Research Interests

I have always been interested in the *algebraic structure of nonassociative systems*, primarily Jordan algebras, superalgebras, triples, and pairs, with excursions into associative and alternative algebras. Jordan algebras arose out of quantum mechanics as a way to capture the algebraic essence of hermitian operators on Hilbert space, where only *hermitian* operators correspond to physical observables. Thus if x, y are observables, so are $\frac{1}{2}(xy + yx)$ and x^2, but not xy. Pascual Jordan found it philosophically unsatisfying to study the algebraic structure using products derived from xy when that product itself was un-observable, so he attempted to develop the entire theory in terms of an observable product $x \cdot y$ (corresponding to the *Jordan product* $\frac{1}{2}(xy + yx)$ for hermitian operators), without ever referring to an un-observable product xy determining the structure from behind the scenes. A *Jordan algebra* is an algebra whose product is *commutative*, $x \cdot y = y \cdot x$, and satisfies the *Jordan identity* $(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x)$.

An associative algebra A always gives rise to a Jordan algebra A^+ via the Jordan product (just as it gives rise to a Lie algebra A^- via the Lie product $[x, y] := xy - yx$). If A carries an involution $*$ then the hermitian or symmetric elements $x^* = x$ form a Jordan (but not an associative) subalgebra (just as the skew elements form a Lie subalgebra). Inside the Clifford algebra C of a bilinear form $\langle v, w\rangle$ on a vector space V over a field F, the defining relation $vw + wv = \langle \cdot, \cdot \rangle 1$ shows that the subspace $F1 \oplus V$ is closed under the Jordan product, and so forms a Jordan subalgebra of C^+. A Jordan algebra is *special* if its product comes from the Jordan product in some associative algebra, i.e. arises as a Jordan subalgebra of some A^+, otherwise it is *exceptional*. Jordan sought an exceptional setting for quantum mechanics. In their classical 1934 paper, Jordan, von Neumann, and Wigner showed that in finite dimensions all formally real Jordan algebras are direct sums of simple algebras, and all simple algebras are either of Clifford type (determined by the dot product on R^n) or of hermitian type ($n \times n$ hermitian matrices over the reals, complexes, quaternions, or (for $n = 3$ only) octonions). In all cases except the 27-dimensional "Albert algebra" of 3×3 hermitian matrices over the nonassociative algebra of octonions, the Jordan algebra was special, its product came from the product xy in some associative algebra, in which case a behind-the-scenes associative product was inescapable. Von Neumann tried to extend this to infinite-dimensional algebras, but was unable to do so. In 1979 Zel'manov finally showed that, even in infinite dimensions, the only simple Jordan algebras which are exceptional (do not come from a lurking associative algebra) are the Albert algebras of dimension 27, too small to encompass quantum mechanics. Indeed, he classified all simple (even prime) Jordan algebras: they were of Clifford type (coming from nondegenerate bilinear forms), Hermitian type (spaces of hermitian elements in an associative algebra with involution), or Albert algebras. Zel'manov went on over the next few years to single-handedly settle most of the outstanding structure theory questions for Jordan systems (algebras, triples, pairs, and superalgbras), then moved on to the Restricted Burnside Theorem and a Fields Medal.

These Jordan algebras (especially the Albert algebras) have unexpectedly turned out to play important roles in many different areas of mathematics: Lie groups and algebras, differential geometry (real symmetric spaces and complex bounded symmetric domains), projective geometry (Moufang planes), probability, and statistics, among others.

I have devoted much of my effort to understanding Jordan structures where there is no scalar $\frac{1}{2}$ available (e.g. Jordan rings over the integers, or over fields of characteristic 2), which necessitate a *quadratic approach* based on the product xyx instead of $\frac{1}{2}(xy+yx)$. Jacobson introduced the ancillary product $U_x y := 2x\cdot(x\cdot y) - x^2\cdot y$ in any Jordan algebra (reducing to xyx in special algebras) to help establish results about inverses and isotopes: x is an invertible element iff U_x is an invertible operator. The product $U_x y$ is not bilinear, it is linear in y but quadratic in x; by linearizing the quadratic term we obtain a trilinear *Jordan triple product* $\{x,y,z\} := \frac{1}{2}U_{x,z}y$ (reducing to $\frac{1}{2}(xyz+zyx)$ in special algebras). Just as any invertible element u in an associative algebra A induces a new associative structure A^u with new product $x_u y = xuy$ and new unit $1_u = u^{-1}$, so any invertible element in a Jordan algebra induces a new Jordan structure $J^{(u)}$ with new product $x\cdot_u y = \{x,u,y\}$ and unit u^{-1}. Jacobson conjectured, but could not prove, the Fundamental Formula $U_{U_x y}z = U_x U_y U_x z$ for these U-operators. This was proven in 1958 by I.G. Macdonald, as part of Macdonald's Theorem that *any* Jordan expression in three variables x,y,z linear in z will vanish identically in all Jordan algebras as soon as it vanishes in all special Jordan algebras. In 1966 I gave an axiomatization of *quadratic Jordan algebras* based on the U-operators: a unital quadratic Jordan algebra is a space with unit element 1 and quadratic product $U_x y$ (think: xyx) satisfying $U_1 = Id,\ U_{U_x y} = U_x U_y U_x,\ V_{x,y}U_x = U_x V_{y,x}$ (also $V_{U_x y,y} = V_{x,U_y x}$) where $V_{x,y}(z) = (U_{x+z} - U_x - U_z)(y)$ is the *Jordan triple product* $\{x,y,z\}$ (think $xyz+zyx$, but beware: in order to avoid $\frac{1}{2}$, the triple product $\sim xyz+zyx$ in quadratic Jordan algebras is twice that $\sim \frac{1}{2}(xyz+zyx)$ in linear Jordan algebras, yet we use exactly the same symbol $\{x,y,z\}$ in both cases). When a scalar $\frac{1}{2}$ is available, there is a 1-1 correspondence between these quadratic Jordan algebras and unital linear Jordan algebras, via $x\cdot y := \frac{1}{2}\{x,1,y\}$ and $U_x y := 2x\cdot(x\cdot y) - x^2\cdot y$.

Jordan triples and *pairs* have a triple product like a quadratic Jordan algebra, but contain no unit and no bilinear products. Examples of Jordan triples are rectangular $m\times n$ matrices over an associative ring with involution with $U_x y := xy^*x$ and $\{x,y,z\} := xy^*z + zy^*x$ (x^* the conjugate transposed matrix), and also a 16-dimensional exceptional triple of 1×2 octonion matrices. The analogous (but more general) Jordan pair would be the pair of spaces $J^+ = Hom(V,W), J^- = Hom(W,V)$ acting on each other (but not on themselves) like Jordan triples via $U_{x^\varepsilon}y^{-\varepsilon} = x^\varepsilon \circ y^{-\varepsilon} \circ x^\varepsilon$ ($\varepsilon = \pm$) in terms of the ordinary composition $\circ$ of linear transformations. In differential geometry it is not Jordan algebras that play a leading role, it is the Jordan triple product which arises naturally from bounded symmetric domains; even if the triple has invertible elements, hence isotopes which are Jordan algebras, it is geometrically artificial to elect one of these as unit, so Jordan triple systems are the natural objects.

Jordan pairs were invented by Ottmar Loos, and were born of the Tits-Kantor-Koecher construction of Lie algebras: if $L_{-1}\oplus L_0\oplus L_1$ is a short 3-graded Lie algebra (L_iL_j falls in L_{i+j}, so vanishes if $|i+j|\geq 2$), then (L_1, L_{-1}) forms a Jordan pair: the Jacobi identity is forces L_1, L_{-1} to act on each other (but not on themselves) in a Jordan triple fashion. Jordan pairs form a crucial tool in understanding ordinary Jordan algebras, as Zelmanov's classification has shown.

Lie superalgebras were invented by physicists to decribe *supersymmetry* between bosons and fermions, encompassing both types of particles within one algebraic framework, and in algebra it has become standard that *superobject* just means *Z_2-graded object*. *Associative superalgebras* had long been known in toplogy: they were simply Z_2-graded associative algebras $A = A_0\oplus A_1$ where

$A_iA_j \subseteq A_{i+j}$ (with indices read modulo 2). A typical example would be $A = End(V)$ for a graded vector space $V = V_0 \oplus V_1$ with $A_0 = End(V_0) \oplus End(V_1)$ consisting of the linear transformations which preserve degree, and $A_1 = Hom(V_0, V_1) \oplus Hom(V_1, V_0)$ the transformations which change degree (as matrices with block decomposition, the even part consists of the two diagonal blocks and the odd part consists of the two off-diagonal blocks).

In contrast, *Lie superalgebras* are also Z_2-graded algebras $L = L_0 \oplus L_1$, but they are not Lie algebras, instead they satisfy a graded version of the Lie axioms: the "even part" L_0 is a Lie algebra, and the "odd part" L_1 is a Lie-bimodule having a Jordan-like symmetric product into L_0. Axiomatically, Lie superalgebras satisfy graded anticommutativity $x_i \bullet y_j = -(-1)^{ij} y_j \bullet x_i$ and the graded Jacobi identity $\sum_{cylic}(-1)^{ik} x_i \bullet (y_j \bullet z_k)$. Similarly, *Jordan superalgebras* are not Jordan algebras, they are graded algebras $J = J_0 \oplus J_1$ satisfying a graded version of the Jordan axioms: graded commutativity $x_i \cdot y_j = +(-1)^{ij} y_j \cdot x_i$ and the graded Jordan identity $\sum_{cyclic}(-1)^{(i+\ell)k}\Big(((x_i \cdot y_j) \cdot w_\ell) \cdot z_k - (x_i \cdot y_j) \cdot (w_\ell \cdot z_k)\Big)$. Thus J_0 is a Jordan algebra with J_1 as bimodule, and J_1 carries a Lie-like skew product $x_1 \cdot y_1$ into J_0. A typical example of a Lie superalgebra would be A^{-s} for an associative superalgebra A with super-Lie product $x_iy_j - (-1)^{ij} y_jx_i$; a Jordan superalgebra is *super-special* if it can be imbedded some A^{+s} with super-Jordan product $\frac{1}{2}(x_iy_j + (-1)^{ij} y_jx_i)$, reducing to the Jordan product if at least one factor is even, and to the Lie bracket if both factors are odd. For example, if $*$ is a *super-involution* $(x_iy_j)^* = (-1)^{ij} y_j^* x_i^*$ on an associative superalgebra A, then the super-skew elements $x_i^* = (-1)^i x_i$ form a Lie sub-superalgebra of A^{-s}, and the super-symmetric elements $x_i^* = (-1)^i x_i$ form a Jordan sub-superalgebra of A^{+s}. Jordan superalgebras form another tool which is important in understanding ordinary Jordan algebras: the "Pchelintsev monsters", degenerate prime Jordan algebras which are not of the 3 classical Zelmanovian types can be understood as the spawn of certain Jordan superalgebras.

As algebraic systems get further away from associativity, they usually become more amorphous and less important. Besides the great classes of Lie and Jordan algebras, certain other classes pay a useful supporting role in algebra. One of the most important is that of the *alternative algebras* are defined axiomatically as those bilinear algebras satisfying the identities $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all elements x, y, equivalently $L_{x^2} = L_x^2$ and $R_{x^2} = R_x^2$. These are natural generalizations of the *composition algebras* (unital algebras carrying a nondegenerate quadratic form Q which "permits composition" with the algebra product, $Q(xy) = Q(x)Q(y)$ and $Q(1) = 1$). The classical Hurwitz Theorem shows that a composition algebra over a field has dimension 1,2,4, or 8, and is either the base field, a quadratic extension thereof, a quaternion algebra, or an octonion algebra, in particular *it is necessarily finite-dimensional and alternative.* Over the field of real numbers the composition algebras with positive-definite quadratic forms are precisely the reals, complexes, Hamilton's quaternions, and Cayley's octonions. The celebrated Bruck-Kleinfeld-Skornyakov Theorem asserts that the only simple alternative algebras (of arbitrary dimension) are either associative or are 8-dimensional octonion algebras over their center. This result presaged Zelmanov's Theorem that the only simple Jordan algebras are special or 27-dimensional Albert algebras over their center.

In studying nonassociative algebras, two convenient notations are the *commutator* $[x, y] := xy - yx$, measuring how far the elements x, y are from commuting, and the *associator* $[x, y, z] := (xy)z - x(yz)$, measuring how far the three elements are from associating. An algebra is commutative iff $[x,y]=0$ for all elements x,y, and is associative iff $[x,y,z]=0$ for all x,y,z. The linear Jordan axioms may be written as $[x,y]=[x^2,y,x]=0$, and the alternative laws as $[x,x,y]=[y,x,x]=0$. A variety which hugs closely to Jordan theory, but sacrifices the commutative law, is the *noncommutative Jordan algebras*: these have a bilinear product satisfying the identities $[x^2,y,x]=0$ and $[x,y,x]=0$ (the *flexible law* $(xy)x=x(yx)$ being a weakened form of commutativity), or equivalently the condition that for each element x the left- and right- multiplication operators $L_x, R_x, L_{x^2}, R_{x^2}$ all commute with each other. All Jordan, associative, and alternative algebras are noncommutative Jordan algebras. The structure theory of finite-dimensional noncommutative Jordan algebras (in characteristic not two) yields as simple algebras (1) the commutative Jordan algebras, (2) the *quasi-associative algebras* $A^{(\lambda)}$ obtained from the product $x\cdot_\lambda y := \lambda xy+(1-\lambda)yx$ in an associative algebra A for a scalar $\lambda \neq \frac{1}{2}$ ($\lambda=\frac{1}{2}$ would just give a special Jordan algebra A^+), (3) the flexible *degree* 2 algebras (where each element satisfies a degree two equation $x^2-t(x)x+n(x)1=0$ for scalars $t(x), n(x)$), and (4) the degree 1 *nodal algebras* (which occur only over a field F of characteristic $p\neq 0$ in the truncated polynomial ring $F[x_1,\ldots,x_n]/\langle x_1^p,\ldots,x_n^p\rangle$).

The most general variety of nonassociative algebras of common interest is that of *power associative* algebras, where every element x generates an associative subalgebra, equivalently $x^n x^m = x^{n+m}$ for all natural numbers n,m. This is the most general setting where one can talk about algebraic elements and their minimum polynomials, including norm and trace. Once one leaves the confines of power-associativity, one enters a different realm.

In what follows I have grouped my publications according to the algebraic structures being investigated.

Linear Jordan Algebras

[LJ1] Jordan algebras of degree 1, Bull. Amer. Math. Soc. 70 (1964), 702. [MR 29 # 2286]
[LJ2] Macdonald's theorem with inverses, Pacific J. Math. 21 (1967), 315–325. [MR 38 # 1138]
[LJ3] A note on reduced Jordan algebras, Proc. Amer. Math. Soc. 19 (1968), 964–970. [MR 37 # 2826]
[LJ4] Jordan algebras with interconnected idempotents, Proc. Amer. Math. Soc. 19 (1968), 1327–1336. [MR 38 # 203]
[LJ5] Nondegenerate Jordan rings are von Neumann regular, J. Algebra 11 (1969), 111–115. [MR 38 # 2180]
[LJ6] Malcev's theorem for Jordan algebras, Comm. Algebra 5 (1977), 937–967. [MR 58 # 28107]
[LJ7] Middle nucleus = center in Jordan algebras, Proc. Amer. Math. Soc. 86 (1982), 21–24 (with Ng Seong Nam). [MR 84a:17013]

The Fundamental Formula made it child's play to establish the basic facts about inverses and isotopes. [LJ1] showed how the U-operator could be used to drastically simplify a complicated result of Jacobson in the structure theory (showing that a central simple Jordan algebra with only one nonzero idempotent (of "degree 1") was just a 1-dimensional copy of the base field. [LJ2] extended Macdonald's Theorem to polynomial expressions in x, x^{-1}, y, y^{-1} linear in z. (Extending this to *rational* expressions, such as Hua's formula $(x+U_x y^{-1})^{-1}+(x+y)^{-1}=x^{-1}$, remains an open problem to this day.)

A Jordan algebra is *nondegenerate* if it has no nonzero *trivial elements* z with $U_z = 0$. Jacobson had introduced this as the natural semisimpliicity condition for his celebrated Artin-Wedderburn structure theory for Jordan algebras with d.c.c. on inner ideals. (Nondegeneracy later proved to be exactly the correct condition for Zelmanov's structure theory for arbitrary algebras.) [LJ5] showed that finite-dimensional nondegenerate Jordan algebras are regular in the sense of von Neumann: all $x \in J$ have $x \in U_x J$ (associatively this means $x = xyx$ for some y, implying x has a "generalized inverse").

Jacobson's structure theory of Jordan algebras made heavy use of his Coordinatization Theorem, that if J had unit $1 = \sum e_i$ a sum of $n \geq 3$ orthogonal idempotents e_i which were pairwise *connected* (each pair e_i, e_j is connected by an element $u_{ij} \in J_{ij}$ of the off-diagonal Peirce subspace, $u_{ij}^2 = u_{ii} + u_{jj}$ for elements u_{kk} invertible in the subalgebra $J_{kk},\ k = i, j$). Connection is a strong condition to impose, though it held in Jacobson's case where the subaglebras J_{kk} were division algebras. [LJ4] introduced a more general notion of *interconnection* (that $e_i \in U_{e_i}(J_{ij} \cdot J_{ij})$, showed that orthogonal idempotents in any simple Jordan algebra are interconnected in this sense, and that interconnected algebras are special if $n \geq 4$.

The *center* of a nonassociative algebra is the set of elements c which commute and associate with all other elements $x, y,\ [c, x] = [c, x, y] = [x, c, y] = [x, y, c] = 0$; the *centroid* is the set of linear operators γ which commute with all multiplications, $\gamma(x \cdot y) = \gamma(x) \cdot y = x \cdot \gamma(y)$, i.e. $[\gamma, R_y] = [\gamma, L_x] = 0$. In a commutative algebra the center condition reduces to $[c, x, y] = [x, c, y] = 0$ and the centroid condition to $[\gamma, L_x] = 0$. It is easy to see that $[c, y, x] = 0$ implies L_c, L_x commute, so L_c lies in the centroid and hence c in the center. In general $[x, c, y] = 0$ does not by itself imply centrality, but in [LJ7] Ng Seong Nam and I showed that it does for semiprime algebras.

Quadratic Jordan Algebras

Foundations of Quadratic Jordan Algebras

[QF1] A general theory of Jordan rings, Proc. Nat. Acad. Sci. U.S.A. 56 (1966), 1071–1079. [MR 34 # 2643]
[QF2] The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras, Trans. Amer. Math. Soc. 139 (1969), 495–510. [MR 39 # 276]
[QF3] The Freudenthal-Springer-Tits constructions revisited, Trans. Amer. Math. Soc. 148 (1970), 293–314. [MR 42 # 6064]
[QF4] Quadratic Jordan algebras and cubing operations, Trans. Amer. Math. Soc. 153 (1971), 265–278. [MR 42 # 3138]
[QF5] Representations of quadratic Jordan algebras, Trans. Amer. Math. Soc. 153 (1971), 279–305. [MR 42 # 3139]
[QF6] Quadratic Jordan algebras of quadratic forms with base points, J. Indian Math. Soc. 35 (1971), 1–45 (with N. Jacobson). [MR 50 # 9999]
[QF7] The generic norm of an isotope of a Jordan algebra, Scripta Math. 29 (1973), 229–236. [MR 53 # 8173]
[QF8] Axioms for inversion in Jordan algebras, J. Algebra 47 (1977), 201–222. [MR 57 # 9776]

[QF9] Localization of Jordan algebras, Comm. Algebra 6 (1978), 911–958 (with N. Jacobson and M. Parvathi). [MR 58 # 28106]
[QF10] Peirce ideals in Jordan algebras, Pacific J. 78 (1978), 397–414. [MR 81i:17011]
[QF11] Adjoints and Jordan algebras, Comm. Algebra 13 (1985), 2567-2596. [MR 87f:17018]

Identities of Quadratic Jordan Algebras

[QI1] Macdonald's theorem for quadratic Jordan algebras, Pacific J. Math. 35 (1970), 681–707 (with R. E. Lewand). [MR 45 # 8696]
[QI2] Koecher's principle for quadratic Jordan algebras, Proc. Amer. Math. Soc. 28 (1971), 39–43. [MR 45 # 8697]
[QI3] A not-so-natural s-identity, Comm. Algebra 15 (1987), 2099–2118. [MR 88k:17024]
[QI4] G_8 and G_9 are equivalent Jordan s-identities, Comm. Algebra 17 (1989), 2871-2891 (with A. D'Amour). [MR 90k:17065]

Quadratic Radicals

[QR1] The radical of the Jordan algebra, Proc. Nat. Acad. Sci. U.S.A. 62 (1969), 671–678. [MR 42 # 3137]
[QR2] A characterization of the radical of a Jordan algebra, J. Algebra 18 (1971), 103–111. [MR 43 # 3316]
[QR3] Solvability and nilpotence for quadratic Jordan algebras, Scripta Math. 29 (1973), 467–483. [MR 53 # 5686]
[QR4] The Zel'manov nilpotence theorem for quadratic Jordan algebras, J. Algebra 63 (1980), 76–97. [MR 81m:17020]
[QR5] The Zel'manov annihilator and nilpotence of the nil radical in quadratic Jordan algebras with chain conditions, J. Algebra 67 (1980), 230–253. [MR 82g:17016]
[QR6] Maximal modular inner ideals and the Jacobson radical of a Jordan algebra, J. Algebra 68 (1981), 155–159 (with L. Hogben). [MR 82g:17015]
[QR7] Strong nilpotence of solvable ideals in quadratic Jordan algebras, J. Algebra 81 (1983), 488–507. [MR 84e:17019]
[QR8] Amitsur shrinkage of Jordan radicals, Comm. Algebra 12 (1984), 777–826. [MR 85k:17022]
[QR9] Invariance of Jordan radicals, Comm. Algebra 12 (1984), 827–855. [MR 85h:17011]
[QR10] Imbedding nondegenerate Jordan algebras in semiprimitive algebras, Proc. Amer. Math. Soc. 103 (1988), 1031–1036 (with W. S. Martindale). [MR 89k:17051]
[QR11] Strict proper nilness modulo an absorber, Comm. in Algebra 27(7) (1999), 3067-3091.

Structure of Quadratic Jordan Algebras

[QS1] Inner ideals in quadratic Jordan algebras, Trans. Amer. Math. Soc. 159 (1971), 445–468. [MR 43 # 4871]
[QS2] Quadratic Jordan algebras whose elements are all invertible or nilpotent, Proc. Amer. Math. Soc. 35 (1972), 309–316. [MR 46 # 7332]

[QS3] Quadratic Jordan algebras whose elements are all regular or nilpotent, Proc. Amer. Math. Soc. 45 (1974), 19–27. [MR 51 # 10402]
[QS4] Zel'manov's prime theorem for quadratic Jordan algebras, J. Algebra 76 (1982), 297–326. [MR 83h:17019]
[QS5] Minimal ideals in quadratic Jordan algebras, Proc. Amer. Math. Soc. 88 (1983), 579–583 (with Ng Seong Nam). [MR 84i:17018]
[QS6] The structure of strongly prime quadratic Jordan algebras, Adv. in Math. 69 (1988), 133–222 (with E. Zel'manov). [MR 89k:17042]
[QS7] Little Jordan Clifford algebras, Comm. in Algebra 27(6) (1999), 2701-2732.
[QS8] Outer inheritance in quadratic Jordan algebras, Comm. in Algebra 27(12) (1999), 6127-6145.

Jordan Triple Systems and Pairs

[JT1] Speciality of Jordan triple systems, Comm. Algebra 5 (1977), 1057–1082 (with O. Loos). [MR 58 # 22214]
[JT2] Peirce ideals in Jordan triple systems, Pacific J. Math. 83 (1979), 415-439. [MR 81i:17012]
[JT3] Compatible Peirce decompositions of Jordan triple systems, Pacific J. Math. 103 (1982), 57–102. [MR 84e:17018]
[JT4] Coordinatization of Jordan triple systems, Comm. Algebra 9 (1981), 1495–1542 (with K. Meyberg). [MR 82k:17012]
[JT5] Strong prime inheritance in Jordan systems, Algebras Groups Geom. 1 (1984), 217–234. [MR 86c:17017]
[JT6] Reduced elements in Jordan triple systems, J. Algebra 97 (1985), 540–564. [MR 87b:17016]
[JT7] A characterization of the nondegenerate radical in quadratic Jordan triple systems, Algebras Groups Geom. 4 (1987), 145–167. [MR 88m:17024]
[JT8] Coordinatization of triangulated Jordan systems, J. Algebra 114 (1988), 411–451 (with E. Neher). [MR 89d:17027]
[JT9] Prime inheritance in Jordan systems, Algebras Groups and Geometries 5 (1988), 191–225. [MR 89k:17053]
[JT10] Local algebras, in S. Gonzalez (ed.) Nonassociative Algebra and its Applications, Math. Appl. 303, Kluwer Acad. Publ., Doordrecht (1994), 279-284. [MR 96f:17037]
[JT11] The local algebras of Jordan systems, J. Algebra 177 (1995), 199-239 (with A. D'Amour). [MR 96k:17047]
[JT12] Jordan centroids, Comm. in Algebra 27(2) (1999), 933-954.[MR 2000b:17041]
[JT13] An elemental characterization of strong primeness in Jordan systems, J. Pure Appl. Algebra 109 (1996), 23-36 (with J.A. Anquela, T. Cortes, O. Loos). [MR 97j:17035]
[JT14] Strong primeness of hermitian Jordan systems, J. Algebra 198 (1997), 311-326 (with J.A. Anquela, T. Cortes, F. Montaner). [MR 98j:17030]
[JT15] Imbedding nondegenerate Jordan systems in semiprimitive systems, Int. J. of Math., Game Th., Algebra 9(1) (1999), 1-13.
[JT16] Properness, strictness, and nilness in Jordan systems, Comm. in Algebra 27(7) (1999), 3041-3066.
[JT17] Involutions of rectangular Jordan pairs, J. Algebra 225 (2000), 885-903.

[JT18] The structure of quadratic Jordan systems of Clifford type, J. Algebra 234(2000), 31-89 (with A. D'Amour).

Jordan Super-Algebras

[SJ1] Specialty and non-specialty of two Jordan superalgebras, J. Algebra 149 (1992), 326–351. [MR 93k:17060]
[SJ2] The Kantor construction of Jordan superalgebras, Comm. Algebra 20 (1992), 109–126 (with D. King). [MR 92j:17032]
[SJ3] Poisson spreads of Lie algebras, Nova J. of Algebra Geom. 1 (1992), 73-110. [MR 93e:17044]
[SJ4] Kaplansky superalgebras, J. Algebra. 164 (1994), 656-694. [MR 95e:17002]
[SJ5] The Kantor doubling process revisited, Comm. Algebra 23(1) (1995), 357-372 (with D. King). [MR 96b:17033]

Associative Algebras and Speciality

[AS1] On Herstein's theorems relating Jordan and associative algebras, J. Algebra 13 (1969), 382–392. [MR 40 # 2721]
[AS2] Speciality of quadratic Jordan algebras, Pacific J. Math. 36 (1971), 761–773. [MR 45 # 5185]
[AS3] Posner's theorem on PI algebras, Kyungpook Math. J. 11 (1971), 53–55. [MR 46 # 3558]
[AS4] Speciality and reflexivity of quadratic Jordan algebras, Comm. Algebra 5 (1977), 903–935. [MR 56 # 8650]
[AS5] Semiprimeness of special Jordan algebras, Proc. Amer. Math. Soc. 96 (1986), 29–33. [MR 87e:17027]
[AS6] The Zel'manov approach to Jordan homomorphisms of associative algebras, J. Algebra 123 (1989), 457–477. [MR 90j:17053]
[AS7] Martindale systems of symmetric quotients, Algebras Groups Geom. 6 (1989), 153–237. [MR 92b:16039]

Power-Associative Algebras

[PA1] Finite power-associative division rings, Proc. Amer. Math. Soc. 17 (1966),1173–1177. [MR 34 # 4319]
[PA2] A note on finite division rings, Proc. Amer. Math. Soc. 23 (1969), 598–600. [MR 40 # 7320]
[PA3] Generically algebraic algebras, Trans. Amer. Math. Soc. 127 (1967), 527–551. [MR 35 # 1644]

[PA 1] showed that *finite* power-associative division rings are fields, generalizing Wedderburn's famous theorem that finite associative division algebras are commutative fields (which has the surprising corollary that all finite projective planes satisfying Desargues' axiom also satisfy Pappus' axiom, a geometric theorem for which no purely geometric proof is known).

The *generically algebraic* algebras introduced by Jacobson are those power-associative algebras where there is a "generic minimum polynomial" $m_x(\lambda) = \lambda^n + a_1(x)\lambda^{n-1} + \ldots + a_n 1$ of finite degree satisfied by every element x, $m_x(x) = 0$ (though the coefficients $a_i(x)$ change with the element, being homogeneous polynomial functions of degree $n-i$ in x). Here the minimum polynomial flows from the *generic norm* $N(x) = (-1)^n a_n(x)$. [PA3] extended the basic result to the case where the algebra need not be finite-dimensional, only the generic minimum polynomial is of finite degree. For example, composition algebras are generically algebraic of degree 2, $x^2 - T(x)x + Q(x)1 = 0$, and the nondegenerate ones turn out to all be finite-dimensional, and the Jordan algebras of cubic forms (such as the Albert algebras) are generically algebraic of degree 3, $x^3 - T(x)x^2 + S(x)x - N(x)1 = 0$.

Noncommutative Jordan Algebras

[NCJA1] Norms and noncommutative Jordan algebras, Pacific J. Math. 15 (1965), 925–956. [MR 34 # 4317]
[NCJA2] Structure and representations of noncommutative Jordan algebras, Trans. Amer. Math. Soc. 121 (1966), 187–199. [MR 32 # 5700]
[NCJA3] On a class of noncommutative Jordan algebras, Proc. Nat. Acad. Sci. U.S.A. 56 (1966), 1–4 (with R. D. Schafer). [MR 34 # 5888]
[NCJA4] A note on quasi-associative algebras, Proc. Amer. Math. Soc. 17 (1966), 1455–1459. [MR 39 # 270]
[NCJA5] A proof of Schafer's conjecture for infinite-dimensional forms admitting composition, J. Algebra 5 (1967), 72–83. [MR 34 # 4318]
[NCJA6] Noncommutative Jordan rings, Trans. Amer. Math. Soc. 158 (1971), 1–33. [MR 46 # 9127]
[NCJA7] Homotopes of noncommutative Jordan algebras, Math. Ann. 191 (1971), 263–270. [MR 47 # 1888]
[NCJA8] Noncommutative Jordan division rings, Trans. Amer. Math. Soc. 163 (1972), 215–224. [MR 46 # 9127]

The graduate students in the Yale Algebra Seminar run by Jacobson in spring 1964 took turns presenting chapters from Koecher's Minnesota lecture notes on Jordan algebras in differential geometry. My turn accidently turned into my dissertation [NCJA1]: it extracted the algebraic essence of Koecher's proof that a real ω-domain gave rise to a Jordan algebra, showing that a semisimple algebraa which carried a nondegenerate norm form N permitting some sort of composition $N(P(x)y) = p(x)N(y)$ on over an arbitrary field (naturally of characteristic not 2 in those days) was necessarily a noncommutative Jordan algebra. This generalized both the Jordan composition $N(U_x y) = N(x)^2 N(y)$ of Jordan algebras (which Jacobson's work on the structure group had shown was so important), and the associative composition $N(xy) = N(x)N(y)$ of alternative algebras. All the partial derivatives, logarithmic derivatives, and chain rule in Koecher's differential geometry made sense for polynomial maps over an arbitrary field, not just for differentiable maps over the reals. I could show that the simple Jordan algebras, quasi-associative algebras, and degree 2 algebras all carried a norm permitting composition, but I couldn't show the nodal algebras did; Marshall Osborn soon gave an ingenious proof for these. Later [NCJA5] extended the results to norms N of finite degree on infinite-dimensional spaces, showing they were all finite-dimensional except for some degree 2 algebras, settling a conjecture of Dick Schafer. Jacobson had always been suspicious of noncommutative Jordan algebras, feeling they were mere axiomatic playthings, but when they arose as generalized composition algebras he felt somewhat more tolerant of them. [NCJA3] with Dick Schafer gave a structure theory for noncommutative Jordan algebras with d.c.c. on inner ideals closely based on Jacobson's then-fresh Artin Wedderburn structure theory for commutative Jordan algebras, leading to the same 4 classes of simple algebras plus the (at the time) unanalyzable division rings.

[NCJA4] gave an instrinsic characterization of forms of quasi-associative algebras.

Alternative Algebras

[A1] Bimodules for composition algebras, Proc. Amer. Math. Soc. 17 (1966), 480–486. [MR 32 # 5699]
[A2] A characterization of the Jacobson-Smiley radical, J. Algebra 18 (1971), 565–573. [MR 43 # 3318]
[A3] Homotopes of alternative algebras, Math. Ann. 191 (1971), 253–262. [MR 47 # 1899]
[A4] Alternative algebras satisfying polynomial identities, J. Algebra 24 (1973), 283–292. [MR 47 # 298]
[A5] Malcev's theorem for alternative algebras, J. Algebra 28 (1974), 484–495. [MR 55 # 446]
[A6] Absolute zero divisors and local nilpotence in alternative algebras, Proc. Amer. Math. Soc. 47 (1975), 293–299. [MR 50 # 7272]
[A7] Finite-dimensional left Moufang algebras, Math. Annalen. 224 (1976), 179–187. [MR 58 # 22217]
[A8] A basic associativity theorem for alternative algebras, Portugal. Math. 38 (1979), 47–53. [MR 85k:17043]
[A9] Composition triples, Algebraist's homage: papers in ring theory and related topics (New Haven, 1981), Contemporary Math. 13 (1982), 279–286. [MR 84f:17002]
[A10] Quadratic forms permitting triple composition, Trans. Amer. Math. Soc. 275 (1983), 107–130. [MR 84d:17003]
[A11] Nonassociative algebras with scalar involution, Pacific J. Math. 116 (1985), 85–109. [MR 86d:17003]
[A12] Derivations and Cayley derivations of generalized Cayley-Dickson algebras, Pacific J. Math. 117 (1985), 163–182. [MR 87c:17002]

I have maintained an interest in alternative algebras as cousins of Jordan algebras. Alternative algebras appear as "coordinates" of Jordan algebras: any composition algebra C (associative or not) gives rise to a Jordan algebra $H_3(C)$ of 3×3 hermitian matrices over C. The ideals and bimodules for the Jordan algebra correspond precisely to the ideals and bimodules for the alternative coordinate algebra. [A1] gave a recipe for obtaining the bimodules for composition algebras C (and hence the related Jordan matrix algebras) in terms of regular and Cayley bimodules C and Ct (the latter coming from the iterative Cayley-Dickson doubling process $C \to \mathcal{CD}(C) = C \oplus Ct$ for composition algebras). Much later [A12] related derivations of $\mathcal{CD}(C)$ to derivations and Cayley-derivations of A. Kaplansky has proposed that to be important an algebra should have a rich autmorphism group and derivation algebra. The Cayley-Dickson doubling process goes on forever, but while the the 8-dimensional octonion algebras have an important simple automorphism group and derivation algebra of type G_2, the algebras past the octonions stop being alternative or permitting composition, and their automorphism group and derivation algebra stop growing, and therefore by this criterion are "uninteresting". [A11] related, over an arbitrary ring of scalars, the existence of a scalar involution (one whose norms $x\bar{x}$ and traces $x + \bar{x}$ are scalar multiples of the unit element 1) to the algebra being alternative with norm permitting composition. [A9] was a talk given at Jacobson's retirement conference in New Haven, announcing the results published later in [A10] on *composition triples*, nondegenerate quadratic forms Q on a space which permit composition with some triple product, $Q(\{x, y, z\}) = Q(x)Q(y)Q(z)$; these turned out to be primarily *isotopes* of ordinary alternative composition algebras with triple composition $x(yz)$.

Long ago A.A. Albert had developed for nonassociative algebras a notion of *isotopies* and *isotopic algebras*, more general than isomorphism: an isotopy $\rho : A' \to A$ is a linear bijection such that there exist associated bijections σ, τ so that $\rho(x \cdot' y) = \sigma(x) \cdot \tau(y)$ for all x, y. For unital algebras an isotopy is just an isomorphism $\rho(x \cdot' y) = \rho(x) \cdot_{u,v} \rho(y)$ of A' with an "elemental isotope" of A (given by $x \cdot_{u,v} y := R_u^{-1}(x) \cdot L_v^{-1}(y)$). Alternative algebras have a smoothly functioning notion of inverse, in particular $L_{u^{-1}} = L_u^{-1}, R_{u^{-1}} = R_u^{-1}$, so $x \cdot_{u,v} y := (xu^{-1}) \cdot (v^{-1}y)$. [A3] developed a notion of homotope $C^{(u,v)}$ for alternative algebras C, introducing a new product $x \cdot_{u,v} y := (xu) \cdot (vy)$ parametrized by two (not necessarily invertible) elements u, v. This generalized the concept of homotopes $J^{(u)}(x \cdot_u y := \{x, u, y\})$ of Jordan algebras. [A2] gave a characterization of the Jacobson radical (the maximal ideal of elements z which are *quasi-invertible* in the sense that $1-z$ is invertible in the unital hull) for alternative algebras, and [A6] related trivial elements (elements z with $zxz = 0$ for all x) to local nilpotency. The difficult analog of this result for Jordan algebras was one of the key steps of Zel'manov's classification of Jordan algebras. [A4] established a theorem on alternative PI algebras (those satisfying a polynomial identity) which extended (and streamlined the proof of) a theorem of Shirshov on associative PI algebras.

[A5] obtained a Malcev Theorem for alternative algebras showing that in the "Wedderburn splitting" $A = S \oplus R$ of a finite-dimensional A into a separable subalgebra S and the radical R, the separable subalgebra S is unique up to "inner" automorphisms for a suitable notion of "inner".

[A7] gave a short proof that finite-dimensional semisimple left Moufang algebras (satisfying the *left Moufang law* $(x(yx))z = x(y(xz))$) are automatically alternative. This was known for simple left alternative algebras ($[x, x, z] = 0$) in characteristic not 2, and there were known examples in characteristic 2 to show that left-alternativity did not suffice to give alternativity. It has become clear that the "proper" notion of left alternativity (e.g. the one which corresponds to "left Moufang planes" of projective geometry) is really the left Moufang law; this is equivalent to left alternativity when there is a scalar $\frac{1}{2}$, and provides enough tools over arbitrary scalars to force alternativity in the semisimple case. (This is analogous to the situation in Jordan algebras, where linear and quadratic Jordan algebras are equivalent in the presence of $\frac{1}{2}$, but the quadratic axioms are needed to obtain the correct theory for general rings of scalars.)

[A8] gave a quick proof that an alternative algebra will be associative as soon as certain "basic associators" vanish (generalizing Artin's theorem that an algebra is alternative iff every subalgebra generated by two elements is associative).

Surveys and Reports

[S1] Quadratic methods in nonassociative algebra, Proceedings of the International Congress of Mathematicians (Vancouver, 1974), Vol. I, 325–330, Canad. Math. Congress, Montreal, 1975. [MR 55 # 12780]
[S2] Jordan algebras and their applications, Bull. Amer. Math. Soc. 84 (1978), 612–627. [MR 57 # 6115]
[S3] The Russian Revolution in Jordan algebras, Algebras Groups Geom. 1 (1984), 1–61. [MR 86j:17028]
[S4] Jordan triple systems: Insights and ignorance, Proc. of Int. Conf. on Algebra, Part 2 (Novosibirsk, 1989), Contemporary Mathematics 131 (1992), 625–637. [MR 93e:17044]
[S5] Jordan supersystems, in Proceedings of Conference on Hadronic Mechanics and Nonpotential Interactions, Part 1 (Cedar Falls, Iowa, 1990), Nova Sci. Publ., Commack N.Y. (1992), 17-36. [MR 95b:17037]
[S6] Jordan Algebras, Proceedings of Oberwolfach Conference (1992), Walter de Gruyter & Co., Berlin (1994) (with W. Kaup, H. Petersson, eds.). [MR 95c:17002]
[S7] *A Taste of Jordan Algebras,* textbook, Springer Verlag.
[S8] Nathan Jacobson (1910-1999) (with. G. Benkart, I. Kaplansky, D. Saltman, G. Seligman), Notices of the A.M.S., 47 (9) (2000), 1061-1071.
[J] Enumeration of the positive rationals, Amer. Math. Monthly 67 (1960), 868. [MR 23 # A65]
[U1] Seminormality and root closure in polynomial rings and algebraic curves, J. Algebra 58 (1979), 217–226 (with J. W. Brewer and D. L. Costa). [MR 80e:13002]
[U2] The range of a structural projection, J. Funct. Anal. 139 (1996), 196-224 (with C.M. Edwards, G.T. Ruettimann). [MR 97d:46085]

On several occasions I have given survey talks about recent results in Jordan theory. My first [S1] was a talk at the Vancouver Congress on the recent quadratic methods in Jordan, associative, and alternative algebras. Next [S2] was a colloquium-type talk to an AMS sectional meeting in Huntsville about Jordan algebras and their applications. Six years later, following Zelmanov's breakthroughs in Jordan structure theory, [S3] surveyed the transformation the Russian Revolution had made on the Jordan landscape. In my first visit to the seat of the revolution in Novosibirsk (when the Soviet Union was just opening up a crack, but before the mass exodus from Novosibirsk), I surveyed Jordan triples [S4]. [S5] discussed some recent activity in Jordan superalgebras and supertriples. The principal lectures at the 1992 Oberwolfach Tagung on Jordan theory were published in book form [S5], to which my name was attached because I was one of the three co-organizers of the conference (though Holger Petersson did most of the editorial work!) The final entry along these lines is my graduate-level textbook [S7], to be published by Springer Verlag, meant to give an historical account of Jordan structure theory and a taste of both the classical and the Zelmanovian methods. The book is dedicated to Nathan Jacobson and his wife Florie; I was able to show them a copy before their deaths. Jake was an important influence on my mathematics and my career, and I (and all Jake's students) am indebted to both of them on a personal as well as professional level. In [S8] I discussed Jake's contributions to Jordan theory in a memorial article in the A.M.S. Notices.

A juvenile work [J] (written when I was 18) gave a simple-minded but till-then-overlooked enumeration of the positive rationals; the usual anti-diagonal method has the disadvantage that it is not 1-1 on the face of it (the integer 1 is counted over and over again as the boustrophedon path crosses the main diagonal at $\frac{n}{n}$), but using the prime factorization of positive rationals reduces the problem to enumerating the *positive and negative integers*, which can easily be done in several ways. Two other accidental pieces, in which I played a minor (and largely uncomprehending) role were [U1], written during Jim Brewer's visit to Charlottesville to work with Doug Costa, where my experience with messy calculations helped at one point in an otherwise-conceptual paper in commutative algebra, and [U2], written during an intensive month spent in Berne with Martin Edwards and the (lamentably late) Freddie Ruettimann; I contributed my algebraic knowledge of the messy intricacies of Peirce decompositions in Jordan triple systems to an otherwise tidy paper in functional analysis.

PhD Students

I have had 10 Ph.D. students: Robert E. Lewand (1971), Bruce Smith (1975), Fu-Shun-Yu (1975), Alain D'Amour (1989), John Magnus (1991), Daniel King (1993), Jacqueline Hall (1994), Daniel Borzynski (1997), Matthew Neal (1998), Bernard Fulgham (2002), James Bowling (2002).