2.3 Submanifolds and Embeddings

The implicit function theorem (see p. 72 of this book) and 140C NOTES sections 25 and 26 deals with subsets of a manifold M that are themselves manifolds in the sense of Definition 2.1.3. Such subsets are called submanifolds of M.

Definition 2.3.1 (**Submanifold**)**.** *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold. A subset $L \subset M$ is called a* **submanifold** *of M of dimension ℓ, if L itself is an ℓ-manifold.*

Definition 2.3.2 (**Embedding**)**.** *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold. A smooth map $f : N \to M$ is called an* **immersion** *if its differential $df(q) : T_qN \to T_{f(q)}M$ is injective* (= one-to-one) *for every $q \in N$. It is called* **proper** *if, for every compact subset $K \subset f(N)$, the preimage $f^{-1}(K) = \{q \in N \mid f(q) \in K\}$ is compact. The map f is called an* **embedding** *if it is a proper injective immersion.*

see 140C NOTES sections 5, 6, 7 pp 10–14

Remark 2.3.3. In our definition of proper maps it is important that the compact set K is required to be contained in the image of f. The literature also contains a stronger definition of *proper* which requires that $f^{-1}(K)$ is a compact subset of M for every compact subset $K \subset N$, whether or not K is contained in the image of f. This holds if and only if the map f is proper in the sense of Definition 2.3.2 and has an M-closed image. (Exercise!)

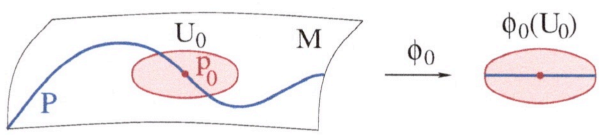


Figure 2.5: A coordinate chart adapted to a submanifold.

Theorem 2.3.4 (**Submanifolds**)**.** *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold.*

(i) *If $f : N \to M$ is an embedding then $f(N)$ is a submanifold of M.*

(ii) *If $P \subset M$ is a submanifold then the inclusion $P \to M$ is an embedding.*

(iii) *A subset $P \subset M$ is a submanifold of dimension n if and only if, for every $p_0 \in P$ there exists a coordinate chart $\phi : U_0 \to \mathbb{R}^m$ defined on an M-open neighborhood $U_0 \subset M$ of p_0 (see Figure 2.5) such that*

$$\phi(U_0 \cap P) = \phi(U_0) \cap (\mathbb{R}^n \times \{0\}).$$

Proof. See page 35. □

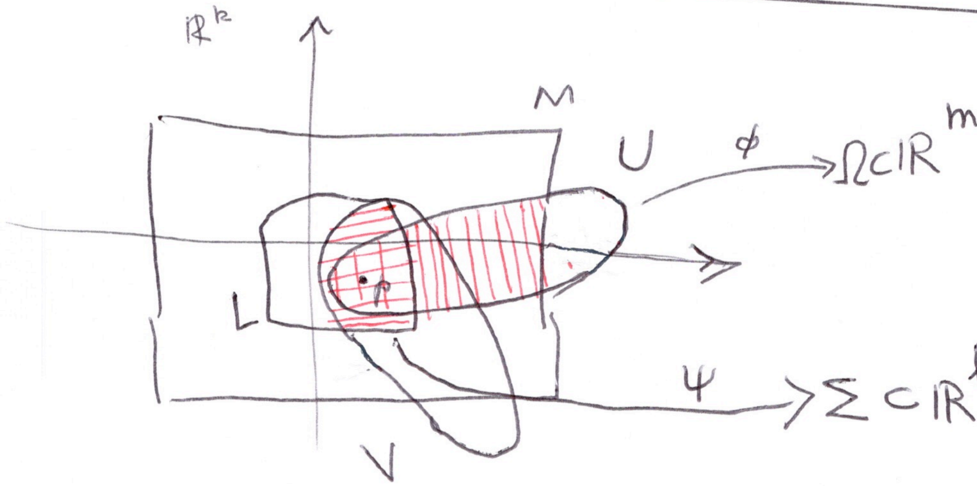
2-8-18

def 2.3.1 $M \subset \mathbb{R}^k$ m-dim'l manifold

(This means $\forall p \in M$ $\exists$ open $U \subset \mathbb{R}^k$ with $U \cap M$ diffeomorphic to open $\Omega \subset \mathbb{R}^m$)

a subset $L \subset M$ is a <u>submanifold of M</u> of dimension l if $L \subset \mathbb{R}^k$ is an l-dim'l manifold.

(so $\forall q \in L$ $\exists$ open $V \subset \mathbb{R}^k$ with $V \cap L$ diffeo to open $\Sigma \subset \mathbb{R}^l$)



<u>def 2.3.2</u>

$$\begin{array}{ccc} \mathbb{R}^l & & \mathbb{R}^k \\ \cup & & \cup \\ N & \xrightarrow{f \text{ smooth}} & M \\ n\text{-dim'l} & & m\text{-dim'l} \\ (\text{so } n \le l) & & (m \le k) \end{array}$$

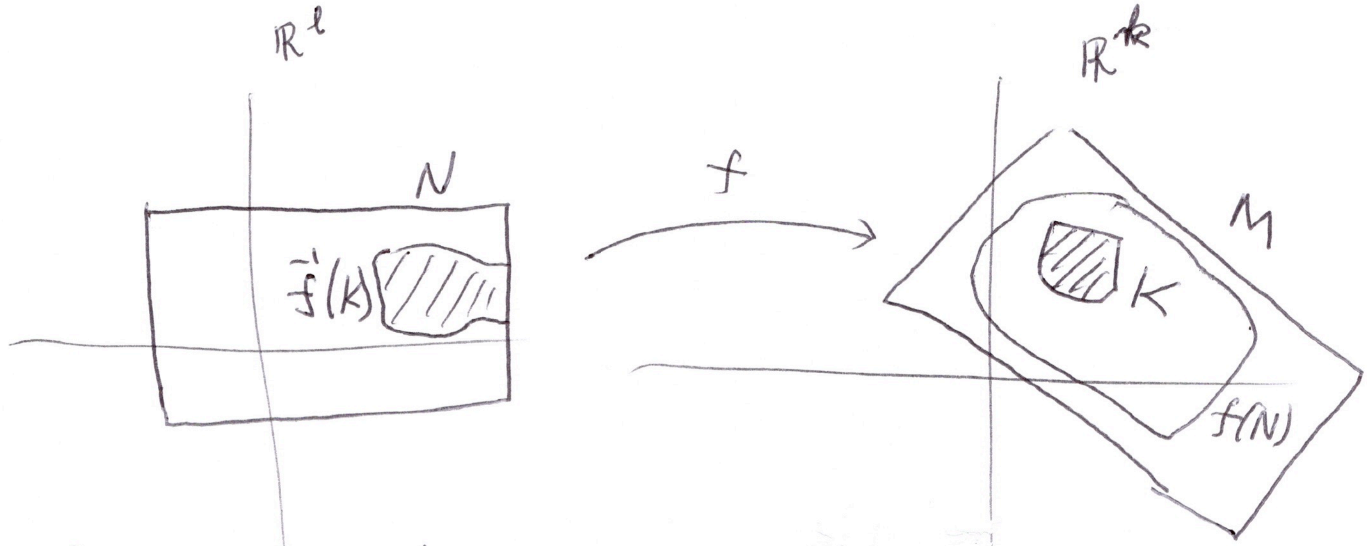
f is <u>immersion</u> if $\forall q \in N$ $df(q) : T_q N \to T_{f(q)} M$ is one-to-one

$$\left(df(q)\, v = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \lim_{h \to 0} \frac{f(\gamma(h)) - f(q)}{h} \in T_{f(q)} M \subset \mathbb{R}^k \right.$$

$\gamma : \mathbb{R} \to N$ $\gamma(0) = q$ $\dot{\gamma}(0) = v$

so $\dim(T_q N) \le \dim(T_{f(q)} M)$)

f is <u>proper</u> if $\forall$ compact $K \subset f(N)$
the pre-image $f^{-1}(K) = \{ q \in N : f(q) \in N \}$ is compact.



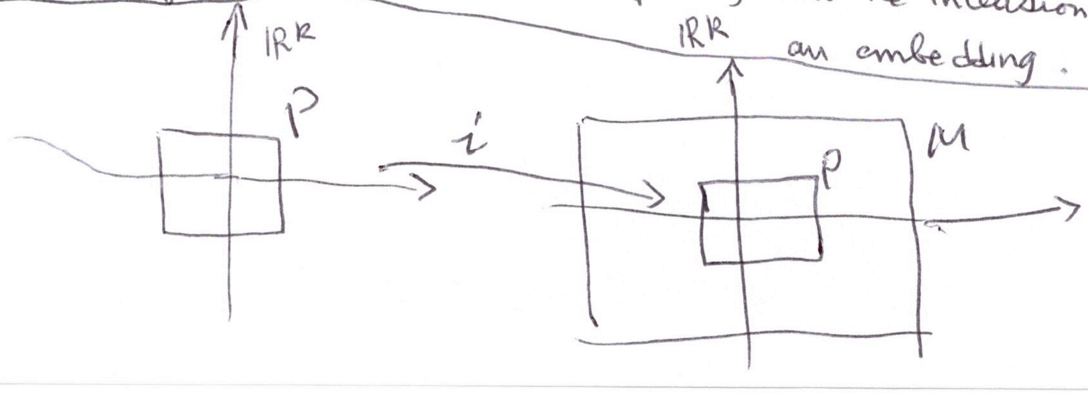
f is an <u>embedding</u> if it is injective (= one-to-one), proper,
and an immersion

Theorem 2.3.4 let $\underset{\text{n-dim'l}}{\overset{\text{smooth}}{f: N}} \xrightarrow{\quad} \underset{\text{m-dim'l}}{M}$, $N \subset \mathbb{R}^l$, $M \subset \mathbb{R}^k$

(i) f an embedding $\Longrightarrow$ $f(N)$ is a submanifold of M

<u>Need to prove</u>: $f(N) \subset M$ and $f(N) \subset \mathbb{R}^k$ is a manifold of some dimension
(it turns out that $f(N)$ is an n-dim'l manifold, same as N)

(ii) If P is a submanifold of M, then the inclusion $P \to M$ is an embedding.



<u>Need to prove</u>:
i is smooth
i is immersion
i is proper
i is injective ?

Lemma 2.3.5 (Embeddings). *Let M and N be as in Theorem 2.3.4, let $f : N \to M$ be an embedding, let $q_0 \in N$, and define*

$$P := f(N), \qquad p_0 := f(q_0) \in P.$$

Then there exists an M-open neighborhood $U \subset M$ of p_0, an N-open neighborhood $V \subset N$ of q_0, an open neighborhood $W \subset \mathbb{R}^{m-n}$ of the origin, and a diffeomorphism $F : V \times W \to U$ such that, for all $q \in V$ and all $z \in W$,

$$F(q,0) = f(q) \tag{2.3.1}$$

and

$$F(q,z) \in P \qquad \iff \qquad z = 0. \tag{2.3.2}$$

Proof. Choose any coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ on an M-open neighborhood $U_0 \subset M$ of p_0. Then the differential

$$d(\phi_0 \circ f)(q_0) = d\phi_0(f(q_0)) \circ df(q_0) : T_{q_0}N \to \mathbb{R}^m$$

is injective. Hence there is a linear map $B : \mathbb{R}^{m-n} \to \mathbb{R}^m$ such that the map

$$T_{q_0}N \times \mathbb{R}^{m-n} \to \mathbb{R}^m : (w, \zeta) \mapsto d(\phi_0 \circ f)(q_0)w + B\zeta \tag{2.3.3}$$

is a vector space isomorphism. Define the set

$$\Omega := \left\{(q,z) \in N \times \mathbb{R}^{m-n} \,|\, f(q) \in U_0,\ \phi_0(f(q)) + Bz \in \phi_0(U_0)\right\}.$$

This is an open subset of $N \times \mathbb{R}^{m-n}$ and we define $F : \Omega \to M$ by

$$F(q,z) := \phi_0^{-1}\left(\phi_0(f(q)) + Bz\right).$$

This map is smooth, it satisfies $F(q,0) = f(q)$ for all $q \in f^{-1}(U_0)$, and the derivative $dF(q_0,0) : T_{q_0}N \times \mathbb{R}^{m-n} \to T_{p_0}M$ is the composition of the map (2.3.3) with $d\phi_0(p_0)^{-1} : \mathbb{R}^m \to T_{p_0}M$ and so is a vector space isomorphism. Thus the Inverse Function Theorem 2.2.15 asserts that there is an N-open neighborhood $V_0 \subset N$ of q_0 and an open neighborhood $W_0 \subset \mathbb{R}^{m-n}$ of the origin such that $V_0 \times W_0 \subset \Omega$, the set $U_0 := F(V_0 \times W_0)$ is M-open, and the restriction of F to $V_0 \times W_0$ is a diffeomorphism onto U_0. Thus we have constructed a diffeomorphism $F : V_0 \times W_0 \to U_0$ that satisfies (2.3.1).

We claim that the restriction of F to the product $V \times W$ of sufficiently small open neighborhoods $V \subset N$ of q_0 and $W \subset \mathbb{R}^{m-n}$ of the origin also satisfies (2.3.2). Otherwise, there exist sequences $q_i \in V_0$ converging to q_0 and $z_i \in W_0 \setminus \{0\}$ converging to zero such that $F(q_i, z_i) \in P$. Hence there exists a sequence $q_i' \in N$ such that $F(q_i, z_i) = f(q_i')$. This sequence converges to $f(q_0)$. Since f is proper we may assume, passing to a suitable subsequence if necessary, that q_i' converges to a point $q_0' \in N$. Then

$$f(q_0') = \lim_{i\to\infty} f(q_i') = \lim_{i\to\infty} F(q_i, z_i) = f(q_0),$$

because f and F are continuous. Since f is injective, this implies $q_0' = q_0$. Hence $(q_i', 0) \in V_0 \times W_0$ for i sufficiently large and $F(q_i', 0) = f(q_i') = F(q_i, z_i)$. This contradicts the fact that the map $F : V_0 \times W_0 \to M$ is injective. Thus we have proved Lemma 2.3.5. □

Proof of Theorem 2.3.4. We prove (i). Let $q_0 \in N$, denote $p_0 := f(q_0) \in P$, and choose a diffeomorphism $F : V \times W \to U$ as in Lemma 2.3.5. Then set $V \subset N$ is diffeomorphic to an open subset of $\mathbb{R}^n$ (after schrinking V if necessry), the set $U \cap P$ is P-open because $U \subset M$ is M-open, and we have $U \cap P = \{F(q, 0) \,|\, q \in V\} = f(V)$ by (2.3.1) and (2.3.2). Hence the map $f : V \to U \cap P$ is a diffeomorphism whose inverse is the composition of the smooth maps $F^{-1} : U \cap P \to V \times W$ and $V \times W \to V : (q, z) \mapsto q$. Hence a P-open neighborhood of p_0 is diffeomorphic to an open subset of $\mathbb{R}^n$. Since $p_0 \in P$ was chosen arbitrary, this shows that P is an n-dimensional submanifold of M.

We prove (ii). The inclusion $\iota : P \to M$ is obviously smooth and injective (it extends to the identity map on $\mathbb{R}^k$). Moreover, $T_pP \subset T_pM$ for every $p \in P$ and the differential $d\iota(p) : T_pP \to T_pM$ is the obvious inclusion for every $p \in P$. That ι is proper follows immediately from the definition. Hence ι is an embedding.

We prove (iii). If a coordinate chart ϕ_0 as in (iii) exists then the set $U_0 \cap P$ is P-open and is diffeomorphic to an open subset of $\mathbb{R}^n$. Since the point $p_0 \in P$ was chosen arbitrary this proves that P is an n-dimensional submanifold of M. Conversely, suppose that P is an n-dimensional submanifold of M and let $p_0 \in P$. Choose any coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ of M defined on an M-open neighborhood $U_0 \subset M$ of p_0. Then $\phi_0(U_0 \cap P)$ is an n-dimensional submanifold of $\mathbb{R}^m$. Hence Theorem 2.1.10 asserts that there are open sets $V, W \subset \mathbb{R}^m$ with $p_0 \in V \subset \phi_0(U_0)$ and a diffeomorphism $\psi : V \to W$ such that

$$\phi_0(p_0) \in V, \qquad \psi(V \cap \phi_0(U_0 \cap P)) = W \cap (\mathbb{R}^n \times \{0\}).$$

Now define $U := \phi_0^{-1}(V) \subset U_0$. Then $p_0 \in U$, the chart ϕ_0 restricts to a diffeomorphism from U to V, the composition $\phi := \psi \circ \phi_0|_U : U \to W$ is a diffeomorphism, and $\phi(U \cap P) = \psi(V \cap \phi_0(U_0 \cap P)) = W \cap (\mathbb{R}^n \times \{0\})$. This proves Theorem 2.3.4. □

(iii) A subset $P \subset M \subset \mathbb{R}^k$ is a submanifold of dimension n $(n \leq k)$

$\Longleftrightarrow$ $\forall\, p_0 \in P$ $\exists$ coordinate chart $\phi : U \to \mathbb{R}^m$,

U an M-open subset of M containing p_0, with

$$\phi(U \cap P) = \phi(U) \cap (\mathbb{R}^n \times \{0\}).$$

Example 2.3.6. Let $S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$ be the unit circle and consider the map $f : S^1 \to \mathbb{R}^2$ given by $f(x, y) := (x, xy)$. This map is a proper immersion but is not injective (the points $(0, 1)$ and $(0, -1)$ have the same image under f). The image $f(S^1)$ is a figure 8 in $\mathbb{R}^2$ and is not a submanifold (Figure 2.6).

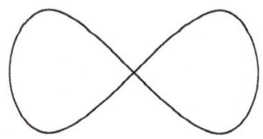


Figure 2.6: A proper immersion.

Example 2.3.7. Consider the restriction of the map f in Example 2.3.6 to the submanifold $N := S^1 \setminus \{(0, -1)\}$. The resulting map $f : N \to \mathbb{R}^2$ is an injective immersion but it is not proper. It has the same image as before and hence $f(N)$ is not a manifold.

Example 2.3.8. The map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) := (t^2, t^3)$ is proper and injective, but is not an embedding (its differential at $x = t$ is not injective). The image of f is the set $f(\mathbb{R}) = C := \{(x, y) \in \mathbb{R}^2 \mid x^3 = y^2\}$ (see Figure 2.7) and is not a submanifold. (Prove this!)

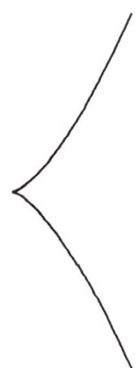


Figure 2.7: A proper injection.

Example 2.3.9. Define the map $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) := (\cos(t), \sin(t))$. This map is an immersion, but it is neither injective nor proper. However, its image is the unit circle in $\mathbb{R}^2$ and hence is a submanifold of $\mathbb{R}^2$. The map $\mathbb{R} \to \mathbb{R}^2 : t \mapsto f(t^3)$ is not an immersion and is neither injective nor proper, but its image is still the unit circle.