Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 22; March 18, 2007

Exercise 32.2

Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.

Solution: For each subinterval $[t_{k-1}, t_k]$ corresponding to a partition P of $[0, b]$, it is obvious that $m(f, [t_{k-1}, t_k]) = 0$. Therefore $L(f, P) = 0$ for every partition P and thus $L(f) = 0$.

We shall show that $U(f) = b^2/2$. This is achieved by showing that for every partition P, $U(f, P) = U(g, P)$ where $g(x) = x$ for every $x \in [0, b]$. (Note that by an argument very similar to that for Example 1 in section 32, g is integrable with $U(g) = b^2/2$.) To prove the assertion above, note that for each subinterval $[t_{k-1}, t_k]$ corresponding to a partition P of $[0, b]$, it is obvious that $M(f, [t_{k-1}, t_k]) = M(g, [t_{k-1}, t_k])$. Therefore $U(f, P) = U(g, P)$ for every partition P and thus $U(f) = \inf_P\{U(f, P)\} = \inf_P\{U(g, P)\} = U(g) = b^2/2$.

(b) Is f integrable on $[0, b]$?

Solution: No; $L(f) \neq U(f)$.

Exercise 32.8

Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subset [a, b]$.

Solution: In this proof, we shall "decorate upper and lower sums so that it will be clear which interval we are dealing with" (See the proof of Theorem 33.6 on page 258).

Given $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that

$$U_a^b(f, P_\epsilon) - L_a^b(f, P_\epsilon) < \epsilon.$$

Let $P'_\epsilon = P_\epsilon \cup \{c, d\}$. Since $P_\epsilon \subset P'_\epsilon$, by Lemma 32.2 we have

$$U_a^b(f, P'_\epsilon) - L_a^b(f, P'_\epsilon) \leq U_a^b(f, P_\epsilon) - L_a^b(f, P_\epsilon) < \epsilon.$$

Now let $P''_\epsilon = P'_\epsilon \cap [c, d]$. Then P''_ϵ is a partition of $[c, d]$ and we have

$$\begin{aligned}
\epsilon &> U_a^b(f, P'_\epsilon) - L_a^b(f, P'_\epsilon) \\
&= U_a^c(f, P'_\epsilon \cap [a, c]) + U_c^d(f, P'_\epsilon \cap [c, d]) + U_d^b(f, P'_\epsilon \cap [d, b]) \\
&\quad -L_a^c(f, P'_\epsilon \cap [a, c]) - L_c^d(f, P'_\epsilon \cap [c, d]) - L_d^b(f, P'_\epsilon \cap [d, b]) \\
&\geq U_a^c(f, P'_\epsilon \cap [a, c]) - L_a^c(f, P'_\epsilon \cap [a, c]) \\
&\quad +U_c^d(f, P'_\epsilon \cap [c, d]) - L_c^d(f, P'_\epsilon \cap [c, d]) \\
&\quad +U_d^b(f, P'_\epsilon \cap [d, b]) - L_d^b(f, P'_\epsilon \cap [d, b]) \\
&\geq 0 + U_c^d(f, P'_\epsilon \cap [c, d]) - L_c^d(f, P'_\epsilon \cap [c, d]) + 0 \\
&= U_c^d(f, P''_\epsilon) - L_c^d(f, P''_\epsilon).
\end{aligned}$$

By Theorem 32.5, f is integrable on $[c, d]$.