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DERIVATION ALGEBRAS AND MULTIPLICATION ALGEBRAS OF SEMI-SIMPLE JORDAN ALGEBRAS

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In this note we investigate the Lie algebra of derivations and the Lie algebra $\mathfrak{L}$ generated by the multiplications in any semi-simple Jordan algebra (with a finite basis) over a field of characteristic 0. We show that the derivation algebra $\mathfrak{D}$ possesses a certain ideal $\mathfrak{J}$ consisting of derivations that we call inner and that $\mathfrak{J}$ is also a subalgebra of the Lie multiplication algebra $\mathfrak{L}$. For semi-simple algebras we prove that $\mathfrak{J} = \mathfrak{D}$. This result is a consequence of a general theorem (Theorem 1) on derivations of semi-simple non-associative algebras of characteristic 0. It can be seen that another easy consequence of our general theorem is the known result that the derivations of semi-simple associative algebras are all inner.[1] Our method can be applied in other cases, too (for example, alternative algebras), but we shall not discuss these here.

An earlier method of obtaining the derivations for the non-exceptional simple Jordan algebras has been given in a forthcoming joint paper by F. D. Jacobson and the present author.[2] We compare the two results and we determine the structure of the derivation algebras. A survey of the results shows that one obtains in this way all the non-exceptional simple Lie algebras. In a forthcoming paper by Chevalley and Schafer the same problems are solved for the exceptional simple Jordan algebra.

1. We recall the definition of a derivation D in a (non-associative) algebra $\mathfrak{A}$ as a linear transformation in $\mathfrak{A}$ that satisfies

$$(xy)D = x(yD) + (xD)y. \tag{1}$$

Thus if we denote the right multiplications $x \rightarrow xa$ by R_a and we replace y by a in (1) we see that this equation is equivalent

$$[R_a , D] \equiv R_aD - DR_a = R_{aD} . \tag{2}$$

Similarly if L_a is the left multiplication $x \rightarrow ax$ then D is a derivation if and only if

$$[L_a , D] = L_{aD} . \tag{3}$$

It is known that the set $\mathfrak{D}$ of derivations is a Lie algebra of linear transformations acting in $\mathfrak{A}$, that is, $\mathfrak{D}$ is closed under addition, under scalar multiplication and under commutation.[3]

If $\mathfrak{A}$ is an associative algebra then

$$R_aR_b = R_{ab} , \qquad L_aL_b = L_{ba} , \qquad [R_a , L_b] = 0 \tag{4}$$

[1] [7] pp. 212–215.
[2] [4]
[3] [7], p. 207.

and these relations show that $D = R_d - L_d$ satisfies (2) for any d. A derivation of this kind is called an *inner* derivation. Their totality is an ideal in the derivation Lie algebra.

Also if $\mathfrak{A}$ is an abstract Lie algebra in the sense that the multiplication satisfies

$$(5) \qquad a^2 = 0, \qquad (ab)c + (bc)a + (ca)b = 0$$

then

$$(6) \qquad R_{ad} = [R_a, R_d]$$

and this shows that $R_d = -L_d$ is a derivation for every d. Derivations of this kind are called *inner* and again it is well known that their totality is an ideal in the derivation algebra.

We consider now the analogous facts for Jordan algebras. Following Albert we define such an algebra by the identities

$$(7) \qquad ab = ba, \qquad (a^2b)a = a^2(ba).$$

The second one of these can be linearized to give

$$(8) \qquad R_{(ab)c} = R_aR_{bc} + R_bR_{ac} + R_cR_{ab} - (R_aR_cR_b + R_bR_cR_a)$$[4]

and this implies that

$$(9) \qquad [R_a, [R_b, R_c]] = R_{A(b,a,c)}$$

where $A(b, a, c)$ is the *associator*

$$(10) \qquad A(b, a, c) = (ba)c - b(ac).$$

The commutative law gives

$$A(b, a, c) = (ab)c - (ac)b = a[R_b, R_c].$$

Hence

$$(11) \qquad [R_a, [R_b, R_c]] = R_{a[R_b,R_c]}.$$

Consequently the mapping $[R_b, R_c]$, or, $x \rightarrow A(b, x, c)$ is a derivation. It follows also that any mapping of the form $x \rightarrow \sum A(b, x, c)$ is a derivation. Such mappings will be called *inner* derivations of the Jordan algebra. Clearly they form a subspace of the derivation algebra. Also it is easy to verify that if we denote $x \rightarrow A(b, x, c)$ by I_{bc} then for any derivation D

$$(12) \qquad [I_{bc}, D] = I_{(bD)c} + I_{b(cD)}.$$

Hence again the inner derivations form an ideal.

In addition to the derivation algebra $\mathfrak{D}$ we shall study in this note the *Lie multiplication algebra* $\mathfrak{L}$ of $\mathfrak{A}$. By this we mean the enveloping Lie algebra, that is, the Lie algebra generated by the multiplications R_a. Because of (11) the set $\mathfrak{A}_r = \{R_a\}$ is a *Lie triple system* of linear transformations, that is, a subspace of

[4] [1] p. 549.

linear transformations that is closed relative to the ternary operation $[A, [B, C]]$.[5] It follows that $\mathfrak{L}$ is the totality of mappings of the form $R_a + \sum[R_b, R_c]$. Thus the subset $\mathfrak{J}$ of inner derivations is a Lie subalgebra of $\mathfrak{L}$.

2. We shall call a non-associative algebra with a finite basis *right* (*left, two-sided*) *semi-simple* if it can be expressed as a direct sum of minimal right (left, two-sided) ideals none of which are right (left, both ways) annihilated by the algebra. Thus $\mathfrak{A}$ is right semi-simple if and only if the set $\mathfrak{A}_r$ of right multiplications is completely reducible and the set of induced mappings in any non-zero invariant subspace (i.e., right ideal) includes non-zero mappings. Similar remarks hold for left semi-simplicity and for (two-sided) semi-simplicity. Of course, all of these concepts coincide for commutative algebras. They coincide also for Lie algebras since multiplication is skew-symmetric in such algebras.

We assume now that $\mathfrak{A}$ is an arbitrary right (left, two-sided) semi-simple algebra that has an identity and a finite basis over a field of characteristic 0. Let $\mathfrak{E}$ and $\mathfrak{L}$, respectively, be the associative and the Lie algebras of linear transformation generated by the right (left, right and left) multiplications in $\mathfrak{A}$. Then $\mathfrak{E}$ and $\mathfrak{L}$ are completely reducible. It follows that $\mathfrak{E}$ is semi-simple and that $\mathfrak{L} = \mathfrak{L}' \oplus \mathfrak{C}$ where $\mathfrak{C}$ is the center and where $\mathfrak{L}'$, the derived algebra, is semi-simple.[6]

Now let D be a derivation in $\mathfrak{A}$. The mapping $\tilde{D}: X \rightarrow [X, D]$ is a derivation in the associative algebra and in the Lie algebra of all linear transformations in the vector space $\mathfrak{A}$. By (2) ((3), (2) and (3)) the subspaces $\mathfrak{E}$ and $\mathfrak{L}$ are mapped into themselves by $\tilde{D}$. Hence $\tilde{D}$ induces derivations in $\mathfrak{E}$ and in $\mathfrak{L}$. Clearly $\tilde{D}$ maps $\mathfrak{L}'$ into itself and it maps the center $\mathfrak{C}$ of $\mathfrak{L}$ into itself. If $C \in \mathfrak{C}$, C is also in the center of $\mathfrak{E}$. Since the center of $\mathfrak{E}$ is a direct sum of separable fields any derivation in $\mathfrak{E}$ maps the elements of the center of $\mathfrak{E}$ into 0. Thus $C\tilde{D} = 0$. Now it is a well-known result of Cartan's that any derivation of a semi-simple Lie algebra $\mathfrak{L}'$ over a field of characteristic 0 is inner.[7] It follows that any derivation of a direct sum $\mathfrak{L}' \oplus \mathfrak{C}$ that maps $\mathfrak{C}$ into 0 is also inner. Thus there exists an element U in $\mathfrak{L}$ such that $[R_a, D] = [R_a, U]$ ($[L_a, D] = [L_a, U]$, both conditions) hold for all a in $\mathfrak{A}$. Consequently

$$\begin{aligned} aD = 1R_{aD} &= 1[R_a, D] = 1[R_a, U] \\ &= aU - (1U)a \end{aligned} \tag{13}$$

or

$$D = U - L_{1U}. \tag{13'}$$

Similarly if $\mathfrak{A}$ is left semi-simple then there is a U in the enveloping Lie algebra of the left multiplications such that

$$D = U - R_{1U}. \tag{14}$$

[5] Systems of this type are considered in [9].

[6] The result on associative algebras is well known. See for example [5] p. 70. The Lie algebra result is given in [6] p. 878.

[7] [2] p. 113 or [3] p. 689.

Finally if $\mathfrak{A}$ is semi-simple then there exists a U in the Lie algebra generated by the R_a and the L_b such that (13′) and (14) hold. Either one implies that D is in $\mathfrak{L}$.

THEOREM 1. *Let $\mathfrak{A}$ be a right (left, two-sided) semi-simple algebra with an identity that has a finite basis over a field of characteristic 0. Then if D is any derivation in $\mathfrak{A}$ there exists an element U in the enveloping Lie algebra of the right (left, right and left) multiplications of $\mathfrak{A}$ such that $D = U - L_{1U}$ ($D = U - R_{1U}$, $D = U$).*

3. Albert has defined the *radical* of a Jordan algebra to be the maximal solvable ideal.[8] Solvability is defined as for Lie algebras; $\mathfrak{S}$ is solvable if the sequence of dyadic powers of $\mathfrak{S}$, namely, $\mathfrak{S}^2 = \mathfrak{S}\mathfrak{S}$, $\mathfrak{S}^4 = \mathfrak{S}^2\mathfrak{S}^2$, $\cdots$ leads to 0. If $\mathfrak{S}$ is solvable it is *nilpotent* in the sense that there exists a positive integer N such that any product of N factors is 0. If the base field has characteristic 0 then Albert has proved that the radical can be determined by a trace condition. Let $\tau(a) = \text{trace } R_a$, then the radical $\mathfrak{R}$ is the totality of elements z such that

$$\tau(xz) = 0 \tag{15}$$

for all x in $\mathfrak{A}$. Also Albert has shown that if $\mathfrak{R} = 0$ then $\mathfrak{A}$ is semi-simple and possesses an identity.

Conversely suppose that $\mathfrak{A}$ is semi-simple in our sense. Since the radical $\mathfrak{R}$ is an ideal, complete reducibility of the set $\{R_a\}$ implies the existence of a complementary ideal $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{B}$. If $\mathfrak{R} \neq 0$ the nilpotencey of $\mathfrak{R}$ implies that there are elements $z \neq 0$ in $\mathfrak{R}$ such that $z\mathfrak{R} = 0$. The set $\mathfrak{Z}$ of these elements satisfies $\mathfrak{Z}\mathfrak{R} = 0$ and $\mathfrak{Z}\mathfrak{B} \subseteq \mathfrak{R} \cap \mathfrak{B} = 0$. Hence $\mathfrak{Z}\mathfrak{A} = 0$ contrary to the definition of semi-simplicity. Thus $\mathfrak{R} = 0$. Also Albert's results now show that $\mathfrak{A}$ has an identity.

If $\mathfrak{A}$ is a Jordan algebra with an identity 1, then the set (1) of multiples of 1 is *characteristic* relative to derivations in the sense that it is mapped into itself by every D in the derivation algebra. For it is clear that $1D = 0$ for any D. Another characteristic subspace is defined by the condition $\tau(y) = 0$. This is clear from (2). The space $\mathfrak{A}_0$ defined by this condition is either the whole space $\mathfrak{A}$ or it has dimension $n - 1$, n the dimension of $\mathfrak{A}$. If $\mathfrak{A}$ has an identity it is clear that $(1) \not\subseteq \mathfrak{A}_0$. Hence $\mathfrak{A} = (1) \oplus \mathfrak{A}_0$.

We can use the trace condition also to prove that the radical $\mathfrak{R}$ is characteristic. For if $z \in \mathfrak{R}$ and D is a derivation then

$$\begin{aligned}\tau(x(zD)) &= \text{trace } R_{x(zD)} = \text{trace } R_{(xz)D-(xD)z} \\ &= \text{trace } R_{(xz)D} - \text{trace } R_{(xD)z} = 0\end{aligned}$$

for all x. Hence $zD \in \mathfrak{R}$.

We consider now the theory of derivations for semi-simple Jordan algebras. We prove first the following

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[8] The results on the structure theory of Jordan algebras that are stated here without proofs are given in Albert's paper [1].

THEOREM 2. *Every derivation of a semi-simple Jordan algebra with a finite basis over a field of characteristic* 0 *is inner.*

PROOF. We have shown that $\mathfrak{A}$ has an identity. Hence Theorem 1 is applicable. Also we know that $\mathfrak{L}$ is the set of mappings of the form $R_d + \sum [R_b, R_c]$. Hence $D = R_d + \sum[R_b, R_c]$. Hence $1D = d + \sum(bc - cb)$. Since $1D = 0$, $d = 0$ and $D = \sum[R_b, R_c]$.

4. We shall determine next the structure of the derivation algebras of semi-simple Jordan algebras. It is easy to see that we have a direct sum reduction to the case of simple Jordan algebras. Hence we restrict our attention to these.

The simple Jordan algebras over a field of characteristic 0 fall into three "great" classes and one exceptional class.[9] The first great class consists of the Jordan algebras that are obtained by using the multiplication $a \cdot b = 1/2(ab + ba)$ in simple associative algebras $\mathfrak{A}$. We denote by $\mathfrak{A}_j$ the Jordan algebra that is obtained from a particular associative algebra $\mathfrak{A}$ by this process. The second great class consists of the Jordan subalgebras $\mathfrak{H}(\mathfrak{A}, J)$ of J-symmetric elements in a simple involutorial algebra $\mathfrak{A}$ with involution J. The third great class is constituted by the algebras that define Clifford systems. These have bases $1, u_1, u_2, \cdots, u_n$ such that 1 is the identity and

$$u_i^2 = \alpha_i \neq 0, \qquad u_i u_j = 0 \quad \text{if} \quad i \neq j. \tag{16}$$

Finally we have the exceptional Jordan algebras corresponding to the system M_3^8 discovered by Jordan, v. Neumann and Wigner. These have 27 dimensions over their centers.

The derivation algebras of the non-exceptional Jordan algebras have been determined in a joint paper by F. D. Jacobson and the present author using the above representations of these algebras.[10] For the first type we showed that any derivation has the form $x \rightarrow [x, d] \equiv xd - dx$. If we denote this mapping as D_d it is clear that $D_d = 0$ if and only if d is in the center. The correspondence $d \rightarrow D_d$ is linear and $D_{[d_1, d_2]} = [D_{d_1}, D_{d_2}]$. It follows that the derivation algebra $\mathfrak{D}$ of $\mathfrak{A}_j$ is isomorphic to $\mathfrak{A}_l/\mathfrak{C}$ where $\mathfrak{A}_l$ is the Lie algebra obtained by replacing ordinary multiplication in $\mathfrak{A}$ by commutation and $\mathfrak{C}$ is the center. It is also easy to see that $\mathfrak{A}_l = \mathfrak{A}_l' \oplus \mathfrak{C}$ where $\mathfrak{A}_l'$ is the derived algebra. Hence $\mathfrak{D} \cong \mathfrak{A}_l'$.

We now compare these results with those that can be obtained from Theorem 2. Applying Theorem 2 we see that any derivation in $\mathfrak{A}_j$ has the form $x \rightarrow \sum A(b, x, c)$. We can verify that

$$A(b, x, c) = (b \cdot x) \cdot c - b \cdot (x \cdot c) = 1/4[x, [b, c]]. \tag{17}$$

Hence the inner derivations $x \rightarrow \sum A(b, x, c)$ has the form $x \rightarrow [x, d]$ noted above.

For the Jordan algebra $\mathfrak{H}(A, \mathfrak{J})$ it has been proved that the derivations have the form $x \rightarrow [x, d]$ where d is J-skew.[11] The enveloping associative algebra of

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[9] [4] and [10].
[10] [4].
[11] [4].

$\mathfrak{H}(\mathfrak{A}, J)$ is $\mathfrak{A}$. Hence if the derivation $D_d = 0$, d is in the center of $\mathfrak{A}$. If J is an involution of first kind the elements of the center are symmetric.[12] It follows that if d is skew and $D_d = 0$ then $d = 0$. It follows that the derivation algebra $\mathfrak{D}$ is isomorphic to the Lie algebra $\mathfrak{S}(\mathfrak{A}, J)$ of J-skew elements (commutation as the product). By Theorem 2 and (17) we see that any derivation in $\mathfrak{H}(A, J)$ has the form

$$x \to [x, \sum [b, c]]$$

where b and c are J-symmetric. Now it is clear that the commutator of two J-symmetric elements is J-skew. Hence this checks with the above. We remark also that it is a consequence of these results that any J-skew element can be expressed as a sum of commutators of J-symmetric elements if J is of the first kind. In a similar manner we see that if J is of second kind, then the derivation algebra of $\mathfrak{H}(\mathfrak{A}, J)$ is isomorphic to $\mathfrak{S}(\mathfrak{A}, J)'$. Moreover, any element of $\mathfrak{S}(\mathfrak{A}, J)'$ can be written as a sum of commutators of J-symmetric elements.

In the case of Jordan algebras defined by (16) any derivation maps the space spanned by the u_i into itself. Thus $u_iD = \sum \mu_{ij}u_j$, $1D = 0$ and the matrix (μ) is J-skew where J is the involution defined by $(\xi) \to (\alpha)(\xi)'(\alpha)^{-1}$ where

$$(\alpha) = \text{diag}\,\{\alpha_1, \alpha_2, \cdots, \alpha_n\}.$$

It follows that the derivation algebra is isomorphic to the Lie algebra of J-skew matrices determined by this J.

The derivation algebra $\mathfrak{D}$ of the exceptional Jordan algebras will be determined in a forthcoming joint paper by Chevalley and Schafer. Their result is that if the basis field is algebraically closed then the derivation algebra of the (single) exceptional Jordan algebra M_3^8 is the exceptional simple Lie algebra F_4 of 52 dimensions. It follows that the derivation algebra of any exceptional Jordan algebra is of *type* F_4 in the sense that if its center is extended to its algebraic closure, then the resulting algebra is F_4.

A survey of these results shows that with the exception of a few easily enumerated cases of low dimensionality the derivation algebras of simple Jordan algebras are simple Lie algebras.[13] Moreover, every non exceptional simple Lie algebra can be obtained in this way. Finally the exceptional Lie algebra of 52 dimensions can be obtained as a derivation algebra of a simple Jordan algebra.

5. We shall determine next the structure of the Lie algebras $\mathfrak{L}$ for the simple Jordan algebras. It will be convenient to denote the Jordan algebra as before by $\mathfrak{A}$. For the first two classes we have $1-1$ imbeddings in associative algebras, that is, we have a $1-1$ mapping $a \to U_a$ where U_a is in the associative algebra such that this mapping is linear and

$$U_{ab} = U_a \cdot U_b \equiv 1/2(U_aU_b + U_bU_a). \tag{18}$$

[12] An involution of a simple associative algebra is of first kind if it leaves the elements of the center fixed and it is of second kind if it induces a non-trivial automorphism in the center.

[13] [8].

For the first class the U_a form a simple associative algebra while in the second the U_a are the J-symmetric elements of a simple involutorial algebra. In the first case we shall replace our original imbedding by a more convenient one. This is obtained by forming the direct sum of the imbedding algebra and an algebra anti-isomorphic to it. The elements of the new algebra have the form $U + \bar{V}$ where multiplication is multiplication of components and where $\bar{U}$ denotes the image of U under a definite anti-isomorphism between the component algebras. The mapping $J: U + \bar{V} \to V + \bar{U}$ is an involution in the direct sum, whose symmetric elements have the form $U + \bar{U}$. The correspondence $a \to S_a = U_a + \bar{U}_a$ is a $1 - 1$ imbedding of the Jordan algebra $\mathfrak{A}$. Thus we now have the same situation as for the second class, that is, we have a $1 - 1$ imbedding in an involutorial associative algebra, now denoted as $\mathfrak{U}$, such that the set of elements corresponding to $\mathfrak{A}$ is the set $\mathfrak{H}(\mathfrak{U}, J)$ of J-symmetric elements. For the sake of uniformity we denote the imbedding in both cases by $a \to S_a$.[14]

Now let Λ denote the enveloping Lie algebra of the elements S_a. We propose to show that $\mathfrak{L}$ *is isomorphic to* Λ. First we shall show that the set $\mathfrak{H}(\mathfrak{U}, J)$ is a Lie triple system isomorphic to the Lie triple system $\mathfrak{A}_r$ of multiplications R_a. Consider the mapping $R_a \to 1/2S_a$. This is $1 - 1$ and linear. Moreover by (9) and (17)

$$\begin{aligned}[R_a, [R_b, R_c]] &= R_{A(b,a,c)} \to 1/2\, S_{A(b,a,c)} \\ &= 1/2((S_b \cdot S_a) \cdot S_c - S_b \cdot (S_a \cdot S_c)) \\ &= 1/8[S_a, [S_b, S_c]] \\ &= [1/2S_a, [1/2S_b, 1/2S_c]].\end{aligned}$$

This proves that the mapping $R_a \to 1/2\, S_a$ is an isomorphism for the Lie triple system.

Suppose now that $R_a + \sum[R_b, R_c] = 0$. Then $1(R_a + \sum [R_b, R_c]) = 0$ and this gives $a = 0$. Hence $R_a = 0$, $S_a = 0$ and $\sum[R_b, R_c] = 0$. This last relation implies that $\sum[R_x, [R_b, R_c]] = 0$ for all x. Hence $\sum[S_x [S_b, S_c]] = 0$. Thus $\sum[S_b, S_c]$ is in the center of $\mathfrak{U}$. Since the enveloping associative algebra of the S_a is $\mathfrak{U}$ and $\mathfrak{U}$ is semi-simple this implies that $\sum[S_b, S_c] = 0$.[15] Thus if $R_a + \sum[R_b, R_c] = 0$ then $1/2\, S_a + 1/4 \sum[S_b, S_c] = 0$. Conversely, suppose that $1/2\, S_a + 1/4 \sum[S_b, S_c] = 0$. Then since S_a is J-symmetric and $\sum[S_b, S_c]$ is skew, $S_a = 0$ and $\sum[S_b, S_c] = 0$. Since $a \to S_a$ is $1 - 1$ this implies that $a = 0$ and hence that $R_a = 0$. Also as before $\sum[R_x, [R_b, R_c]] = 0$. By (9) $\sum A(b, x, c) = 0$. Hence $x(\sum[R_b, R_c]) = 0$. Thus $\sum[R_b, R_c] = 0$. It is now immediate that the mapping

$$R_a + \sum[R_b, R_c] \to 1/2\, S_a + 1/4 \sum[S_b, S_c] \tag{19}$$

is $1 - 1$ and linear. Since commutation in $\mathfrak{L}$ and in Λ are determined by the

[14] The imbedding $a \to S_a$ is a "universal" imbedding in the sense of [4].

[15] [6] Lemma 4, p. 877.

ternary commutation in $\mathfrak{A}_r$ and in $\mathfrak{H}(\mathfrak{U}, J)$ respectively it follows that (19) is an isomorphism of $\mathfrak{L}$ into Λ.

It is fairly straightforward to determine the elements of Λ. If $\mathfrak{A}$ is in the second class and J is an involution of first kind then we know that any J-skew element is a sum of commutators of J-symmetric elements. Hence any J-skew element as well as any J-symmetric element is in Λ. Thus $\Lambda = \mathfrak{U}_l$.

If J is of second kind ($\mathfrak{A}$ in the second great class) then we assert $\Lambda = \mathfrak{U}_l' \oplus \mathfrak{C}_0$ where $\mathfrak{C}_0$ is the set of symmetric elements of the center. Clearly $\Lambda \supseteqq \mathfrak{C}_0$. Moreover since any element of $\mathfrak{U}$ is a sum of J-symmetric element and a J-skew element any element of $\mathfrak{U}_l'$ is a sum of elements of the forms $[H, H']$, $[H, S]$, $[S, S']$ where H, H' are J-symmetric and S, S' are J-skew. Since $[H, S]$ is J-symmetric such an element belongs to Λ. Evidently $[H, H']$ is in Λ and as we noted before any element of the form $[S, S']$ is a sum of commutators of elements of $\mathfrak{H}(\mathfrak{U}, J)$. Hence any $[S, S']$ is in Λ. Thus $\Lambda \supseteqq \mathfrak{U}_l'$. Hence $\Lambda \supseteqq \mathfrak{U}_l' + \mathfrak{C}_0$. If $\Lambda \supset \mathfrak{U}_l' + \mathfrak{C}_0$ the decomposition $\mathfrak{U} = \mathfrak{U}_l' \oplus \mathfrak{C}$, $\mathfrak{C}$ the whole center, shows that Λ contains an element of $\mathfrak{C}$ that is not symmetric. It follows that Λ contains a non-zero skew element of the center. On the other hand the elements of Λ that are J-skew are in $\mathfrak{U}_l'$ and this contradicts $\mathfrak{U}_l' \cap \mathfrak{C} = 0$. Thus $\Lambda = \mathfrak{U}_l' \oplus \mathfrak{C}_0$.

In a similar manner we can treat the case in which $\mathfrak{A}$ is in the first class. We observe first that a commutator of J-skew elements is also a commutator of J-symmetric elements. For let $U - \bar{U}$ and $V - \bar{V}$ be skew. Then

$$[U - \bar{U}, V - \bar{V}] = [U, V] + [\bar{U}, \bar{V}] = [U + \bar{U}, V + \bar{V}].$$

It follows from this that $\Lambda \supseteqq \mathfrak{U}_l'$. Also Λ contains $\mathfrak{C}_0$ the totality of elements of the form $C + \bar{C}$, C in the center of the first component algebra. It follows as in the preceding case that $\Lambda = \mathfrak{U}_l' \oplus \mathfrak{C}_0$.

In all of the cases considered thus far Λ is a direct sum of a commutative Lie algebra and one or two simple Lie algebras of type A.

We consider next the Lie multiplication algebras for the Jordan algebras defined by (16). If we use the basis $(1, u_1, u_2, \cdots, u_n)$ the matrix of R_b, $b = \beta_0 1 + \sum \beta_i u_i$ is

$$\begin{Bmatrix} \beta_0 & \beta_1 & \cdot & \cdot & \beta_n \\ \beta_1\alpha_1 & \beta_0 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \beta_n\alpha_n & & & & \beta_0 \end{Bmatrix}. \tag{20}$$

As before we denote by J the involution $(\xi) \to (\alpha)(\xi)'(\alpha)^{-1}$ in the $n \times n$ matrix algebra. If we recall that the derivations of $\mathfrak{A}$ have as matrices relative to $(1, u_1, \cdots, u_n)$ the matrices

$$\begin{pmatrix} 0 & \\ & (\mu) \end{pmatrix} \tag{21}$$

where (μ) is J-skew, we see that $\mathfrak{L}$ is isomorphic to the Lie algebra of matrices that are sums of matrices of the form (20) and of the form (21). The latter Lie algebra is a direct sum of a commutative Lie algebra and the Lie algebra of matrices of the form

$$(22) \qquad \begin{Bmatrix} 0 & \beta_1 & \cdot & \cdot & \cdot & \beta_n \\ \beta_1\alpha_1 & & & & & \\ \cdot & & & (\mu) & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \beta_n\alpha_n & & & & & \end{Bmatrix}.$$

It can be verified that this algebra can be characterized as the set of K-skew matrices of $(n + 1)$ rows where K is the involution $(\xi) \to (\gamma)(\xi)'(\gamma)^{-1}$ where

$$(\gamma) = \text{diag } \{-1, \alpha_1, \alpha_2, \cdots, \alpha_n\}.$$

This is a Lie algebra of type B or D. It may be noted also that by Theorem 2 any matrix of the form (21) is a sum of commutators of matrices of the form (20).

The determination of $\mathfrak{L}$ for the exceptional Jordan algebra will be given in the paper by Chevalley and Schafer referred to above. Here it will be shown that if the base field is algebraically closed then $\mathfrak{L} = (1) + E_6$. Where E_6 is the third exceptional simple Lie algebra (of 78 dimensions).

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