C. R. Acad. Sci. Paris, t. 325, Série I, p. 21-26, 1997
Analyse mathématique/*Mathematical Analysis*

Hankel operators in the Dixmier class

Song-Ying LI[1] and Bernard RUSSO

Department of Mathematics, University of California Irvine,
California 92697-3875, U.S.A.
E-mail : sli@math.uci.edu
brusso@math.uci.edu

Abstract. We study holomorphic functions f in the unit ball for which the small Hankel operator h_f belongs to the Dixmier class.

Les opérateurs de Hankel dans la classe de Dixmier

Résumé. *Nous étudions les fonctions holomorphes f pour lesquelles l'opérateur de Hankel h_f appartient à la classe de Dixmier.*

Version française abrégée

Soient H un espace de Hilbert sur $\mathbb{C}$ et T un opérateur compact sur H. Pour $0 < p < \infty$, on dit que $T \in S_p(H)$ (la classe de Schatten-von Neumann) si $(\mu_n(T))_{n=1}^{\infty} \in \ell^p$, où $\mu_n(T)$ sont les valeurs propres de $|T| = (T^* T)^{1/2}$.

Soient D la boule unité dans $\mathbb{C}^n$ et $\mathcal{H}^2(D)$ l'espace holomorphe de Hardy sur D. $S : L^2(\partial D, d\sigma) \to \mathcal{H}^2(D)$ désigne le projecteur de Szegö avec noyau de Szegö $S(\cdot, \cdot)$. Pour chaque $\alpha > 0$, posons $dv_\alpha(z) = c_\alpha (1 - |z|^2)^{\alpha-1} dv(z)$ tel que $\int_D dv_\alpha = 1$. Soient $A_\alpha^2(D)$ l'espace de Bergman avec poids sur D et $P_\alpha : L^2(D, dv_\alpha) \to A_\alpha^2(D)$ le projecteur de Bergman avec noyau de Bergman $K_w^\alpha(z) = K^\alpha(z, w) = c_\alpha (1 - \langle z, w\rangle)^{-n-\alpha}$. Quand $\alpha = 1$, nous écrivons $K(z, w) = K^1(z, w)$ pour le noyau de Bergman habituel. Il est facile de voir que la limite faible de dv_α lorsque $\alpha \to 0^+$ est $d\sigma$. Donc, pour unifier notre notation, nous posons $A_0^2 = \mathcal{H}^2(D)$ et $P_0 = S$. Pour $\alpha \geq 0$ quelconque et $f \in A_\alpha^2(D)$, soit $h_f^\alpha(u) = \overline{P_\alpha(f\bar{u})}$ l'opérateur de Hankel défini sur $A_\alpha^2(D)$. Pour $1 \leq p < \infty$, soit $B^p(D)$ l'espace de Besov holomorphe sur D avec semi-norme $\|\cdot\|_{B^p}$ définie par :

$$(1) \quad \|f\|_{B^p}^p = \int_D |f^{(n+1)}(z)|^p K(z, z)^{1-p}\, dv(z) \quad \text{avec} \quad |f^{(n+1)}(z)| = \sum_{|\beta|=n+1} \left| \frac{\partial^{n+1} f}{\partial z^\beta}(z) \right|.$$

Le problème de déterminer une condition nécessaire et suffisante sur la fonction holomorphe f sur le domaine $\Omega \subset \mathbb{C}^n$ pour que h_f appartienne à $S_p(A_\alpha^2)$ a longtemps occupé l'esprit de plusieurs

Note présentée par Alain CONNES.

auteurs. Un théorème de Peller (*voir* [1]) montre que si $n = 1$ et $\alpha = 0$, alors $h_f \in S_p(A_0^2(D))$ si et seulement si $f \in B^p(D)$ pour $1 \leq p < \infty$. Un théorème analogue à été obtenu quand D est le demi-plan $\operatorname{Im} z > 0$ dans $\mathbb{C}$ par Coifman et Rochberg (*voir* [2]) pour $p = 1$, et par Rochberg (*voir* [3]) pour $p > 1$. Le résultat analogue pour $n = 1$ et $\alpha > 0$ (espace de Bergman à poids) est démontré pour $1 \leq p < \infty$ dans [4].

En dimensions plus grandes, ce théorème de caractérisation a été prolongé à la boule unité par Feldman et Rochberg (*voir* [5]) et Zhang (*voir* [6]) pour l'espace de Hardy ($\alpha = 0$), et par Burbea (1988 non-publié) pour l'espace de Bergman à poids.

Une classe importante d'opérateurs qui se trouve entre $S_1(H)$ et $S_{1+\varepsilon}(H)$ est l'idéal de Macaev $S_1^+(H)$ (ou $\mathcal{L}^{(1,\infty)}(H)$), que nous appellerons la *classe de Dixmier* (*voir* [7]). Nous disons que $T \in S_1^+(H)$ si $(\sigma_n/\log n)_{n=2}^{\infty} \in \ell^\infty$, où $\sigma_n = \sum_{j=1}^n \mu_j(T)$. Cette classe a été utilisée en 1966 par Dixmier (*voir* [8], et aussi [7], p. 303 ou [9], p. 5408), pour répondre négativement à la question de l'unicité de la trace sur $\mathcal{L}(H)$. Notons que S_1^+ est un espace de Banach pour la norme $\|T\|_{S_p^+} = \sup_{n\geq 2}\{\sigma_n(T)/\log n\}$.

Plus récemment, Bellissard et ses collaborateurs ont étudié le rapport entre les opérateurs de Hankel sur l'espace de Hardy du disque unité et le « quantum Hall effect » (*voir* [9]). La question suivante nous a été posée par Bellissard : Quelles sont les fonctions holomorphes dans le disque unité qui se présentent comme symboles des opérateurs de Hankel appartenant à la classe de Dixmier S_1^+ ?

Le but de cette Note est de répondre à cette question. Notre résultat principal, Théorème 1 ci-après, est valable pour plusieurs variables et la démonstration peut être modifiée facilement pour d'autres domaines habituels. Donc, il suffit d'énoncer et de démontrer notre résultat pour la boule unité.

Pour un domaine $D \subset \mathbb{C}^n$, nous disons qu'une fonction holomorphe f sur D appartient à $B_+^1(D)$ si, avec la notation de (1),

$$(2)\quad \|f\|_{B_+^1(D)} = \int_D \frac{|f^{(n+1)}(z)|}{|\log F(f)(z)|}\, dv(z) < \infty \quad \text{avec } F(f)(z) = (1 + |f^{(n+1)}(z)|)\, K(z, z)^{-1}.$$

Nous avons le théorème suivant :

THÉORÈME 1. – *Soient $\alpha \geq 0$ et $f \in A_\alpha^2(D)$, où D est la boule unité. Alors,*

(i) $h_f^\alpha \in S_1^+(A_\alpha^2(D))$ *si et seulement si* $\sup_{1<p\leq 2}\{(p-1)\|f\|_{B^p}^p\} < \infty$.

(ii) *Si* $f \in B_+^1(D)$, *alors* $h_f^\alpha \in S_1^+(A_\alpha^2(D))$.

(iii) *Si* $h_f^\alpha \in S_1^+(A_\alpha^2(D))$, *alors, pour* $p \in (1, 2)$,

$$(3)\qquad \int_D |f^{(n+1)}(z)|(1 + |\log F(f)|)\,(1 + \log(1 + |\log F(f)|))^{-p}\, dv(z) < \infty.$$

Nous donnons un exemple d'une fonction $f \in B^2(D)$ telle que $h_f^\alpha \in S_1^+(A_\alpha^2)$ et $f \notin B_+^1(D)$. En plus, (3) est faux quand $p = 1$.

---

1. Introduction

Let H be a Hilbert space over $\mathbb{C}$. For $0 < p < \infty$, we say that $T \in S_p(H)$ (the Schatten-von Neumann p-class) if $\{\mu_n(T)\}_{n=1}^\infty \in \ell^p$, where the $\mu_n(T)$ are the eigenvalues of $|T| = (T^* T)^{1/2}$.

Let D be the unit ball in $\mathbb{C}^n$ and $\mathcal{H}^2(D)$ be the usual holomorphic Hardy space on D. Let $S : L^2(\partial D, d\sigma) \to \mathcal{H}^2(D)$ be the Szegö projection with Szegö kernel $S(\cdot, \cdot)$. For each $\alpha > 0$, let $dv_\alpha(z) = c_\alpha(1 - |z|^2)^{\alpha-1}\, dv(z)$, where dv is the Lebesgue volume measure, and $\int_D dv_\alpha = 1$.

Let $A^2_\alpha(D)$ denote the weighted Bergman space on D and $P_\alpha : L^2(D, dv_\alpha) \to A^2_\alpha(D)$ the Bergman projection with Bergman kernel $K^\alpha_w(z) = K^\alpha(z, w) = c_\alpha(1 - \langle z, w\rangle)^{-n-\alpha}$. When $\alpha = 1$, $K(z, w) = K^1(z, w)$ denotes the usual Bergman kernel. It is easy to see that the weak limit of dv_α as $\alpha \to 0^+$ is $d\sigma$. Let $A^2_0 = \mathcal{H}^2(D)$ and $P_0 = S$ to unify our notation. For any $\alpha \geq 0$ and $f \in A^2_\alpha(D)$, let $h^\alpha_f(u) = \overline{P_\alpha(f\bar{u})}$ be the small Hankel operator defined on $A^2_\alpha(D)$. For $1 \leq p < \infty$, let $B^p(D)$ denote the holomorphic Besov space over D with the seminorm $\|\cdot\|_{B^p}$ defined as follows:

$$(1)\qquad \|f\|^p_{B^p} = \int_D |f^{(n+1)}(z)|^p K_z(z)^{1-p}\, dv(z), \qquad |f^{(n+1)}(z)| = \sum_{|\beta|=n+1} \left|\frac{\partial^{n+1} f}{\partial z^\beta}\right|.$$

The problem of characterizing holomorphic functions f on domains $\Omega \subset \mathbb{C}^n$ so that h_f belongs to $S_p(A^2_\alpha)$ draw the attention of several authors. A theorem of Peller (*see* [1]) states that if $n = 1$ and $\alpha = 0$, $h_f \in S_p(A^2_0(D))$ if and only if $f \in B^p(D)$ for $1 \leq p < \infty$. A similar theorem was obtained when D is the upper half plane in $\mathbb{C}$ by Coifman and Rochberg (*see* [2]) for $p = 1$, and by Rochberg (*see* [3]) for $p > 1$. The corresponding result for $n = 1$ and $\alpha > 0$ (weighted Bergman space) is proved for $1 \leq p < \infty$ in [4].

In higher dimensions, the above characterization theorem for h_f, was proved for the unit ball in $\mathbb{C}^n$ by Feldman and Rochberg (*see* [5]) and Zhang (*see* [6]) for the Hardy space ($\alpha = 0$), and by Burbea (preprint, 1988) for the weighted Bergman space ($\alpha > 0$).

An important class of operators which lies between $S_1(H)$ and $S_{1+\varepsilon}(H)$ is the Macaev ideal $S^+_1(H)$ (also denoted by $\mathcal{L}^{(1,\infty)}(H)$), which we shall call the Dixmier class (*see* [7]). We say that $T \in S^+_1(H)$ if $\{\sigma_n/\log n\}^\infty_{n=2} \in \ell^\infty$, where $\sigma_n = \sum^n_{j=1} \mu_j(T)$. This class was used in 1966 by Dixmier (*see* [8], and also [7], p. 303 or [9], p. 5408), to settle in the negative the question of the uniqueness of the trace on $\mathcal{L}(H)$). We mention that S^+_1 is a Banach space under the norm: $\|T\|_{S^+_p} = \sup_{n\geq 2}\{\sigma_n(T)/\log n\}$.

More recently, J. Bellissard and co-workers have connected Hankel operators on the Hardy space of the unit disk with their study of the quantum Hall effect (*see* [9]). The following question was posed by Bellissard to us: What is the holomorphic function space in the unit disc which consists of precisely the symbols of Hankel operators belonging to the Dixmier class S^+_1?

The aim of this Note is to answer this question. Our main result, Theorem 1 below, is valid in several complex variables and the proof can be extended almost trivially for several standard domains. For convenience we shall state and prove our result for the unit ball.

For a domain $D \subset \mathbb{C}^n$, we say that a holomorphic function f over D belongs to $B^1_+(D)$ if

$$(2)\qquad \|f\|_{B^1_+(D)} = \int_D \frac{|f^{(n+1)}(z)|}{1 + |\log F(f)|}\, dv(z) < \infty \quad \text{with } F(f) = \frac{1 + |f^{(n+1)}(z)|}{K_z(z)}.$$

Then we prove the following theorem.

THEOREM 1. – *Let $\alpha \geq 0$ and let $f \in A^2_0(D)$, where D is the unit ball. Then*

(i) $h^\alpha_f \in S^+_1(A^2_\alpha(D))$ *if and only if* $\sup_{1<p\leq 2}\{(p-1)\|f\|^p_{B^p}\} < \infty$.

(ii) *If $f \in B^1_+(D)$, then $h^\alpha_f \in S^+_1(A^2_\alpha(D))$.*

(iii) *If $h^\alpha_f \in S^+_1(A^2_\alpha(D))$, then, for any $p \in (1, 2)$,*

$$(3)\qquad \int_D |f^{(n+1)}(z)|(1 + |\log F(f)|)(1 + \log(1 + |\log F(f)|))^{-p}\, dv(z) < \infty.$$

We provide an example of a function $f \in B^2(D)$ such that $h^\alpha_f \in S^+_1(A^2_\alpha)$ but $f \notin B^1_+(D)$. Moreover, (3) fails when $p = 1$.

Remark. – We point out here that, by using the results on the boundedness and compactness of h_f in [10] and [11], and the asymptotic expansion of the Bergman and Szegö kernels given in [12], one can prove Theorem 1 in the case where D is a smoothly bounded strictly pseudo-convex domain in $\mathbb{C}^n$. This remark can also apply to other domains in $\mathbb{C}^n$, such as bounded symmetric domains, by using the results proved in [13].

The authors would like to thank M. Ho for various conversations.

2. Proof of Part (i) in Theorem 1

Let H be a Hilbert space over $\mathbb{C}$, T be a compact opertator on H, and let $\{\mu_j(T)\}$ denote the singular numbers of T in decreasing order. For each s with $\mathrm{Re}(s) > 1$, we define the zeta function as follows: $\xi_T(s) = \sum_{n=1}^{\infty} \mu_n(T)^s$.

The following lemma is a qualitative version of a statement (*see* [7], Prop. 4, p. 306).

LEMMA 1. – *With the above definition for* $\xi_T(s)$, *we have:*

(a) *If* $(s-1)\zeta_T(s)$ *is bounded on* $(1, 2)$, *then* $T \in S_1^+$.

(b) *If* $T \in S_1^+$, *then* $(s-1)\zeta_T(s)$ *is bounded on* $(1, 2)$.

Proof. – It is not difficult to prove (a); (b) can be proved by using Abel's identity and detailed analysis. We omit the detail here.

LEMMA 2. – *Let* D *be the unit ball in* $\mathbb{C}^n$, *and let* $\alpha \geq 0$. *If* $f \in A_\alpha^2(D)$ *then*

$$h_f^\alpha \in S_+^1(A_\alpha^2(D)) \Leftrightarrow \sup_{1<p\leq 2} \{(p-1)\|f\|_{B^p}^p\} < \infty. \tag{4}$$

Proof. – It was proved in [5] and [6] (for $\alpha = 0$), and in Burbea (preprint, 1988 for $\alpha > 0$), that there is a $C_{p,\alpha} > 1$ such that

$$\frac{1}{C_{p,\alpha}}(\|f\|_{A_\alpha^2} + \|f\|_{B^p(D)}) \leq \|h_f^\alpha\|_{S_p(A_\alpha^2)} \leq C_{p,\alpha}(\|f\|_{A_\alpha^2} + \|f\|_{B^p(D)}) \tag{5}$$

for all $1 \leq p < \infty$. The method of the complex interpolation tells us that

$$C_{p,\alpha} \leq C_{1,\alpha}^{1-\theta} C_{2,\alpha}^{\theta}, \qquad \frac{1}{2} = 1 - \frac{\theta}{2}, \qquad \theta \in [0, 1],$$

so that $C_{p,\alpha}$ is bounded from above for $p \in (1, 2)$. A similar argument shows that it is bounded from below independently of $p \in [1, 2]$. Thus, if we set $C_\alpha = \sup_{1\leq p\leq 2} C_{p,\alpha}$, (5) holds uniformly for $p \in [1, 2]$ if $C_{p,\alpha}$ is replaced by C_α. Combining this fact with Lemma 1 completes the proof of Lemma 2, and hence the proof of Part (i) of Theorem 1.

3. Completion of the proof of Theorem 1

3.1. Proof of Part (ii) of Theorem 1

Proof. – Let $f \in B_+^1(D) \cap A_\alpha^2(D)$. We shall prove that $h_f^\alpha \in S_+^1$. Let $\beta(z, w)$ be the Bergman metric for D, and for $0 < \varepsilon \ll 1$, we let $B_\varepsilon(z) = \{w \in D : \beta(w, z) < \varepsilon\}$. Then there is a constant $C_\varepsilon > 1$ so that $C_\varepsilon^{-1} K_z(z) \leq |B_\varepsilon(z)|^{-1} \leq C_\varepsilon K_z(z)$.

For any $z_0 \in D$, $K(z, z) \approx K(z_0, z_0)$ for all $z \in B_\varepsilon(z_0)$. So, without loss of generality, we may replace $K(z, z)$ in the definition of $F(f)$ by $K(z_0, z_0)$ for all $z \in B_\varepsilon(z_0)$. Let $F_0(f)(z) = (1 + |f^{(n+1)}(z)|) K(z_0, z_0)^{-1}$. Then

$$\frac{F(f)(z)}{1 + \log F(f)(z)} \approx \frac{F_0(f)(z)}{1 + \log F_0(f)(z)} := G_1(z).$$

It is not difficult to show that $G_1(z)$ is plurisubharmonic on $D(z_0) = \{z \in B_\varepsilon(z_0);\ F_0(f)(z) > 1\}$. If $\{z \in D(z_0) : G_1(z) < 2\}$ is empty, since $G_1(z)$ is continuous on $D(z_0)$, either $F_0(f)(z)[1 + |\log F_0(f)(z)|]^{-1} \le 2$ on $B_\varepsilon(z_0)$ or $D(z_0) = B_\varepsilon(z_0)$, and $G_1(z)$ is plurisubharmonic in $B_\varepsilon(z_0)$. Otherwise,

$$G(z) := \begin{cases} \max\{2,\ G_1(z)\}, & \text{if } z \in D(z_0), \\ 2, & \text{if } z \in B_\varepsilon(z_0) \backslash D(z_0), \end{cases}$$

is plurisubharmonic on $B_\varepsilon(z_0)$. By using the sub-mean value property for G, one can prove

$$F(f)(z)[1 + |\log F(f)(z)|]^{-1} \le CC_\varepsilon(1 + \|f\|_{B^1_+}).$$

Therefore, we have $F(f)(z) \le C[1 + \|f\|_{B^1_+}]^2$ for all $z \in D$. Moreover, it is easy to show

$$(p-1)[F(f)(z)]^{p-1} \le C(1 + \|f\|_{B^1_+})^3[1 + |\log F(f)(z))|]^{-1}.$$

Hence

$$\sup_{1<p\le 2}\{(p-1)\int_D |f^{(n+1)}|F(f)^{p-1}\,dv\} \le C(1 + \|f\|_{B^1_+})^3 \int_D \frac{|f^{(n+1)}|}{1 + |\log F(f)|}\,dv,$$

and the proof of Part (ii) of Theorem 1 is complete.

3.2. Proof of Part (iii) of Theorem 1

Proof. – Suppose that $h^\alpha_f \in S^1_+$. Since $S^1_+ \subset S_p$ for $p > 1$, Lemma 2 implies that $f \in B^p$ for every $p > 1$. In particular, $f \in \mathcal{B}_0(D)$, the little Bloch space. Thus $\lim_{|z|\to 1^-} F(f)(z) = 0$, so that in the proof of (3), we may assume $0 < F(f)(z) \le e^{-\varepsilon}$ for $z \in D$.

For $t > 1$, if $g(s) = (t-1)s^{-1}(-\log s)^{-t}$, then $\int_0^{1/e} g(s)\,ds = 1$. Without loss of generality, we assume $\|h^\alpha_f\|_{S_\infty} = 1$. We have easily that

$$\begin{aligned} C\|h^\alpha_f\|_{S^+_1} &\ge \int_0^{1/e} s\int_D |f^{(n+1)}|F(f)^s\,dv\,g(s)\,ds \\ &\ge -C + \frac{t-1}{e}\int_D \frac{|f^{(n+1)}(z)|}{-\log F(f)}\left(\log\log\frac{1}{F(f)}\right)^{-t} dv. \end{aligned}$$

Therefore, the proof of Part (iii) of Theorem 1 is complete. This completes the proof of Theorem 1.

Finally, we provide an example on the unit disc. This same construction works for other domains including the unit ball.

3.3. Example

Let $f(z) = \log(i\pi + \log(1-z))$, $z \in \Delta$. One can show that $h_f \in S^+_1(A^2_\alpha(\Delta))$ by proving $\sup_{p\in(1,2]}\{(p-1)\|f\|^p_{B^p}\} < \infty$. On the other hand, $f \notin B^+_1(\Delta)$ since $\|f\|_{B^+_1} = \int_\Delta |f''(z)|[1 + |\log(|f''(z)|(1-|z|)^2)|]^{-1}\,dA(z) = \infty$.

[1] Author supported in part by National Science Foundation grant DMS-9500758.

Note remise le 2 mai 1997, acceptée le 5 mai 1997.

References

[1] **Peller V. V., 1980, 1982.** Hankel operators of class S_p and applications, *Math. Sb.*, 113, pp. 538-581; *Translation in Math. USSR Sbornik*, 4, pp. 443-479

[2] **Coifman R. and Rochberg R., 1980.** Representation theorems for holomorphic and harmonic functions in L^p, *Astérisque*, 77, pp. 11-60.

[3] **Rochberg R., 1982.** Trace ideal criteria for Hankel operators and commutators, *Indiana U. Math. J.*, 31, pp. 913-925.

[4] **Janson S., Peetre J. and Rochberg R., 1987.** Hankel forms and the Fock space, *Revista Math. Iberoamericano*, 3, pp. 61-138.

[5] **Feldman M. and Rochberg R., 1990.** Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group, *Analysis and PDE*, Sadosky C. ed., *A collection of papers dedicated to M. Cotlar*, Dekker, New York, pp. 121-160.

[6] **Zhang G., 1991.** Hankel operators on Hardy space and Schatten classes, *Chinese Ann. of Math.*, 12B, 3, pp. 282-293.

[7] **Connes A., 1994.** *Noncommutative geometry*, Academic Press.

[8] **Dixmier J., 1966.** Existence de traces non normales, *C. R. Acad. Sci. Paris*, 262, pp. 1107-1108.

[9] **Bellissard J., van Elst A. and Schulz-Baldes, 1994.** The noncommutative geometry of the quantum Hall effect, *J. Math. Phys.*, 35, pp. 5373-5451.

[10] **Krantz S. G. and Li S.-Y., 1995.** On the Decomposition and Factorization Theorems for Hardy Spaces in Domains in $\mathbb{C}^n$, *J. Fourier Analysis and Applications*, 2, pp. 65-107.

[11] **Symesak F., 1996.** Hankel operators on pseudoconvex domains of finite type in $\mathbb{C}^n$, *preprint*, Université de Poitiers.

[12] **Fefferman C., 1974.** The Bergman kernel and biholomorphic mappings, *Invent. Math.*, 26, pp. 1-65.

[13] **Zhu K., 1995.** Holomorphic Besov spaces on bounded symmetric domains, *Indiana U. Math. J.*, 44, pp. 1017-1031.