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Operads and triangulation of Loday's diagram on Leibniz algebras

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Abstract We factor the classical functors $As \xrightarrow{-} Lie$ and $Dias \xrightarrow{-} Leib$ through the categories *Pre-Lie* and *Pre-Leib* of two new types of algebras. Thanks to Koszul duality for binary quadratic operads, we deduce two more categories of algebras *Perm* and *Ricod* giving rise to other factorizations. This yields a *triangulation* of Loday's commutative diagram of functors on Leibniz algebras and associated operads. As an application, we define a notion of *extended Leibniz algebras*.

Keywords Perm algebra · Koszul duality · Operad · Lie algebra · Extended Leibniz algebra · Dialgebra · Dendriform algebra

Mathematics Subject Classification 17A30 · 17A40 · 17B60 · 17C05 · 17C40 · 17C50 · 17C99 · 18G40 · 18G50 · 18G60 · 18G99

1 Introduction

Discovered by Loday [5], *Leibniz algebras* are a non-commutative variation of usual Lie algebras. He also introduced the notion of *diassociative algebras* (or *dialgebras* for short) which are a dichotomization of associative algebras, in the sense that there exists a functor $Dias \xrightarrow{-} Leib$ analogous to the classical Liezation functor $As \xrightarrow{-} Lie$. Using the machinery of Koszul duality for binary quadratic operads, J.-L. Loday constructed two new types of algebras: *Zinbiel* (resp. *dendriform*) algebras governed by an operad *Zinb* (resp. *Dend*) which is dual to the quadratic operad *Leib* (resp. *Dias*) of Leibniz algebras (resp. dialgebras), see [7,8]. The categories of these new algebraic objects are connected by functors making commutative the following diagram whose symmetry (with respect to the median axis of

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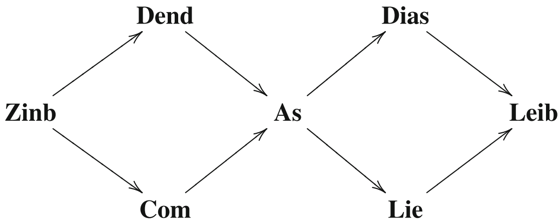
the self-dual operad *As* of non-unital associative algebras) reflects the Koszul duality of the corresponding operads:



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In this article, we show that there is a factorization

$$As \hookrightarrow Pre\text{-}Lie \xrightarrow{-} Lie$$

of the Liezation functor $As \xrightarrow{-} Lie$ through a category *Pre-Lie* of algebras characterized by the identity:

$$(ab)c - a(bc) = (ac)b - a(cb).$$

We also give a factorization

$$Dias \hookrightarrow Pre\text{-}Leib \xrightarrow{-} Leib$$

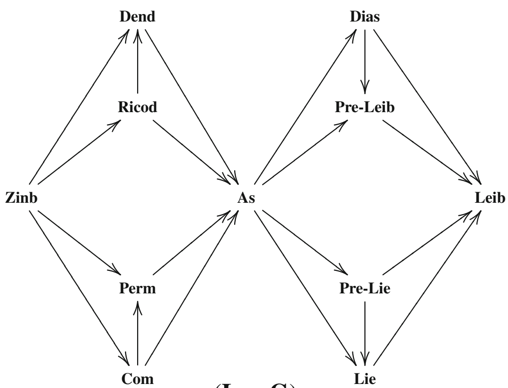
of Loday's functor $Dias \xrightarrow{-} Leib$ through a certain category *Pre-Leib* of algebras with two products, $\dashv$ and $\vdash$, satisfying some axioms weaker than those of dialgebras (see Proposition 2.2 below).

It turns out that pre-Lie algebras are governed by a quadratic operad *Pre-Lie* whose Koszul dual is the operad *Perm* of *right-commutative and associative* algebras subjected to the identities:

$$(ab)c = a(bc) = a(cb).$$

Similarly, the quadratic operad *Pre-Leib* admits for dual the operad *Ricod* of *right-commutative and dendriform* algebras which are dendriform algebras subjected to two more identities (see Theorem 2.6 below).

Once more, the functor $Zinb \longrightarrow Dend$ factors through the category *Ricod*, and there is a functor $Zinb \longrightarrow Perm$ given by the composition of the *plus-functor* $Zinb \xrightarrow{+} Com$ with the natural inclusion $Com \hookrightarrow Perm$. All the categories assemble in a *triangulation* of the above Loday's commutative diagram of functors:



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Observe that for any Leibniz algebra $\mathcal{G}$ and any perm algebra R, the $\mathbb{K}$-module $R \otimes \mathcal{G}$ is a Leibniz algebra for the bracket given by

$$[r \otimes x, s \otimes y] := (rs) \otimes [x, y], \quad \forall r, s \in R, \quad \forall x, y \in \mathcal{G}.$$

We call such an algebra an *extended Leibniz algebra*, in a way similar to the notion of *extended Lie algebras* (see [4] and references inside). On another hand, as a consequence of Koszul duality, if R is a perm algebra and P is a pre-Lie algebra, then the $\mathbb{K}$-module $R \otimes P$ is a Lie algebra for the bracket given by

$$[r \otimes p, r' \otimes p'] := rr' \otimes pp' - r'r \otimes p'p, \quad \forall r, r' \in R, \quad \forall p, p' \in P.$$

Similarly, if $(R_d, \prec, \succ)$ is a ricod algebra and $(D_p, \dashv, \vdash)$ is a pre-Leibniz algebra, then the $\mathbb{K}$-module $R_d \otimes D_p$ is a Lie algebra for the bracket given by

$$\begin{aligned}[r \otimes d, r' \otimes d'] := (r \prec r') \otimes (d \dashv d') - (r' \succ r) \otimes (d' \vdash d) \\ - (r' \prec r) \otimes (d' \dashv d) + (r \succ r') \otimes (d \vdash d').\end{aligned}$$

In the whole paper $\mathbb{K}$ denotes a field of characteristic zero over which all tensor products are taken, except explicitly otherwise stated. We refer to the papers [3,6–8] for the constructions on operads, Leibniz algebras and dialgebras; we denote by *As* (resp. *Com*, resp. *Lie*) the binary operad of non-unital associative (resp. non-unital associative and commutative, resp. Lie) algebras.

2 New types of algebras and some factorizations

2.1 Pre-Lie and pre-Leibniz algebras

A *pre-Lie algebra* is a $\mathbb{K}$-module P equipped with a product $(-,-)\colon P \otimes P \to P$ satisfying the identity

$$(ab)c - a(bc) = (ac)b - a(cb), \quad \forall a, b, c \in P. \tag{2.1}$$

It is clear that any associative algebra is a pre-Lie algebra. Another non-trivial example of pre-Lie algebra is a $\mathbb{K}$-module of derivations $\mathrm{Der}(\mathbb{K}[[x_1^{\pm}, \ldots, x_n^{\pm}]])$ under the product given by $u\delta_i.v\delta_j := v\delta_j(u)\delta_i$, see [2].

Proposition 2.1 *The classical Liezation functor $As \xrightarrow{-} Lie$ factors through the category Pre-Lie of pre-Lie algebras.*

Proof For any product $(-,-) : A \otimes A \to A$, the bracket given by $[a, b] := ab - ba$ is clearly skew-symmetric and satisfies

$$\begin{aligned}[[a, b], c] + [[b, c], a] + [[c, a], b] = [(ab)c - a(bc) - (ac)b + a(cb)] \\ + [(bc)a - b(ca) - (ba)c + b(ac)] \\ + [(ca)b - c(ab) - (cb)a + c(ba)].\end{aligned}$$

Therefore if $(A, (-,-))$ is a pre-Lie algebra, then $A_{Lie} := (A, [-,-])$ is a Lie algebra; from whence the factorization $As \hookrightarrow \textit{Pre-Lie} \xrightarrow{-} \textit{Lie}$. □

Recall that a *Leibniz algebra* is a $\mathbb{K}$-module $\mathcal{G}$ equipped with a bracket $[-,-]\colon \mathcal{G}\otimes\mathcal{G} \to \mathcal{G}$ satisfying the identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \tag{2.2}$$

and that a *dialgebra* is a $\mathbb{K}$-module D equipped with two associative products, $\dashv$ and $\vdash$, satisfying the axioms

$$\begin{cases} x \dashv (y \dashv z) = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = (x \vdash y) \vdash z. \end{cases} \tag{2.3}$$

In fact, it is a *dichotomization* of the classical associativity condition. J.-L. Loday showed that any dialgebra $(D, \dashv, \vdash)$ gives rise to a Leibniz algebra $D_{Leib} := (D, [-,-])$ whose bracket is defined by

$$[x,y] := x \dashv y - y \vdash x. \tag{2.4}$$

Proposition 2.2 *The functor Dias $\xrightarrow{-}$ Leib factors through the category Pre-Leib of algebras with two operations, $\dashv$ and $\vdash$, satisfying the axioms*

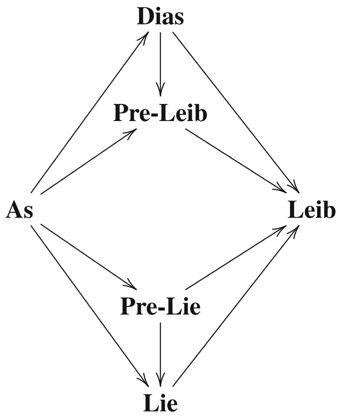
$$\begin{cases} x \dashv (y \dashv z) - (x \dashv y) \dashv z = x \dashv (z \vdash y) - (x \dashv z) \dashv y, \\ x \vdash (y \dashv z) - (x \vdash y) \dashv z = x \vdash (z \vdash y) - (x \dashv z) \vdash y, \\ x \vdash (y \vdash z) - (x \vdash y) \vdash z = x \vdash (z \dashv y) - (x \vdash z) \dashv y. \end{cases} \tag{2.5}$$

Proof Let $\dashv, \vdash\colon A \otimes A \to A$ be two any products; then the bracket (2.4) satisfies

$$\begin{aligned} &[x,[y,z]] - [[x,y],z] + [[x,z],y] \\ &\quad = x \dashv (y \dashv z - z \vdash y) - (y \dashv z - z \vdash y) \vdash x - (x \dashv y - y \vdash x) \dashv z \\ &\qquad + z \vdash (x \dashv y - y \vdash x) + (x \dashv z - z \vdash x) \dashv y - y \vdash (x \dashv z - z \vdash x) \\ &\quad = [x \dashv (y \dashv z) - (x \dashv y) \dashv z - x \dashv (z \vdash y) + (x \dashv z) \dashv y] \\ &\qquad - [y \vdash (x \dashv z) - (y \vdash x) \dashv z - y \vdash (z \vdash x) - (y \dashv z) \vdash x] \\ &\qquad + [z \vdash (x \dashv y) - (z \vdash x) \dashv y - z \vdash (y \vdash x) + (z \vdash y) \vdash x]. \end{aligned}$$

Therefore it is clear that if the products $\dashv$ and $\vdash$ satisfy axioms (2.3), then the bracket (2.4) defines a Leibniz algebra structure; from whence a factorization $\textit{Dias} \hookrightarrow \textit{Pre-Leib} \xrightarrow{-} \textit{Leib}$. □

In [8], Loday gives a lot of examples of dialgebras thanks to which one can construct some examples of pre-Leib algebras (by inclusion) and pre-Lie algebras (by identifying the products $\dashv = \vdash$). It is clear that we have a commutative diagram of functors between the corresponding operads:



(Diag.1)

2.2 Perm and ricod algebras

A *perm algebra* is a $\mathbb{K}$-module P equipped with a product $(-,-) : P \otimes P \to P$ satisfying the identities

$$(ab)c = a(bc) = a(cb), \quad \forall a, b, c \in P. \tag{2.6}$$

Example 2.3 (a) Any associative and commutative algebra is a perm algebra. Conversely, a perm algebra with a unit element is nothing but a unital associative and commutative algebra. But in the sequel, we shall deal with non-unital perm algebras.

(b) If (A, d) is a commutative differential algebra [i.e. $d : A \to A$ satisfies $d^2 = 0$ and $d(ab) = ad(b) + bd(a)$], then the product given by $(a, b) \mapsto ad(b)$ satisfies identities (2.6) of perm algebras.

(c) A general procedure to construct perm algebras is the following:

Proposition 2.4 *Let A be an associative and commutative algebra, and let R be a right A-module equipped with an A-module morphism $f : R \to A$. Then the product given by $(r, r') \mapsto rf(r')$ endows the $\mathbb{K}$-module R with a perm algebra structure that we denote by R_f.*

Proof Indeed, for any $r, r', r'' \in R$, we have

$$\begin{aligned}(rr')r'' &= (rf(r'))f(r'') = r(f(r')f(r'')) = rf(r'f(r'')) = rf(r'r'') = r(r'r'') \\ &= r(f(r'')f(r')) = rf(r''f(r')) = rf(r''r') = r(r''r').\end{aligned}$$

□

2.3 Free perm algebras

Assume now that V is a $\mathbb{K}$-module and let $\mathrm{S}(V) := \bigoplus_{n\geq 0} \mathrm{S}_n(V)$ be the free symmetric algebra on V. Then we have

Theorem 2.5 *The free perm algebra on V is the $\mathbb{K}$-module* $\mathrm{Perm}(V) := V \otimes \mathrm{S}(V)$ *equipped with the product defined by*

$$(u \otimes a).(v \otimes b) := u \otimes (avb) \quad \forall u, v \in V, \ \forall a, b \in \mathrm{S}(V). \tag{2.7}$$

Proof One easily checks that formula (2.7) defines a product satisfying identities (2.6). Now, given a $\mathbb{K}$-linear map $\phi\colon V \to P$ where P is a perm algebra, then the $\mathbb{K}$-linear map $\Phi\colon \mathrm{Perm}(V) \to P$ defined on generators by

$$\Phi(v) := \phi(v) \quad \text{and} \quad \Phi(v_0 \otimes v_1 \dots v_n) := \phi(v_0)\phi(v_1)\dots\phi(v_n), \quad v, v_i \in V,$$

is a perm algebra morphism, the unique one such that $\Phi \circ \iota = \phi$ where $\iota\colon V \cong V \otimes \mathbb{K} \hookrightarrow V \otimes \mathrm{S}(V)$ is the natural inclusion. This proves the universality of the perm algebra $\mathrm{Perm}(V)$. □

Observe that the free perm algebra $\mathrm{Perm}(V) := V \otimes \mathrm{S}(V)$ is a special case of Proposition 2.4 where $A := \mathrm{S}(V)$ and $P := V \otimes \mathrm{S}(V)$ with the obvious right A-module structure, and $f\colon P \to A$ is the *fusion map* that is,

$$f(v) := v \quad \text{and} \quad f(v_0 \otimes v_1 \dots v_n) := v_0 v_1 \dots v_n, \quad v, v_i \in V.$$

Recall that a *Zinbiel algebra* (or a dual Leibniz algebra) is a $\mathbb{K}$-module Z equipped with a product satisfying the identity

$$(ab)c = a(bc) + a(cb). \tag{2.8}$$

A *dendriform algebra* is a $\mathbb{K}$-module D equipped with two products, $\prec$ and $\succ$, satisfying the identities (see [8])

$$\begin{cases} (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \\ a \succ (b \prec c) = (a \succ b) \prec c, \\ a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c. \end{cases} \tag{2.9}$$

Any Zinbiel algebra Z gives rise to a dendriform algebra $Z_{Dend} := (Z, \prec, \succ)$ with

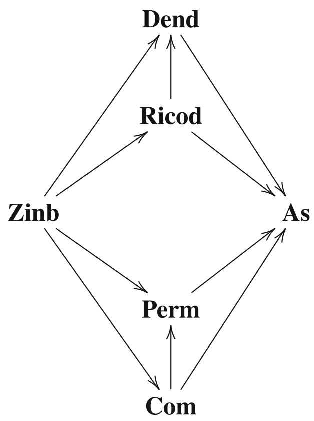
$$x \prec y := xy \quad \text{and} \quad x \succ y := yx. \tag{2.10}$$

This defines a functor *Zinb* $\to$ *Dend* where *Zinb* (resp. *Dend*) denotes the category of Zinbiel (resp. dendriform) algebras. It is straightforward to check that

Theorem 2.6 *The functor Zinb $\to$ Dend factors through the category Ricod of algebras with two products, $\prec$ and $\succ$, satisfying the identities*

$$\begin{cases} (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \\ a \prec (b \prec c) = a \prec (c \succ b), \\ a \succ (b \prec c) = (a \succ b) \prec c, \\ a \succ (b \succ c) = a \succ (c \prec b), \\ a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c. \end{cases} \tag{2.11}$$

Obviously, the *plus-functor Zinb* $\xrightarrow{+}$ *Com* (the commutative product being given by $x * y := xy + yx$) composed with the natural inclusion *Com* $\hookrightarrow$ *Perm* yields a well-defined functor *Zinb* $\to$ *Perm*. When all is said and done, we have another commutative diagram of functors between the correponding operads:



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3 Koszul duality

In this paragraph we aim at showing that the two last diagrams are dual to each other (in the sense of Koszul duality for binary operads) with respect to the median axis of the self-dual operad *As* governing the category of non-unital associative algebras. Remind first the well-known dualities between *Leib* and *Zinb*, *Dias* and *Dend*, *Com* and *Lie*, and $As^! = As$, see [8]. The following theorem has a first version due to F. Chapoton and M. Livernet (see [1]):

Theorem 3.1 *The quadratic operads Perm and Pre-Lie are dual to each other.*

Proof Since the underlying vector space of the free perm algebra on the $\mathbb{K}$-module V is $\mathrm{Perm}(V) := V \otimes \mathrm{S}(V)$, the operad *Perm* is such that $Perm(n) = \mathbb{K}^n$, $n \geq 1$, and is generated by the elements

$$X_i := x_i(x_1 \dots \widehat{x_i} \dots x_n), \quad i = 1, \dots, n$$

on which the symmetric group Σ_n acts by

$$\sigma.X_i := X_{\sigma(i)} = x_{\sigma(i)}(x_1 \dots \widehat{x_{\sigma(i)}} \dots x_n), \quad \sigma \in \Sigma_n.$$

In fact the operad *Perm* is quadratic and we have $Perm = \mathcal{P}(\mathbb{K}[\Sigma_2], R_{perm})$ where R_{perm} is the $\mathbb{K}$-module freely generated by the relators

$$r_\sigma := x_{\sigma,1} - x_{\sigma,2}, \quad r'_\sigma := x_{\sigma,1} - x_{\bar{\sigma},1}, \quad \sigma \in \Sigma_3.$$

Here, for $\sigma := (ijk) \in \Sigma_3$, we have put

$$\bar{\sigma} := (ikj), \quad x_{\sigma,1} := x_i(x_j x_k) \quad \text{and} \quad x_{\sigma,2} := (x_i x_j)x_k.$$

Thanks to the scalar product characterized by the orthogonality of the basis $(x_{\sigma,i})$ and the relations

$$\langle x_{\sigma,i}, x_{\sigma,i} \rangle := (-1)^{i+1} \mathrm{sgn}(\sigma),$$

we show that the orthogonal to R_{perm} is the $\mathbb{K}$-module generated by the relators

$$x_{\sigma,1} - x_{\sigma,2} - x_{\bar{\sigma},1} + x_{\bar{\sigma},2}, \quad \sigma \in \Sigma_3,$$

which are the generators of the $\mathbb{K}$-module $R_{pre\text{-}Lie}$ of relations defining the operad of pre-Lie algebras. From whence the duality

$$Perm^! = Pre\text{-}Lie = \mathcal{P}(\mathbb{K}[\Sigma_2], R_{pre\text{-}Lie}).$$

□

Similarly, the operad of pre-Leib algebras is quadratic and characterized by

$$Pre\text{-}Leib = \mathcal{P}(2.\mathbb{K}[\Sigma_2], R_{pre\text{-}Leib})$$

where "2.$\mathbb{K}[\Sigma_2]$" stands for $\mathbb{K}[\Sigma_2] \oplus \mathbb{K}[\Sigma_2]$ and is freely generated by the elements $x_{\sigma(1)} \dashv x_{\sigma(2)}$ and $x_{\sigma(1)} \vdash x_{\sigma(2)}$, $\sigma \in \Sigma_2$. Here the space $R_{pre\text{-}Leib}$ is the $\mathbb{K}$-module freely generated by the relators

$$\begin{aligned}
&a_{\sigma,1} - a_{\sigma,2} - c_{\bar{\sigma},1} + a_{\bar{\sigma},2},\\
&d_{\sigma,1} - d_{\sigma,2} - b_{\bar{\sigma},1} + c_{\bar{\sigma},2},\\
&b_{\sigma,1} - b_{\sigma,2} - d_{\bar{\sigma},1} + d_{\bar{\sigma},2}
\end{aligned}$$

where

$$\begin{aligned}
a_{\sigma,1} &:= x_{\sigma(1)} \dashv (x_{\sigma(2)} \dashv x_{\sigma(3)}), \quad a_{\sigma,2} := (x_{\sigma(1)} \dashv x_{\sigma(2)}) \dashv x_{\sigma(3)},\\
b_{\sigma,1} &:= x_{\sigma(1)} \vdash (x_{\sigma(2)} \vdash x_{\sigma(3)}), \quad b_{\sigma,2} := (x_{\sigma(1)} \vdash x_{\sigma(2)}) \vdash x_{\sigma(3)},\\
c_{\sigma,1} &:= x_{\sigma(1)} \dashv (x_{\sigma(2)} \vdash x_{\sigma(3)}), \quad c_{\sigma,2} := (x_{\sigma(1)} \dashv x_{\sigma(2)}) \vdash x_{\sigma(3)},\\
d_{\sigma,1} &:= x_{\sigma(1)} \vdash (x_{\sigma(2)} \dashv x_{\sigma(3)}), \quad d_{\sigma,2} := (x_{\sigma(1)} \vdash x_{\sigma(2)}) \dashv x_{\sigma(3)}.
\end{aligned}$$

The operad of ricod algebras is quadratic and characterized by

$$Ricod = \mathcal{P}(2.\mathbb{K}[\Sigma_2], R_{ricod})$$

where "2.$\mathbb{K}[\Sigma_2]$" stands for $\mathbb{K}[\Sigma_2] \oplus \mathbb{K}[\Sigma_2]$ and is freely generated by the elements $x_{\sigma(1)} \prec x_{\sigma(2)}$ and $x_{\sigma(1)} \succ x_{\sigma(2)}$, $\sigma \in \Sigma_2$. Here the space R_{ricod} is the $\mathbb{K}$-module freely generated by the relators

$$\begin{aligned}
&\alpha_{\sigma,1} - \alpha_{\sigma,2} + \gamma_{\sigma,1},\\
&\alpha_{\sigma,1} - \gamma_{\sigma,1},\\
&\delta_{\sigma,1} - \delta_{\sigma,2}\\
&\beta_{\sigma,1} - \delta_{\sigma,1},\\
&\gamma_{\sigma,2} - \beta_{\sigma,1} + \beta_{\sigma,2}
\end{aligned}$$

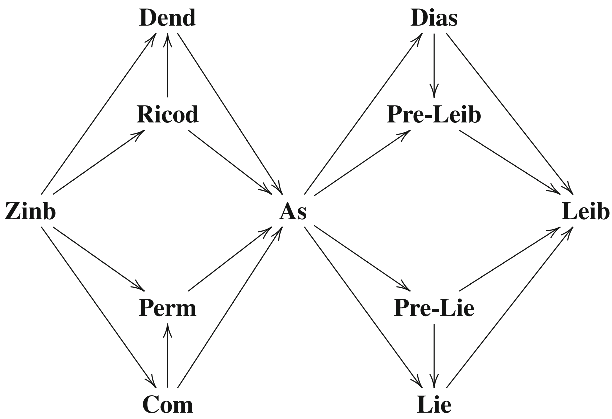
where we have put

$$\begin{aligned}
\alpha_{\sigma,1} &:= x_{\sigma(1)} \prec (x_{\sigma(2)} \prec x_{\sigma(3)}), \alpha_{\sigma,2} := (x_{\sigma(1)} \prec x_{\sigma(2)}) \prec x_{\sigma(3)}),\\
\beta_{\sigma,1} &:= x_{\sigma(1)} \succ (x_{\sigma(2)} \succ x_{\sigma(3)}), \beta_{\sigma,2} := (x_{\sigma(1)} \succ x_{\sigma(2)}) \succ x_{\sigma(3)}),\\
\gamma_{\sigma,1} &:= x_{\sigma(1)} \prec (x_{\sigma(2)} \succ x_{\sigma(3)}), \gamma_{\sigma,2} := (x_{\sigma(1)} \prec x_{\sigma(2)}) \succ x_{\sigma(3)}),\\
\delta_{\sigma,1} &:= x_{\sigma(1)} \succ (x_{\sigma(2)} \prec x_{\sigma(3)}), \delta_{\sigma,2} := (x_{\sigma(1)} \succ x_{\sigma(2)}) \prec x_{\sigma(3)}).
\end{aligned}$$

By the same arguments as before, one can show that

Theorem 3.2 *The quadratic operads Ricod and Pre-Leib are dual to each other.*

This yields a commutative diagram of functors whose symmetry (with respect to the median axis of the self dual operad *As* of non-unital associative algebras) reflects the Koszul duality between the corresponding operads:



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Remark 3.3 Since $\dim Perm(n) = n$, the Poincaré series of the operad of perm algebras is

$$g_{perm}(x) = \sum_{n\geq 0} (-1)^n n x^n / n! = -x \exp(-x)$$

which is invertible (for the composition of series) and has for inverse the series

$$g_{pre-Lie} = \sum_{n\geq 0} (-1)^n n^{n-1} x^n / n!,$$

as pointed out by Chapoton–Livernet who showed moreover that both the operads *Perm* and *Pre-Lie* are Koszul, see [1].

4 Extended Leibniz algebras and tensor products

We now define the notion of *extended Leibniz algebras* for which perm algebras were introduced.

Proposition 4.1 *Let R be a perm algebra and let $\mathcal{G}$ be a Leibniz algebra. Then the $\mathbb{K}$-module $R \otimes \mathcal{G}$ equipped with the bracket given by*

$$[r \otimes x, s \otimes y] := (rs) \otimes [x, y], \quad \forall r, s \in R, \quad \forall x, y \in \mathcal{G}, \tag{4.1}$$

is a Leibniz algebra that we call "extended Leibniz algebra".

Proof In fact, for any $r, s, t \in R$ and any $x, y, z \in \mathcal{G}$, we have

$$\begin{aligned}
&[r \otimes x, [s \otimes y, t \otimes z]] - [[r \otimes x, s \otimes y], t \otimes z] + [[r \otimes x, t \otimes z], s \otimes y] \\
&\quad = r(st) \otimes [x, [y, z]] - (rs)t \otimes [[x, y], z] + (rt)s \otimes [[x, z], y] \\
&\quad = (r(st) - (rs)t) \otimes [[x, y], z] - (r(st) - (rt)s) \otimes [[x, z], y].
\end{aligned}$$

Therefore, to get Leigniz identity (2.2) for the bracket on $R \otimes \mathcal{G}$, it is enough to have the relations $r(st) = (rs)t$ and $r(st) = r(ts)$ which are equivalent to the perm identities $(rs)t = r(st) = r(ts)$. □

On another hand, as consequence of the above Koszul dualities, one readily checks that

Proposition 4.2 *If R is a perm algebra and if P is a pre-Lie algebra, then the $\mathbb{K}$-module $R \otimes P$ is a Lie algebra for the bracket given by*

$$[r \otimes p, r' \otimes p'] := (rr') \otimes (pp') - (r'r) \otimes (p'p), \quad \forall r, r' \in R, \quad \forall p, p' \in P. \tag{4.2}$$

If $(R_d, \prec, \succ)$ is a ricod algebra and if $(D_p, \dashv, \vdash)$ is a pre-Leibniz algebra, then the $\mathbb{K}$-module $R_d \otimes D_p$ is a Lie algebra with the bracket given by

$$[r \otimes d, r' \otimes d'] := (r \prec r') \otimes (d \dashv d') - (r' \succ r) \otimes (d' \vdash d) - (r' \prec r) \otimes (d' \dashv d)$$
$$+ (r \succ r') \otimes (d \vdash d'), \quad \forall r, r' \in R_d, \quad \forall d, d' \in D_p. \tag{4.3}$$

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