A, B, C, \ldots = 1, 2, \ldots N$, then the total dimension of this algebra is

$$D(N) = 1 + 2N + 3N^2 + \frac{2(N^3 - N)}{3} = \frac{2N^3 + 9N^2 + 4N + 3}{3}.$$

These algebras are the most natural Z_3-graded generalizations of usual Z_2-graded algebras of fermionic (anticommuting) variables. Similarly, cubic Clifford algebras can be defined if their generators Q^b are supposed to satisfy the following ternary commutation relations:

$$Q^a Q^b Q^c = jQ^b Q^c Q^a + j^2 Q^c Q^a Q^b + 3\eta^{abc}\mathbf{1} \tag{4.6}$$

instead of usual binary constitutive relations

$$\gamma^\mu\gamma^\lambda = (-1)\gamma^\lambda\gamma^\mu + g^{\mu\lambda}\mathbf{1}.$$

A *conjugate* ternary Clifford algebra isomorphic with the above is readily defined if we introduce the conjugate matrices $\bar{Q}^a$ satisfying similar ternary condition with j replacing j^2 and vice versa [13].

Applying cyclic permutation operator π to all triplets of indexes on both sides of the definition (4.6), one easily arrives at the condition that must be satisfied by the tensor η^{abc}, corresponding to the symmetry condition on the metric tensor $g^{\mu\lambda}$ in the usual (binary) case:

$$\eta^{abc} + j\eta^{bca} + j^2\eta^{cab} = 0.$$

This equation has two independent solutions,

$$\eta^{abc} = \eta^{bca} = \eta^{cab}, \quad \text{and} \quad \rho^{abc} = j^2\rho^{bca} = j\rho^{cab}.$$

The second, non-trivial solution defines a *cubic matrix* ρ^{abc}; its conjugate, satisfying complex conjugate ternary relations, provides a Z_3-conjugate matrix $\bar{\rho}^{abc}$.

These two non-trivial solutions, denoted by $\rho^{(1)}$ and $\rho^{(2)}$; form an interesting non-associative ternary algebra with ternary multiplication rule defined as follows [14, 4]:

$$(\rho^{(i)} * \rho^{(k)} * \rho^{(m)})_{abc} = \sum_{d,e,f} \rho^{(i)}_{fad}\rho^{(k)}_{dbe}\rho^{(m)}_{ecf}. \tag{4.7}$$

A Z_3-graded analogue of usual commutator as readily defined as

$$\{\rho^{(i)}, \rho^{(j)}, \rho^{(k)}\} := \rho^{(i)} * \rho^{(j)} * \rho^{(k)} + j\rho^{(j)} * \rho^{(k)} * \rho^{(i)} + j^2\rho^{(k)} * \rho^{(i)} * \rho^{(j)}. \tag{4.8}$$

We shall explore the general properties of cubic matrices in Sec. 7; before that, let us present a general setting of ternary algebras and modules, and corresponding differential structures.

5. Ternary Algebras and Tri-Modules

Most of the material covered in this and the next section can be found in the paper written in common with Borowiec and Bazunowa [39].

Let us start with definitions. By *ternary* (associative) algebra $(\mathcal{A}, [\ ])$ we mean a linear space $\mathcal{A}$ (over a field $\mathbb{K}$) equipped with a linear map $[\ ] : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ called a (ternary) multiplication (or product), which satisfies the following strong associativity condition:

$$[[abc]de] = [a[bcd]e] = [ab[cde]].$$

Weaker versions of ternary associativity, when only one of the above identities is satisfied, can be called *left* (respectively, *right* or *central*) associativity.

We look at associative ternary algebras as a natural generalization of binary one: If $(\mathcal{A}, \cdot)$ is the usual (binary, associative) algebra then an induced ternary multiplication can be, of course, defined by $[abc] = (a \cdot b) \cdot c = a \cdot (b \cdot c)$. In what follows, such ternary algebras will be called trivial; from now on we shall study exclusively non-trivial ternary algebras. It is known that unital ternary algebras are trivial. Later on we shall show that any finitely generated ternary algebra is a ternary subalgebra of some trivial ternary algebra, which is a ternary generalization of Ado's theorem for finite-dimensional Lie algebras.

Many notions known in the binary case can be directly generalized to the ternary case. For example, the notion of ternary $\star$-algebra is defined by $[abc]^* = [c^* b^* a^*]$, where the star operation $* : \mathcal{A} \to \mathcal{A}$ is, as it should be, (anti-) linear anti-involution which means $(a^*)^* = a$ and $(ab)^* = b^* a^*$. By the way, the very concept of involution can be generalized so that it becomes adapted to ternary structures. A ternary involution should satisfy $((a^*)^*)^* = a$, as an example, we can introduce the operation $*$ such that $[abc] = [b^* c^* a^*]$. In some applications, an important role is played by ternary algebras with a different associativity law:

$$[[abc]de] = [a[dcb]e] = [ab[cde]]. \tag{5.1}$$

Such associativity is sometimes called "type B-associativity" or "2nd kind" [41]. In the case of ternary $\star$-algebras both types of associativity are related to each other. Assuming that $(\mathcal{A}, [\ ], *)$ is a ternary $\star$-algebra, one can introduce another ternary multiplication $[\ ]_*$ such that

$$[abc]_* \stackrel{def}{=} [ab^*c], \quad \forall\ a,\ b,\ c \in \mathcal{A}.$$

The algebra $(\mathcal{A}, [\ ]_*, *)$ becomes an associative ternary $\star$-algebra of B-type. The converse statement is also true: any ternary $\star$–algebra of B-type gives rise to a standard ternary $\star$-algebra. Observe that in the case of algebras over the field of complex numbers one has to assume anti-linearity of ternary multiplication in the middle factor instead of linearity.

The concept of tri-module is a particular case of the concept of module over an algebra over an operad defined in [48]. It was considered in a more general context of n-ary algebras in [49]. Here we define the tri-module structure over a ternary algebra $\mathcal{A}$ on a vector space $\mathcal{M}$ by the following three linear mappings:

$$\text{left} \quad [\ ]_L : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad \text{right} \quad [\ ]_R : \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{M},$$
$$\text{and central} \quad [\ ]_C : \mathcal{A} \otimes \mathcal{M} \otimes \mathcal{A} \to \mathcal{M}$$

multiplication, respectively (see also [41, 53]). They are assumed to satisfy the following compatibility conditions

$$[ab[cdm]_L]_L = [[abc]dm]_L = [a[bcd]m]_L, \tag{5.2}$$
$$[[mab]_R cd]_R = [ma[bcd]]_R = [m[abc]d]_R, \tag{5.3}$$
$$[a[b[cmx]_C y]_C z]_C = [[abc]m[xyz]]_C, \tag{5.4}$$
$$[a[bcm]_L d]_C = [ab[cmd]_C]_L = [[abc]md]_C, \tag{5.5}$$
$$[a[mbc]_R d]_C = [[amb]_C cd]_R = [am[bcd]]_C, \tag{5.6}$$
$$[[abm]_L cd]_R = [ab[mcd]_R]_L = [a[bmc]_C d]_C, \tag{5.7}$$
$$\forall\ a,\ b,\ c,\ d,\ x,\ y,\ z \in \mathcal{A},\ m \in \mathcal{M}.$$

In the case of tri-module $\mathcal{M}$ over an algebra $\mathcal{A}$ of type B the conditions (5.2), (5.3), (5.7) remain unchanged while (5.4)–(5.6) have to be replaced correspondingly by

$$[a[b[cmx]_C y]_C z]_C = [[ayc]m[xbz]]_C, \tag{5.8}$$
$$[a[cbm]_L d]_C = [[amb]_C cd]_R, \quad [a[mbc]_R d]_C = [ab[cmd]_C]_L, \tag{5.9}$$
$$\forall\, a,\ b,\ c,\ d,\ x,\ y,\ z \in \mathcal{A},\ m \in \mathcal{M}.$$

Much in the same way as binary algebra is a trivial ternary algebra, the notion of tri-module generalizes the notion of bi-module. More exactly, let $\mathcal{A}$ be a (binary) algebra and $\mathcal{M}$ a bi-module over it. Thus defining $[abm]_L = a \cdot (b \cdot m) = (a \cdot b) \cdot m$, $[amb]_C = a \cdot (m \cdot b) = (a \cdot m) \cdot b$ and $[mab]_R = m \cdot (a \cdot b) = (m \cdot a) \cdot b$ we see that $\mathcal{M}$ becomes a tri-module over the same algebra considered as a trivial ternary algebra.

Analogously, we can obtain an enveloping module $\mathcal{U}_{\mathcal{M}}$ over the enveloping algebra $\mathcal{U}_{\mathcal{A}}$ of ternary module $\mathcal{M}$ over ternary algebra $\mathcal{A}$. Let us denote by $\mathcal{U}_{\mathcal{M}}$ a $\mathbb{Z}_2$-graded vector space

$$\mathcal{U}_{\mathcal{M}} = \mathcal{M}_1 \oplus \mathcal{M}_0, \tag{5.10}$$

where the odd part $\mathcal{M}_1 \equiv \mathcal{M}$. The even part is defined as a quotient vector space

$$\mathcal{M}_0 = (\mathcal{A} \otimes \mathcal{M} \oplus \mathcal{M} \otimes \mathcal{A})/\mathrm{lin}\langle S\rangle,$$

where S is a set of elements in $\mathcal{A} \otimes \mathcal{M} \oplus \mathcal{M} \otimes \mathcal{A}$ ($a,\ b,\ c, \in \mathcal{A},\ m\ \in \mathcal{M}$.)

$$[abc] \otimes m - a \otimes [bcm]_L, \quad [abm]_L \otimes c - a \otimes [bmc]_C,$$
$$[amb]_C \otimes c - a \otimes [mbc]_R, \quad [mab]_R \otimes c - m \otimes [abc],$$

which generate the subspace $\mathrm{lin}\langle S\rangle$.

As previously, denote by $a \circledast m$ or $m \circledast a$ the corresponding equivalence classes, elements of $\mathcal{M}_0$. Define left and right multiplication $\overline{\circledast}$ between elements from $\mathcal{U}_{\mathcal{A}}$ and those from $\mathcal{U}_{\mathcal{M}}$ in the following way:

$$a \overline{\circledast} m \stackrel{def}{=} a \circledast m; \quad m \overline{\circledast} a \stackrel{def}{=} m \circledast a;$$
$$(a \circledast b) \overline{\circledast} m = a \overline{\circledast} (b \circledast m) \stackrel{def}{=} [abm]_L;$$
$$m \overline{\circledast} (c \circledast d) = (m \circledast c) \overline{\circledast} d \stackrel{def}{=} [mcd]_R;$$
$$a \overline{\circledast} (m \circledast b) \stackrel{def}{=} [amb]_C; \quad (a \circledast m) \overline{\circledast} b \stackrel{def}{=} [amb]_C;$$
$$\forall\ a,\ b,\ c,\ d,\ e,\ f\ \in \mathcal{A},\ m\ \in \mathcal{M}.$$

One can check the following properties of the action of algebra $\mathcal{A}$ on module $\mathcal{M}$

$$[abc] \circledast m = a \circledast [bcm]_L; \quad m \circledast [bcd] = [mbc]_R \circledast d;$$
$$[abc] \circledast (m \circledast d) = [ab[cmd]_C]_L; \quad (a \circledast m) \circledast [bcd] = [[amb]_C cd]_R;$$
$$(a \circledast [bcd]) \circledast m = ([abc] \circledast d) \circledast m = [abc] \circledast (d \circledast m) = [ab[cdm]_L]_L;$$
$$m \circledast ([cde] \circledast f) = m \circledast (c \circledast [def]) = (m \circledast c) \circledast [def] = [[mcd]_R ef]_R;$$
$$(a \circledast [bcd]) \circledast (m \circledast e) = ([abc] \circledast d) \circledast (m \circledast e) = [abc] \circledast [dme]_C;$$
$$(a \circledast m) \circledast ([bcd] \circledast e) = (a \circledast m) \circledast (b \circledast [cde]) = [[amb]_C \circledast [cde].$$

Thus $\mathcal{U}_{\mathcal{M}}$ becomes a $\mathbb{Z}_2$-graded bi-module over $\mathcal{U}_{\mathcal{A}}$ since

$$\mathcal{A}_i \overline{\circledast} \mathcal{M}_j \subseteq \mathcal{M}_{i+j(mod2)}, \quad \mathcal{M}_j \overline{\circledast} \mathcal{A}_i \subseteq \mathcal{M}_{i+j(mod2)}, \quad i,\ j \in \{0,1\}.$$

In particular, $\mathcal{M} \equiv \mathcal{M}_1$ and $\mathcal{M}_0$ are $\mathcal{A}_0$-bi-modules.

Further on, we shall use the same symbol $\circledast$ to denote the equivalence class in $\mathcal{U}_{\mathcal{M}}$, its bi-module structure and for the multiplication in $\mathcal{U}_{\mathcal{A}}$.

Let us stress again that any bi-module over a (binary) algebra becomes automatically a tri-module over the same algebra considered as a trivial ternary algebra.

What we have shown above is that any tri-module is a sub-tri-module of some universal bi-module $\mathcal{U}_{\mathcal{M}}$ over $\mathcal{U}_{\mathcal{A}}$. Conversely, if $\mathcal{N}$ is a $\mathbb{Z}_2$-graded bi-module over $\mathcal{U}_{\mathcal{A}}$, then its odd part $\mathcal{N}_1$ is a tri-module over $\mathcal{A}$.

6. Universal Differentiation of Ternary Algebras

A first order differential calculus (differential calculus in short) of ternary algebra $\mathcal{A}$ is a linear map from ternary algebra a into tri-module over it, i.e. $d : \mathcal{A} \to \mathcal{M}$, such that a ternary analog of the Leibniz rule takes place:

$$d([fgh]) = [dfgh]_R + [fdgh]_C + [fgdh]_L, \quad \forall f, g, h \in \mathcal{A}. \tag{6.1}$$

In particular, if $\mathcal{M} = \mathcal{A}$, then we shall call so defined differential *ternary derivation* of $\mathcal{A}$. An interesting example is provided by natural ternary derivation arising in metrix spaces.

As an example, one can define ternary derivative in Hilbert (or metric) vector space. As we already noticed, any Hilbert space $(\mathcal{H}, \langle , \rangle)$ inherits a canonical ternary 2nd type associative structure given by $\{abc\} = \langle a, b \rangle c$.

For a linear operator being a ternary derivation $D : \mathcal{H} \to \mathcal{H}$ one calculates:

$$D\{abc\} = \{Dabc\} + \{aDbc\} + \{abDc\},$$

Now, taking into account that $D\{abc\} = \langle a, b\rangle Dc = \{abDc\}$ it implies

$$\langle Da, b\rangle = -\langle a, Db\rangle \Rightarrow D^+ = -D, \quad \text{i.e. } (iD)^+ = iD$$

i.e. that ternary derivations are in one-to-one correspondence with hermitian operators in $\mathcal{H}$. This makes possible a link with Quantum Mechanics, especially the version introduced by Nambu [8].

Let us refer again to the classical (binary) case. First order differential calculus from an algebra into bi-module can be automatically interpreted as a ternary differential calculus from trivial ternary algebra into a trivial tri-module over it. It can be easily seen from

$$d(fgh) = d((fg)h) = d(fg)h + fgd(h) = dfgh + fdgh + fgdh.$$

The converse statement is, in general, not true. A ternary Leibniz rule for differential calculus from an algebra into bi-module does not necessarily imply, in case of non-unital algebras, the existence of a standard (binary) Leibniz rule. In particular, the set of ternary derivations of non-unital algebra should be an extension of the set of standard (binary) derivations.

Let $(\mathcal{A}, \mathcal{M}, d)$ be our ternary differential calculus from ternary algebra into tri-module. By the Leibniz rule

$$\tilde{d}(a \circledast b) = (da) \circledast b + a \circledast (db)$$

it can be uniquely extended to a 0-degree differential $\tilde{d} : \mathcal{U}_{\mathcal{A}} \to \mathcal{U}_{\mathcal{M}}$. Conversely, any 0-degree first order differential calculus from $\mathcal{U}_{\mathcal{A}}$ into $\mathcal{U}_{\mathcal{M}}$, such that $\tilde{d}|_{\mathcal{A}} \subset \mathcal{M}$ gives rise to ternary $\mathcal{M}$-valued differential calculus on $\mathcal{A}$.

The universal first order differential calculus on non-unital algebras is described in [44, 45, 51]. Let us recall this construction shortly. Determine a vector space $\Omega^1_u(\hat{\mathcal{A}}) = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ is a non-unital (binary) algebra. Any element from $\Omega^1_u(\hat{\mathcal{A}})$ can be written in the form: $(a, b \otimes c)$, where $a,\ b,\ c\ \in \hat{\mathcal{A}}$. Define left and right multiplications by elements from $\hat{\mathcal{A}}$:

$$x(a, b \otimes c) = (0, x \otimes a + xb \otimes c),$$
$$(a, b \otimes c)y = (ay, -a \otimes y + b \otimes cy - bc \otimes y).$$

In this way, $\Omega^1_u(\hat{\mathcal{A}})$ becomes a $\hat{\mathcal{A}}$-bi-module since

$$(x(a, b \otimes c))y = x((a, b \otimes c)y).$$

Let $D : \hat{\mathcal{A}} \to \hat{\mathcal{A}} \oplus \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$, $Da = (a, 0), \forall\ a \in \hat{\mathcal{A}}$ be a canonical embedding. Because it satisfies the Leibniz rule:

$$D(a)b + aD(b) = (a, 0)b + a(b, 0) = (ab, -a \otimes b) + (0, a \otimes b) = (ab, 0) = D(ab).$$

D is a differential. We shall call it the universal differential for a given algebra $\hat{\mathcal{A}}$.

For unital algebras, there exists an alternative construction of $\Omega^1_u(\hat{\mathcal{A}})$ as a kernel of multiplication map [43] (see also [40]). Since our ternary algebras have no unit element, we cannot use such construction here.

An analogous construction can be proposed in the case of ternary algebra.

From a $\mathbb{Z}_2$-graded $\mathcal{U}_\mathcal{A}$-bi-module $\Omega^1_u(\mathcal{U}_\mathcal{A}) = \mathcal{U}_\mathcal{A} \oplus \mathcal{U}_\mathcal{A} \otimes \mathcal{U}_\mathcal{A}$, let us extract its odd subspace $\mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$ with elements

$$(a, \beta \otimes b, c \otimes \gamma), \quad \forall\ a,\ b,\ c,\ \in \mathcal{A},\ \beta,\ \gamma \in \mathcal{A}_0.$$

We shall denote it as $\Omega^1_T(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}_0 \otimes \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A}_0$. As we already know from our previous considerations, $\Omega^1_T(\mathcal{A})$ is a tri-module over $\mathcal{A}$. Thus we have defined the left, central and right ternary multiplications:

$$\begin{aligned}
[xy(a, \beta \otimes b, c \otimes \gamma)]_L &= (0, (x \circledast y) \otimes a + (x \circledast [y\beta]) \otimes b, [xyc] \otimes \gamma);\\
[x(a, \beta \otimes b, c \otimes \gamma)y]_C &= (0, -(x \circledast a) \otimes y - ([x\beta] \circledast \gamma) \otimes y + (x \circledast c) \otimes [\gamma y]\\
&\quad - (x \circledast [c\gamma]) \otimes y, x \otimes (a \circledast y) + [x\beta] \otimes (b \circledast y));\\
[(a, \beta \otimes b, c \otimes \gamma)xy]_R &= ([axy], \beta \otimes [bxy], -a \otimes (x \circledast y) - [\beta b] \otimes (x \circledast y)\\
&\quad + c \otimes ([\gamma x] \circledast y) - [c\gamma] \otimes (x \circledast y)).
\end{aligned} \tag{6.2}$$

The canonical embedding $D : \mathcal{A} \to \Omega^1_T\mathcal{A}$:

$$D(a) = (a, 0, 0), \quad \forall\ a \in \mathcal{A} \tag{6.3}$$

defines a ternary differential (6.1). In fact, one has

$$\begin{aligned}
&[(a, 0, 0)bc]_R + [a(b, 0, 0)c]_C + [ab(c, 0, 0)]_L\\
&\quad = ([abc], 0, -a \otimes (b \circledast c)) + (0, -(a \circledast b) \otimes c, a \otimes (b \circledast c)) + (0, (a \circledast b) \otimes c, 0)\\
&\quad = ([abc], 0, 0).
\end{aligned}$$

This ternary differential calculus is universal because for any tri-module E and any ternary E-valued differential calculus $d : \mathcal{A} \to E$, there exists one and only one covering tri-module homomorphism $\tilde{\varphi}_d$ such that $d = \tilde{\varphi}_d \circ D$. Moreover, if the tri-module E is spanned by the elements $d\mathcal{A}$, $[\mathcal{A}\ d\mathcal{A}\ \mathcal{A}]_C$ and $[d\mathcal{A}\ \mathcal{A}\ \mathcal{A}]_R$, then $\tilde{\varphi}_d$ is an epimorphism and $E = \Omega^1_T(\mathcal{A})/Ker(\tilde{\varphi}_d)$.

In this way, the problem of classification of all first order differential calculi over $\mathcal{A}$ can be translated into the problem of classification of all sub-tri-modules in $\Omega^1_T(\mathcal{A})$. Remember that $\Omega^1_T(\mathcal{A})$ is an odd part of $\Omega^1_u(\mathcal{U}_\mathcal{A})$ and our ternary differential (6.3) is, in fact, a restriction of the universal differential.

As it is well known [44, 45, 51] the bi-module $\Omega^1_u(\mathcal{U}_\mathcal{A})$ extends, by means of the graded Leibniz rule, to the universal graded differential algebra with $d^2 = 0$. This leads to higher order differential calculi. Another universal extension with $d^N = 0$, still for the case of binary (unital) algebras, has been considered in [21, 22]. These universal extensions have been provided by means of the q-Lebniz rule, for q being primitive N-degree root of the unity, i.e. $q = e^{\frac{2\pi i}{N}}$ (see also [50] in this context). However, the so-called N-ary case ($d^N = 0$) seems also to be specially well adopted for N-ary algebras.

7. From Hilbert Space to Ternary Algebras

Let $x, y \in V, A \in L(V, V)$, and let $A(x) = y$, where V is a Hilbert space, and A a linear operator defined on it.

Then in a given orthonormal basis $\{e_i\}$, with the usual identification of the dual basis (i.e. the basis of linear functionals on V) by means of a scalar product in H: $e^{*k}(e_i) = \langle e_k, e_i\rangle = \delta^k_i$, we may write $A \sim \sum_{k,i} A^k_i e_k e^{*i}$, so that

$$A(x) = \sum_{k,i} A^k_i e_k \otimes e^{*i} \left(\sum_m x^m e_m\right) = \sum_{k,m} A^k_m x^m e_k = y^k e_k = y. \tag{7.1}$$

The linear operator A defined by its matrix elements A^k_m can be also identified with the element of the tensor product $V \otimes V$ as

$$A \Rightarrow \sum_{k,m} A_{km} e_k \otimes e_m, \quad \text{with } A_{km} := A^k_m. \tag{7.2}$$

In the above picture the rôle of the two copies of the Hilbert space that have been used to produce general linear operators (containing the algebra of observables) by means of the tensor product is utterly different from the rôle of the third copy serving as the *space of states*, as may be seen below:

$$(L(V, V) \oplus V) \sim (V \otimes V) \oplus V.$$

In some sense it is also analogous with a strange and unusual summation that is performed on the indices of the Riemann tensor in order to obtain the Ricci tensor: let us recall that although all the indices of the Riemann tensor vary within the same range, which is the dimension of the Riemannian manifold V_n on which the metric and the Riemann tensor are defined, their nature is totally different. The curvature, which is a Lie algebra valued 2-form, can be written in a given coordinate system as:

$$\Omega = \Omega_{ij} dx^i \wedge dx^j = R^k_{ijm} E^m_k dx^i \wedge dx^j \tag{7.3}$$

where the *matrices* E^m_k span the basis of the n^2-dimensional Lie algebra of the linear group $GL_n(\mathrm{R})$, satisfying the commutation relations

$$E^m_k E^i_l - E^i_l E^m_k = \delta^m_l E^i_k - \delta^i_k E^m_l.$$

Here again, the matrices E^m_k can be put in a one-to-one correspondence (by lowering one of the indices with the metric tensor g_{lm}, $E^m_k \to g_{lm} E^m_k = E_{lk}$) with the elements of the tensor product of 1-forms, $V^* \otimes V^*$ In the definition of the Ricci tensor, $R_{ik} = \sum_j R^j_{ijk}$, we perform the summation over the indices belonging to different realms: one comes from the vector space, while another is a part of the *multi-index* labeling the elements of the space of *linear transformations* (*matrices*) acting on this space.

In some sense these examples look like a discrete version of the choice of a *local section* in a fibre bundle, whose total space would be $V \otimes V \otimes V$.

This suggests that one could restore the full symmetry between the three copies of the vector space V by embedding $L(V,V) \oplus V$ into the tensor cube $V \otimes V \otimes V$, which can be then "reduced" or "projected" in *six different ways* onto $L(V,V) \oplus V$.

An arbitrary element $a \in V \otimes V \otimes V$ can be written in the basis $\{e_k\}$ as

$$a = \sum_{k,l,m} a_{klm} e_k \otimes e_l \otimes e_m,$$

which defines a one-to-one correspondence between the elements of the triple product space $V \otimes V \otimes V$ and the three-tensors a_{klm}, which we will call also *cubic matrices* from now on.

It might be that one of the reasons for the non-observability of quarks is related to the fact that they belong to this kind of "mixed" space, in which there is no clear distinction between the state and the observable, both being included in a 3-form (or a "*cubic matrix*").

The symmetric group S_3 (of permutations of three elements) acts in an obvious way on the complex cubic matrices by permuting their indices. It is therefore natural to distinguish separate subspaces of $V \otimes V \otimes V$ that provide the irreducible representations of S_3; e.g. there is the subspace of the totally symmetric cubic matrices satisfying $a_{klm} = a_{lmk} = a_{mlk} = \ldots$, the subspace of *j-skew-symmetric* cubic matrices satisfying $b_{klm} = j b_{lmk} = j^2 b_{mlk} = \bar{b}_{mlk}$, etc.

Curiously enough, the most natural internal composition law that generalizes the multiplication of ordinary matrices (or of the elements of $V \otimes V$) is *ternary* and is given by the following rule:

$$(a \oslash b \oslash c)_{ijk} := \sum_{p,q,r} a_{ipq} b_{pjr} c_{qrk}. \tag{7.4}$$

In contrast with ordinary matrix multiplication this composition is non-associative, in the sense that

$$(a \oslash (b \oslash c \oslash d) \oslash e) \neq (a \oslash b \oslash c) \oslash d \oslash e \neq a \oslash b \oslash (c \oslash d \oslash e).$$

Note that the group S_3 acts also on the so defined ternary algebra, so that both actions (the permutation of factors in the product and the permutation of the indices in the resulting cubic algebra) can compensate each other thus defining *invariant classes* in our algebra.

Such ternary products have been introduced in [34] and [17], and studied also by Lawrence [35] and Vainerman and the author [4], and are, in fact, a particular case of a more general *n-fold* multiplication defined on the n-tensors as follows:

$$m(a^{(1)}, a^{(2)}, \ldots, a^{(n)})_{i_1 i_2 \ldots i_n} = \sum_{j_{kr}=1 (k<r)}^{l} a^{(1)}_{i_1 j_{12} \cdots j_{1n}} a^{(2)}_{j_{12} i_2 j_{23} \cdots j_{2n}} \times \cdots \times$$
$$\times\, a^{(k)}_{j_{1k} \cdots j_{k-1k} i_k j_{kk+1} \cdots j_{kn}} \times \cdots \times a^{(n)}_{j_{1n} \cdots j_{n-1n} i_n}. \tag{7.5}$$

We believe that the *ternary* case is exceptional because it involves the symmetry group S_3 (permutations of three objects, or indices), and this group is the last one that possesses an exact and faithful representation in the field of complex numbers, the next one, S_4, has a representation with a double degeneracy, while starting from S_5 there are no representations in **C**.

8. The 3-Algebra of Cubic Matrices: The Cubic Chessboard

Let us concentrate now on a more detailed study of the ternary algebra of complex-valued cubic matrices. Such a study seems to be particularly important in view of the pertinence of these matrices (which are isomorphic with the elements of $V \otimes V \otimes V$) to a possible generalization of quantum mechanics and field theory.

Also, from a purely mathematical point of view, a ternary composition law in a linear space over complex numbers is particularly interesting because it can be decomposed in an irreducible way with respect to the permutation group S_3, which is the last of the permutation groups that has a *faithful* representation in the complex plane [36]. Later on we shall give further arguments suggesting the exceptional rôle of cubic matrices and their ternary algebra; for the time being, it suffices to draw attention to the fact that the cubic matrices can be visualized in three dimensions (like the "Rubik's cube", for example), and they are probably the last case that can be still treated and analyzed in a finite time, although even in this case the use of the computer becomes crucial.

We start by fixing the notation and conventions. Let the indices $i, k, l, m, \ldots$ run from 1 to N. Let the elements of a (complex-valued) matrix a in a given basis be a_{ikm}. The multiplication introduced previously is defined as:

$$(a \oslash b \oslash c)_{ijk} := \sum_{p,q,r} a_{ipq} b_{pjr} c_{qrk}. \tag{8.1}$$

The ternary "multiplication table" is like a cubic chessboard with dimensions $(N^3) \times (N^3) \times (N^3)$, which amounts to 512 different entries for the case $N = 2$ and to $27^3 = 3^9 = 19683$ different entries for the case $N = 3$ — some chessboard, indeed! That is why in what follows we shall restrain ourselves to the cases $N = 2$ and $N = 3$ only.

One of the natural bases in the space of cubic matrices is the set defined as:

$$e_{ikm} := 1 \text{ at the intersection of } i\text{-th, } k\text{-th and } m\text{-th rows, 0 elsewhere.}$$

Let us show that in the case $N = 2$ it is possible to obtain a decomposition of the 8-dimensional ternary algebra (as a linear space) into the direct sum of its three special subalgebras. In fact there are 8 matrix units in the whole algebra. Three of them: $e_{111}, e_{222}, e_{333}$ generate a subalgebra Diag of the diagonal matrices which is evidently S_n-commutative. Using the ternary multiplication formula for the considered partial case, one can compute that the subalgebra generated by the matrix units $e_{112}, e_{121}, e_{122}$ has a zero multiplication and consequently it is abelian.

The same is true for the subalgebra generated by the matrix units $e_{221}, e_{212}, e_{211}$. Thus, we have a decomposition

$$\mathrm{Mat}(2,3;C) = \mathrm{Diag} \oplus \{e_{112}, e_{121}, e_{122}\} \oplus \{e_{221}, e_{212}, e_{211}\},$$

in which the first summand is S_n-commutative and the two others are abelian subalgebras. This decomposition looks like a decomposition of 2×2-matrices on the diagonal and two triangular subalgebras. But it is not unique; one can get at least two similar decompositions:

$$\mathrm{Mat}(2,3;C) = \mathrm{Diag} \oplus \{e_{112}, e_{212}, e_{211}\} \oplus \{e_{121}, e_{122}, e_{221}\} \quad \text{and}$$
$$\mathrm{Mat}(2,3;C) = \mathrm{Diag} \oplus \{e_{121}, e_{221}, e_{211}\} \oplus \{e_{112}, e_{122}, e_{212}\}.$$

But it is another decomposition, connected with the representation properties with respect to the group Z_3 (eventually S_3) that will be important in our forthcoming study of ternary algebra of cubic matrices.

Let J be the cyclic permutation operator acting on cubic matrices as:

$$(Ja)_{ikl} := a_{kli}; \quad \text{obviously } (J^2 a)_{ikl} := a_{lik}, \quad \text{and} \quad J^3 = Id. \tag{8.2}$$

In the articles [34, 4] we have introduced an alternative multiplication law for cubic matrices defined below:

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \tag{8.3}$$

in which any *cyclic* permutation of the matrices in the product is equivalent to the same permutation on the indices:

$$(a * b * c)_{ikl} = (b * c * a)_{kli} = (c * a * b*)_{lik}. \tag{8.4}$$

It is easy to see that the two multiplication laws are related as follows:

$$a \oslash b \oslash c = (Ja) * b * (J^2 c), \tag{8.5}$$

and neither of the two is associative. Let us denote by j the cubic root of unity, $j = e^{\frac{2\pi i}{3}}$; we have $j + j^2 + 1 = 0$, and $\bar{j} = j^2$.

The complex square $N \times N$-matrices can be divided into subspaces with particular representation properties with respect to the group of permutations S_2 (isomorphic with Z_2), thus defining *symmetric*, *anti-symmetric*, *hermitian* and *anti-hermitian* matrices: let T be the transposition operator, $(Ta)_{ik} = a_{ki}$; then we can define the aforementioned types of matrices as the ones that have the following transformation laws under the action of T:

$$Ta = a, \quad Ta = -a, \quad Ta = \bar{a}, \quad Ta = -\bar{a}$$

which gives in index notation the usual definitions:

$$a_{ik} = a_{ki}, \quad a_{ik} = -a_{ki}, \quad a_{ik} = \bar{a}_{ki}, \quad a_{ik} = -\bar{a}_{ki}.$$

Similarly, the complex cubic matrices can be divided into classes according to the representations of the group S_3. With J defined as above (9) and T the operator of odd transposition, $(Ta)_{ikm} = a_{mki}$, we can define cubic matrices with the following non-equivalent representation properties under the action of J and T:

$$\begin{array}{c} Ja = a, \quad Ta = a; \quad Ja = ja, \quad Ta = a; \quad Ja = j^2 a, \quad Ta = a, \\ \text{or} \quad Ja = a, \quad Ta = \bar{a}; \quad Ja = ja, \quad Ta = \bar{a}; \quad Ja = j^2 a, \quad Ta = \bar{a}. \end{array} \tag{8.6}$$

From now on we shall concentrate on the class of matrices displaying well-defined properties with respect to the group of cyclic permutations Z_3 only, i.e. supposing that there is no particular relation between a and Ta.

This type of decomposition is important in the analysis of the possible representations of ternary algebras of cubic matrices in terms of associative matrix algebras. We shall follow the well known example of Ado's theorem for finite-dimensional Lie groups, which states that for such groups an associative *enveloping* algebra can be found, such that the skew-symmetric, non-associative composition law satisfying the Jacobi identity can be faithfully represented by a *commutator* of the corresponding elements.

Although at this stage we do not know if an analogue of the Jacobi identity exists for ternary algebra of cubic matrices, we shall show that at least for the simplest cases, certain ternary algebras with a non-associative composition law displaying particular symmetries can be represented in the algebra of associative matrices. Let us decompose the algebra of cubic matrices into the direct sum of the following linear subspaces:

Diagonal, containing N diagonal cubic matrices $\omega^{(k)}$:

$$\omega^{(k)}_{kkk} = 1, \quad \text{all other elements} = 0;$$

Symmetric, containing $(N^3 - N)/3$ *traceless*, *totally symmetric* cubic matrices

$$\pi^{(\alpha)}_{klm} = \pi^{(\alpha)}_{lmk} = \pi^{(\alpha)}_{mkl}, \quad \alpha = 1, 2, \ldots, (N^3 - N)/3$$

j-Skew-symmetric, containing $(N^3 - N)/3$ cubic matrices satisfying

$$\rho^{(\alpha)}_{klm} = j\rho^{(\alpha)}_{lmk} = j^2 \rho^{(\alpha)}_{mkl};$$

and j^2-*Skew-symmetric*, containing $(N^3 - N)/3$ cubic matrices satisfying

$$\kappa^{(\alpha)}_{klm} = j^2 \kappa^{(\alpha)}_{lmk} = j\kappa^{(\alpha)}_{mkl}.$$

Only the diagonal matrices form a 3-subalgebra with respect to the ternary multiplication law;

$$\omega^{(k)} * \omega^{(l)} * \omega^{(m)} = 0 \quad \text{if } k \neq l \neq m \quad \text{and} \quad \omega^{(k)} * \omega^{(k)} * \omega^{(k)} = \omega^{(k)}. \tag{8.7}$$

This 3-subalgebra is associative and commutative and is easily represented by ordinary (square) matrices $\omega^{(k)}$, whose only non-vanishing element 1 is found at the intersection of the k-th line with the k-th column.

With eight independent generators the ternary algebra's multiplication table is also a *cubic* array, and in order to define it completely we must display as many as $8 \times 8 \times 8 = 512$ different ternary products. Because of the non-associativity of ternary law, it is impossible to find a realization of these multiplication rules by means of a set of finite $n \times n$ matrices.

This situation is not new, and could be observed in the case of binary non-associative algebras. The well known Ado's theorem states that a class of finite dimensional non-associative algebras with particular symmetry of the composition law, $\{X, Y\} = -\{Y, X\}$ and satisfying the Jacobi identity (Lie algebras) can always be represented by a subset of some bigger *associative* algebra, called the enveloping algebra.

Let us show on a simple example of $2 \times 2 \times 2$ cubic matrices that a representation in the associative binary algebra of 2×2 ordinary matrices can be found provided that the ternary composition law is endowed with a particular symmetry that generalizes the skew symmetry of the ordinary Lie algebra.

In the multiplication table for the cubic matrices in the particular basis of $\omega^{(k)}, \pi^{(\alpha)}, \rho^{(\beta)}$ and $\kappa^{(\gamma)}$ it is difficult to find any subalgebras except for the obvious "central" one containing the $\omega^{(k)}$. Usually a 3-product of three matrices will decompose into a linear combination of the matrices belonging to various symmetry types, e.g.

$$\rho^{(1)} * \rho^{(1)} * \rho^{(2)} = \omega^{(1)} - \frac{1}{3}\pi^{(2)} + \frac{2}{3}j^2\rho^{(2)} - \frac{1}{3}j\kappa^{(2)}, \quad \text{etc.}$$

The situation changes if we introduce a new composition law that follows the particular symmetry of the given type of cubic matrices. For example, let us define:

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\} := \rho^{(\alpha)} * \rho^{(\beta)} * \rho^{(\gamma)} + j\rho^{(\beta)} * \rho^{(\gamma)} * \rho^{(\alpha)} + j^2\rho^{(\gamma)} * \rho^{(\alpha)} * \rho^{(\beta)}. \tag{8.8}$$

Because of the symmetry of the ternary *j-bracket* one has

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{ikm} = j\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{kmi},$$

so that it becomes obvious that with respect to the *j-bracket* composition law the matrices $\rho^{(\alpha)}$ form a ternary subalgebra. Indeed, we have

$$\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\} = -\rho^{(2)}; \quad \{\rho^{(2)}, \rho^{(1)}, \rho^{(2)}\} = -\rho^{(1)}; \tag{8.9}$$

all other combinations being proportional to the above ones with a factor j or j^2, whereas the j-brackets of three identical matrices obviously vanish.

Our aim is to find the simplest representation of this ternary algebra in terms of a *j-commutator* defined in an associative algebra of matrices $M_2(\mathbf{C})$ as follows:

$$[A, B, C] := ABC + jBCA + j^2CAB. \tag{8.10}$$

It is easy to see that the trace of any j-bracket of three matrices must vanish; therefore, the matrices that would represent the cubic matrices $\rho^{(\alpha)}$ must be traceless. Then it is a matter of simple exercise to show that any two of the three

Pauli sigma-matrices divided by $\sqrt{2}$ provide us with a representation of the ternary j-skew algebra of the ρ-matrices; e.g.

$$\sigma^1\sigma^2\sigma^1 + j\sigma^2\sigma^1\sigma^1 + j^2\sigma^1\sigma^1\sigma^2 = -2\sigma^2, \quad \sigma^2\sigma^1\sigma^2 + j\sigma^1\sigma^2\sigma^2 + j^2\sigma^2\sigma^2\sigma^1 = -2\sigma^1.$$

Thus, it is possible to find a representation in the associative algebra of finite matrices for the non-associative j-bracket ternary algebra. A similar representation can be found for the two cubic matrices $\kappa^{(\alpha)}$ with the j^2-skew bracket.

It is also worthwhile to note that the ordinary Lie algebras with the skew-symmetric composition law can be found in the representation of the ternary j-bracket algebra in the associative algebra, provided the latter one is endowed with a central (unit) element. Indeed, we have:

$$[A, \mathbf{1}, C] = A\mathbf{1}C + j\mathbf{1}CA + j^2CA\mathbf{1} = AC + (j + j^2)CA = AC - CA. \tag{8.11}$$

The fact that Pauli matrices did appear in a quite natural way is encouraging. It suggests that although we start here from a ternary algebra with j-skew 3-commutator, more familiar notions such as the Lorentz group and spin can be encoded in some way in this unusual rules, and appear sooner or later as secondary features of a purely algebraic theory. The following exercise reinforces this hope.

A natural question to ask now concerns the nature of all the automorphisms of this simple ternary algebra. The most general homogeneous transformation of the cubic matrices $\rho^{(\alpha)}$ involves all their indices:

$$\tilde{\rho}^{(\alpha)}_{ikm} = \Lambda^{\alpha}_{\beta} U^p_i U^r_k U^s_m \rho^{(\beta)}_{prs}, \quad \alpha, \beta, i, k, \ldots = 1, 2 \tag{8.12}$$

with (invertible) matrices Λ^{α}_{β}, U^p_i chosen in such a way that the ternary relations between the transformed cubic matrices $\tilde{\rho}^{(\alpha)}$ remain the same as defined above.

Let us show that even in a simplified case when we choose $U^p_q = \delta^p_q$, the condition of invariance of the ternary algebra leads to non-trivial solutions for the group of matrices Λ^{α}_{β}. As a matter of fact, we get the following system of equations for Λ^{α}_{β}:

$$\Lambda^1_1(\Lambda^2_2\Lambda^1_1 - \Lambda^1_2\Lambda^2_1) = \Lambda^2_2; \quad \Lambda^1_2(\Lambda^2_1\Lambda^1_2 - \Lambda^1_1\Lambda^2_2) = \Lambda^2_1, \quad \text{and} \tag{8.13}$$

$$\Lambda^2_2(\Lambda^1_1\Lambda^2_2 - \Lambda^2_1\Lambda^1_2) = \Lambda^1_1; \quad \Lambda^2_1(\Lambda^1_2\Lambda^2_1 - \Lambda^2_2\Lambda^1_1) = \Lambda^1_2 \tag{8.14}$$

from which follows that $[\det(\Lambda)]^2 = 1$, so that either

$$\det(\Lambda) = 1, \quad \text{and} \quad \Lambda^1_1 = \Lambda^2_2, \quad \Lambda^1_2 = -\Lambda^2_1, \quad \text{or} \tag{8.15}$$

$$\det(\Lambda) = -1, \quad \text{and} \quad \Lambda^1_1 = -\Lambda^2_2, \quad \Lambda^1_2 = \Lambda^2_1. \tag{8.16}$$

This group has two disjoint components; the simply connected component of the unit element is a subgroup, whereas the second component can be obtained from the first one by multiplication by the $2 \otimes 2$ matrix $\text{diag}(1, -1)$.

The simply connected subgroup is an abelian, (real) two-dimensional Lie group of matrices whose general form is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \text{with } a, b \text{ complex numbers satisfying } a^2 + b^2 = 1 \tag{8.17}$$

which can be decomposed into a simple product of two matrices:

$$\begin{pmatrix} \cosh\psi & i\sinh\psi \\ -i\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{8.18}$$

This group is easily identified as the simple product of Euclidean rotations and translations. It can be realized as the isometry group of a cylindrical Minkowski space parameterized with two variables τ and ϕ, $[0 \leq \phi \leq 2\pi] \times [-\infty \leq \tau \leq \infty]$, with one "boost" and one angular translation. When embedded in a many-dimensional Minkowski space, this object looks like a motionless closed string. This invariance group reduces to $U(1)$ if we impose the reality condition on the matrices $\rho(\alpha)$ requiring that $\rho^{(\alpha)}_{ikl} = \bar{\rho}^{(\alpha)}_{lki}$.

The ternary algebra of complex cubic matrices in three dimensions, $\mathrm{Mat}(3, 3, \mathbf{C})$, has a very rich structure; its multiplication table (in three dimensions, too) can be visualized as a cubic matrix with $27 \times 27 \times 27 = 3^9$ entries. Here again, subsets displaying a particular Z_3-symmetry can be defined, containing eight independent matrices each, so that the whole algebra of 27 independent matrices decomposes as $M = \mathrm{Diag} \oplus M_0 \oplus M_1 \oplus M_2$.

Let us denote these cubic matrices by: $O^{(a)}_{bcd}$ (the diagonal part); $R^{(A)}_{abc}$, with $A = 1, 2, \ldots 8$, and $a, b = 1, 2, 3$ spanning the subset M_1; $K^{(A)}_{abc}$ spanning the subset M_2, and $P^{(A)}_{abc}$ spanning the totally Z_3-symmetric traceless subset M_0. The cubic matrices denoted by capital Latin letters display the same symmetries as their prototypes belonging to $\mathrm{Mat}(2, 3, \mathbf{C})$ denoted by the corresponding Greek letters ω, ρ, κ and π. It is easy to see that the component M_1 containing the matrices $R^{(A)}$ satisfying

$$R^{(A)}_{abc} = jR^{(A)}_{bca} = j^2 R^{(A)}_{cab}$$

consists of three two-dimensional ternary subalgebras, each of them isomorphic with the algebra of ρ-matrices shown above. The three subalgebras are spanned by (we just give the only non-vanishing elements):

$$\{R^{(1+)}_{232}, R^{(1-)}_{323}\}; \quad \{R^{(2+)}_{313}, R^{(2-)}_{131}\} \quad \text{and} \quad \{R^{(3+)}_{121}, R^{(3-)}_{212}\}$$

besides, there are two more independent generators,

$$R^{(7)}_{123} \quad \text{and} \quad R^{(8)}_{321}.$$

This situation is similar to the one observed in the examples of the Lie algebras $su(2)$ and $su(3)$, where the algebra $su(2)$ can be embedded in three different ways in the algebra $su(3)$. It is also clear that among the automorphisms of the ternary algebra spanned by $R^{(A)}$, with the j-skew ternary commutator, we will find three copies of the automorphisms of the simple ternary algebra of $\rho^{(\alpha)}$ cubic matrices, which means that we shall have *three* independent Lorentzian boosts, and *three* independent rotations of a plane, which is exactly what is needed to generate the 6-parameter Lorentz group in 4-dimensional space-time. Similar observation can be made concerning the cubic matrices $K^{(A)}$. This does not exclude the possibility of finding other interesting subgroups in the group of automorphisms of ternary relations between the cubic matrices $R^{(A)}$ or $K^{(A)}$, e.g. the group $SU(3)$ in its adjoint representation, although it may be intertwined with the elements of the Lorentz group in a very tricky way.

To find a maximal ternary subalgebra of $M_1 \subset \text{Mat}(3,3;\mathbf{C})$ that can be represented in a finite associative algebra with the $j - skew$ commutator as the composition law is not an easy task, and we do not know the full answer to this problem. However, the fact that the traceless part of the 3-algebra of cubic matrices splits naturally into three equal parts suggests that its representation by means of an associative enveloping algebra can be naturally Z_3-graded, with three grades $0, 1, 2$ adding up modulo 3. Such algebras are also very interesting, and we were able to investigate them to some extent.

9. Dreaming about Possible Future Developments

Classical gauge fields appear as the necessary device that maintains the covariance of the Dirac equation with respect to *unitary* transformations of the spinor wave function, $\psi \to e^{iS}\psi$ when S becomes a function of the space-time coordinates. Then we replace the free 4-momentum operator p_μ by its covariant counterpart $p_\mu - ieA_\mu$; simultaneously with a gauge transformation $\psi \to e^{iS}\psi$, we have (in the simplest, abelian case) $A_\mu \to A_\mu + \partial_\mu S$.

It is natural to ask what is the generalization of Dirac's equation that would lead to the modified gauge theory, with curvature 3-form Ω introduced above. The answer is quite obvious: if in the classical case the curvature 2-form $F_{\mu\nu}$ containing the *first* derivatives of A did appear naturally when we diagonalized the Dirac equation, applying once more the conjugate Dirac operator, here we must introduce a Schrödinger-like equation linear in the momentum operator, only the *third power* of which would become diagonal.

This leads naturally to a ternary generalization of Clifford algebras. Instead of the usual binary relation defining the usual Clifford algebra,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}, \quad \text{with } g^{\mu\nu} - g^{\nu\mu} = 0$$

we should introduce its ternary generalization, which is quite obvious (see also Abramov [38]):

$$Q^aQ^bQ^c + Q^bQ^cQ^a + Q^cQ^bQ^a = 3\eta^{abc}\mathbf{1}, \tag{9.1}$$

where the tensor η^{abc} must satisfy $\eta^{abc} = \eta^{bca} = \eta^{cab}$. The lowest-dimensional representation of such an algebra is given by complex 3×3 matrices:

$$Q^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \quad Q^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{9.2}$$

These matrices are given the Z_3-grade 1; their hermitian conjugates $Q^{*a} = (Q^a)^\dagger$ are of Z_3-grade 2, whereas the diagonal matrices are of Z_3-grade 0; it is easy to verify that the so defined grades add up modulo 3.

The matrices Q^a $(a = 1, 2, 3)$ satisfy the ternary relations (49) with η^{abc} a totally-symmetric tensor, whose only non-vanishing components are $\eta^{111} = \eta^{222} = \eta^{333} = 1$, $\eta^{123} = \eta^{231} = \eta^{321} = j^2$, and $\eta^{321} = \eta^{213} = \eta^{132} = j$.

Therefore, the Z_3-graded generalization of Dirac's equation should read:

$$\frac{\partial \psi}{\partial t} = Q^1 \frac{\partial \psi}{\partial x} + Q^2 \frac{\partial \psi}{\partial y} + Q^3 \frac{\partial \psi}{\partial z} + mB\psi \tag{9.3}$$

where ψ stands for a triplet of wave functions, which can be considered either as a column, or as a grade 1 matrix with three non-vanishing entries uvw, and B is the diagonal 3×3 matrix with the eigenvalues $1j$ and j^2. It is interesting to note that this is possible only with *three* spatial coordinates.

In order to diagonalize this equation, we must act three times with the same operator, which will lead to the same equation of *third order*, satisfied by each of the three components u, v, w, e.g.:

$$\frac{\partial^3 u}{\partial t^3} = \left[\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - \frac{\partial^3}{\partial x \partial y \partial z} \right] u + m^3 u. \tag{9.4}$$

This equation can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1 e^{\omega t} + A_2 e^{j\omega t} + A_3 e^{j^2 \omega t}, \quad B_1 e^{\mathbf{k.r}} + B_2 e^{j\mathbf{k.r}} + B_3 e^{j^2 \mathbf{k.r}};$$

their nine independent products can be represented in a basis of real functions as

$$\begin{pmatrix} A_{11} e^{\omega t + \mathbf{k.r}} & A_{12} e^{\omega t - \frac{\mathbf{k.r}}{2}} \cos \xi & A_{13} e^{\omega t - \frac{\mathbf{k.r}}{2}} \sin \xi \\ A_{21} e^{-\frac{\omega t}{2} + \mathbf{k.r}} \cos \tau & A_{22} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \cos \tau \cos \xi & A_{23} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \cos \tau \sin \xi \\ A_{31} e^{-\frac{\omega t}{2} + \mathbf{k.r}} \sin \tau & A_{32} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \sin \tau \cos \xi & A_{33} e^{-\frac{\omega t}{2} - \frac{\mathbf{k.r}}{2}} \sin \tau \sin \xi \end{pmatrix} \tag{9.5}$$

where $\tau = \frac{\sqrt{3}}{2} \omega t$ and $\xi = \frac{\sqrt{3}}{2} \mathbf{kr}$. The parameters $\omega, \mathbf{k}$ and m must satisfy the cubic dispersion relation:

$$\omega^3 = k_x^3 + k_y^3 + k_z^3 - 3k_x k_y k_z + m^3. \tag{9.6}$$

This relation is invariant under the simultaneous change of sign of ω, $\mathbf{k}$ and m, which suggests the introduction of another set of solutions constructed in the same manner, but with minus sign in front of ω and $\mathbf{k}$, which we shall call *conjugate* solutions.

Although neither of these functions belongs to the space of tempered distributions, on which a Fourier transform can be performed, their ternary skew-symmetric products contain only trigonometric functions, depending on the combinations $2(\tau - \xi)$ and $2(\tau + \xi)$. As a matter of fact, not only the *determinant*, but also each of the *minors* of the above matrix is a combination of the trigonometric functions only. The same is true for the binary products of "conjugate" solutions, with the opposite signs of ωt and $\mathbf{k}.\mathbf{r}$ in the exponentials.

This fact suggests that it is possible to obtain via linear combinations of these products the solutions of *second* or *first order* differential equations, like Klein–Gordon or Dirac equation.

Still, the parameters ω and $\mathbf{k}$ do not satisfy the proper mass shell relations; however, it is possible to find new parameters, which are linear combinations of these, that will satisfy quadratic relations that may be interpreted as a mass shell equation. We can more readily see this if we use the following parameterization: let us put

$$\zeta = (k_x + k_y + k_z), \quad \chi = Re(jk_x + j^2 k_y + k_z), \quad \eta = Im(jk_x + j^2 k_y + k_z),$$
$$\text{and} \quad r^2 = \chi^2 + \eta^2 \quad \phi = \text{Arc}tg(\eta/\chi).$$

In these coordinates the cubic mass hyperboloid equation becomes

$$\omega^3 - \zeta r^2 = m^3. \tag{9.7}$$

Two obvious symmetries can be immediately seen here, the rotation around the axis $[1, 1, 1]$ $(\phi \to \phi + \delta\phi)$, and simultaneous dilatation of ζ and r:

$$r \to \lambda r, \quad \zeta \to \lambda^{-2}\zeta.$$

The same relation can be factorized as

$$(\omega + \zeta)(\omega^2 - r^2) + (\omega - \zeta)(\omega^2 + r^2) = 2m^3.$$

We can define a one-dimensional subset of the above 3-dimensional hypersurface by requiring

$$\omega^2 - r^2 = [2m^3 - (\omega - \zeta)(\omega^2 + r^2)]/(\omega + \zeta) = M^2 = \text{Const.}$$

If we have *three* hypersurfaces (corresponding to the dispersion relations of three quarks satisfying the 3-rd order differential equation), which are embedded in the 12-dimensional space $M_4 \times M_4 \times M_4$, then the resulting 3-dimensional hypersurface defined by the above constrained applied to each of the three dispersion relations independently will produce the ordinary mass hyperboloid

$$\omega_1^2 + \omega_2^2 + \omega_3^2 - r_1^2 - r_2^2 - r_3^2 = \Omega^2 - r_1^2 - r_2^2 - r_3^2 = 3M^2.$$

Another way to achieve a similar result is to observe that we need not multiply the solutions of our third-order differential equation *pointwise,* i.e. with the same argument; we should rather multiply the solutions with the same value of t, but with different values of $\mathbf{k}_a$ et $\mathbf{r}_b$. Then we must impose supplementary conditions on the parameters ω_a, $\mathbf{k}_a$ and $\mathbf{r}_c$ in order to cancel all real exponentials in these products. In terms of these variables the resulting constraints amount to something very close to *confinement*, because our solutions will be subjected to the conditions of the general type

$$\mathbf{k}_1.\mathbf{r}_1 - \frac{1}{2}\mathbf{k}_2.\mathbf{r}_2 - \frac{1}{2}\mathbf{k}_3.\mathbf{r}_3 = 0;$$

which will at the same time factorize the cubic dispersion relation producing (although not in a unique way) a relativistic mass hyperboloid for certain linear combinations of ω_a and $\mathbf{k}_b$.

The solutions of our third-order differential equation do not belong to the space of tempered distributions and their Fourier transform is not well defined; also their products cannot be represented as inverse Fourier transforms of the convolution of their Fourier transforms. Nevertheless, as in classical field theory, we can do this if their supports are restricted to positive frequencies only. Then one can write symbolically the convolution of three quark field propagators as follows:

$$\frac{1}{\omega^3 - \mathbf{k}^3 - m^3} * \frac{1}{\omega^3 - \mathbf{k}^3 - m^3} * \frac{1}{\omega^3 - \mathbf{k}^3 - m^3}$$

where $\mathbf{k}^3$ stands for the cubic form $\eta^{abc}k_a k_b k_c$, and the integral is taken over the cubic hyperboloid (;); to the product of wave functions of quark with anti-quark corresponds *one* convolution of two factors of this type.

According to our hypothesis, the convolution of the Fourier transforms of three quark (or anti-quark) propagators should generate the propagator of the corresponding composed particle, i.e. a fermion, whereas the convolution of two such propagators should give the propagator of a boson.

A simple power counting gives the dimension of the Fourier transform of the resulting propagator:

$$(-3) \times 3 + 2 \times D = -9 + 2D \text{ in the first case, and}$$
$$(-3) \times 2 + D = -6 + D \text{ in the second case,}$$

where D is the dimension of the space-time. It is only when the dimension $D = 4$ that we get the resulting propagator of dimension -1 for a ternary combination, and of dimension -2 for a binary combination, which is what is observed indeed for *fermions* and *bosons*, i.e. fields obeying the first and second order wave equations, respectively.

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