Growth-Optimality against Underperformance

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Large Deviation Theory (Markov's Inequality)

If $X \geq 0$ is a random variable and $x > 0$, then the Markov inequality is:

$$x \cdot \mathbb{P}[X \geq x] \leq \mathbb{E}[X]$$

$$\mathbb{P}[X \geq x] \leq \frac{\mathbb{E}[X]}{x}$$

Now let X_i be random variables, so that for any $\theta > 0$ and $x \in \mathbb{R}$ we have

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \leq x\right] = \mathbb{P}\left[e^{-\theta \sum_{i=1}^{T} X_i} \geq e^{-\theta T x}\right] \leq \frac{\mathbb{E}\left[e^{-\theta \sum_{i=1}^{T} X_i}\right]}{e^{-\theta T x}}$$

Large Deviation Theory (Chernoff bounds)

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \leq x\right] = \mathbb{P}\left[e^{-\theta\sum_{i=1}^{T} X_i} \geq e^{-\theta T x}\right] \leq \frac{\mathbb{E}\left[e^{-\theta\sum_{i=1}^{T} X_i}\right]}{e^{-\theta T x}}$$

If the X_i are independent, we can split the expectation into the product, so it follows that

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \leq x\right] \leq e^{\theta T x}\prod_{i=1}^{T}\mathbb{E}\left[e^{-\theta\sum_{i=1}^{T} X_i}\right] = e^{\theta T x + \sum_{i=1}^{T}\log\mathbb{E}\left[e^{-\theta X_i}\right]}$$

Large Deviation Theory (Chernoff bounds)

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \le x\right] \le e^{\theta T x}\prod_{i=1}^{T}\mathbb{E}\left[e^{-\theta\sum_{i=1}^{T} X_i}\right] = e^{\theta T x + \sum_{i=1}^{T}\log\mathbb{E}\left[e^{-\theta X_i}\right]}$$

The function

$$\lambda_i(\theta) = \log\mathbb{E}\left[e^{\theta X_i}\right] = \sum_{n=1}^{\infty}\frac{\kappa_n}{n!}\theta^n$$

is called the cumulant generating function for X_i, where $\kappa_1 = \mathbb{E}[X_i]$ and $\kappa_2 = \mathrm{var}(X_i)$. For $\theta > 0$ and X_i independent our bound is now

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \le x\right] \le e^{\theta T\left(x + \frac{1}{T}\sum_{i=1}^{T}\lambda_i(-\theta)\right)}$$

Large Deviation Theory (Chernoff bounds)

Since this holds for all $\theta > 0$, we have

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \leq x\right] \leq \inf_{\theta>0} e^{\theta T\left(x+\frac{1}{T}\sum_{i=1}^{T}\lambda_i(-\theta)\right)}$$

If the X_i are i.i.d. then we have

$$\mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} X_i \leq x\right] \leq \inf_{\theta>0} e^{\theta T\left(x+\lambda_1(-\theta)\right)} = \left(e^{\inf_{\theta>0}\left[(x-\kappa_1)\theta+\frac{\kappa_2}{2}\theta^2-\dots\right]}\right)^T$$

If $x - \kappa_1 = x - \mathbb{E}[X_1] < 0$ and the power series for $\lambda(\theta)$ has nonzero radius then this bound is not trivial. In fact, if the X_i are normal then $\kappa_n = 0$ for all $n \geq 3$.

Gambling

- Let W_0 be our initial wealth.
- We choose to bet $0 \leq p \leq 1$ fraction of our wealth on a gamble with odds $\pi > 1/2$.
- After T rounds our wealth is

$$W_{p,T} = W_0 \prod_{i=1}^{T} R_{p,i} = W_0 \exp\left(\sum_{i=1}^{T} \log R_{p,i}\right)$$

where

$$\mathbb{P}\left[R_{p,i} = 1 + p\right] = \pi, \quad \mathbb{P}\left[R_{p,i} = 1 - p\right] = 1 - \pi$$

- The Kelly criterion says to pick p so as to maximize

$$\max_{0 \leq p \leq 1} \mathbb{E}\left[\log R_{p,i}\right] = \max_{0 \leq p \leq 1} \log\left((1+p)^{\pi}(1-p)^{1-\pi}\right)$$

Underperforming a benchmark

Suppose we are now concerned about underperforming some benchmark rate $a > 1$.

$$\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] = \mathbb{P}\left[\frac{1}{T}\sum_{i=1}^{T} \log R_{p,i} \leq \log a\right]$$

Using large deviations we immediately have

$$\begin{aligned}\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] &\leq \inf_{\theta>0} \exp\left(\theta T\left(\log a + \log \mathbb{E}\left[e^{-\theta \log R_{p,1}}\right]\right)\right)\\ &= \left(\inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]\right)^T\end{aligned}$$

Underperforming a benchmark

$$\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] \leq \left(\inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]\right)^T$$

As T grows, suppose we want to minimize our chances of underperforming the benchmark. Our goal is to pick a $0 \leq p \leq 1$ so as to minimize

$$\min_{0\leq p\leq 1} \quad \inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]$$

Suppose $\pi = 0.6$ and the benchmark is 1%, then this becomes

$$\min_{0\leq p\leq 1} \inf_{\theta>0} \left[0.6\left(\frac{1+p}{1.01}\right)^{-\theta} + 0.4\left(\frac{1-p}{1.01}\right)^{-\theta}\right]$$

Kelly Criterion

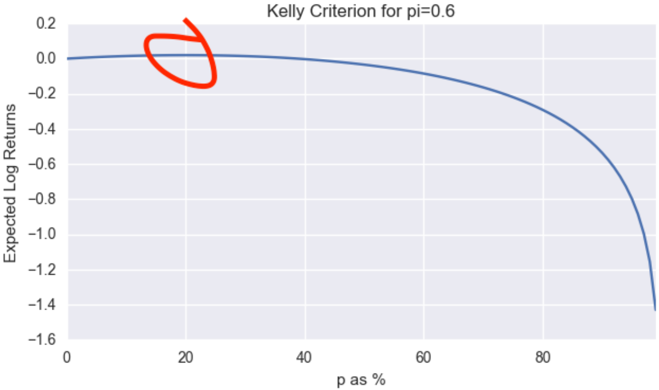
For $\pi = 0.6$ Kelly is

$$\max_{0 \le p \le 1} \mathbb{E}\left[\log R_{p,1}\right] = \max_{0 \le p \le 1} 0.6 \log(1+p) + 0.4 \log(1-p)$$
$$= \max_{0 \le p \le 1} \log\left((1+p)^{0.6}(1-p)^{0.4}\right)$$

which is realized when $p = \pi - (1-\pi) = 2\pi - 1 = 0.2$.

Kelly Criterion

$$\max_{0 \le p \le 1} \log \left((1+p)^{0.6} (1-p)^{0.4} \right)$$



Underperforming a benchmark

$$\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] \leq \left(\inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]\right)^T$$

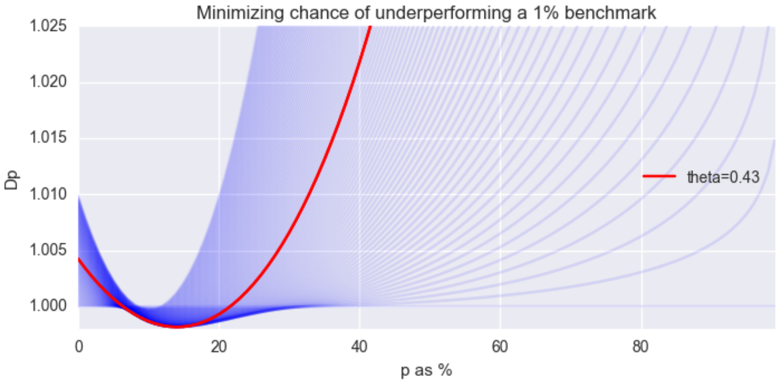
Now suppose our goal is minimizing the probability of underperforming the benchmark 1%. We want to minimize

$$\min_{0\leq p\leq 1,\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]$$

$$= \min_{0\leq p\leq 1,\theta>0} \left[0.6\left(\frac{1+p}{1.01}\right)^{-\theta} + 0.4\left(\frac{1-p}{1.01}\right)^{-\theta}\right]$$

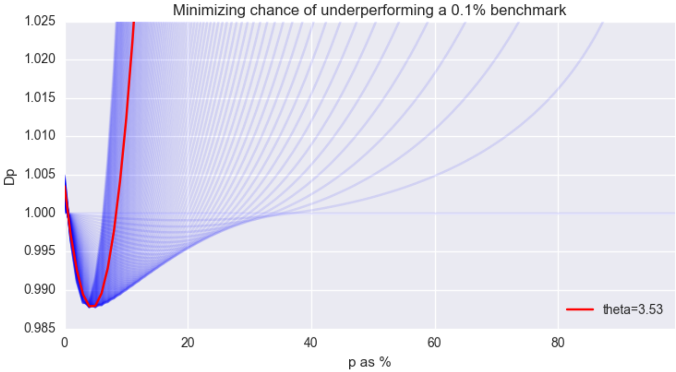
Underperforming a benchmark of 1%

$$D_{p,\theta} = 0.6\left(\frac{1+p}{1.01}\right)^{-\theta} + 0.4\left(\frac{1-p}{1.01}\right)^{-\theta}$$



Underperforming a (smaller) benchmark of 0.1%

$$D_{p,\theta} 0.6 \left(\frac{1+p}{1.001}\right)^{-\theta} + 0.4 \left(\frac{1-p}{1.001}\right)^{-\theta}$$



Underperforming a benchmark

- Our goal of minimizing the asymptotic probability

$$\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] \leq \left(\inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]\right)^T$$

leads us to consider a dual optimization in terms of p and θ.

$$\min_{0\leq p\leq 1} \min_{\theta>0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]$$

- $1+\theta$ plays the role of the bettor's risk aversion. It is not exogenous, but rather determined by the inner maximization. For instance, a bettor who is concerned with outperforming returns of 1% exhibits risk aversion of $1+\theta = 1.43$.

Isoelastic Utility

- Note that our problem

$$\min_{0 \leq p \leq 1} \min_{\theta > 0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]$$

can be rephrased to appear similar to maximizing the isoelastic utility of our returns:

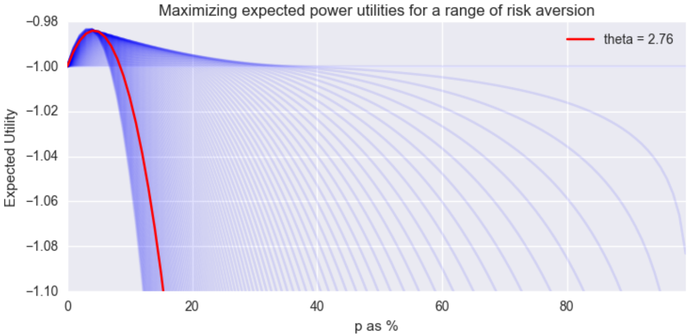
$$\max_{0 \leq p \leq 1} \max_{\gamma > 1} \mathbb{E}\left[-\left(\frac{R_{p,1}}{a}\right)^{1-\gamma}\right]$$

where γ is risk aversion.

Risk Aversion

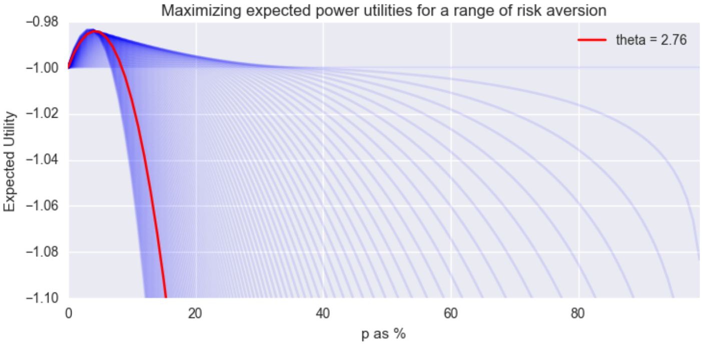
- Consider the expected utility for the Blackjack game with $\pi = 0.6$ and varying risk aversion $1 + \theta = \gamma$.

$$\max_{0 \le p \le 1} \mathbb{E}\left[-R_{p,1}^{-\theta}\right] = \max_{0 \le p \le 1}\left[-0.6(1+p)^{-\theta} - 0.4(1-p)^{-\theta}\right]$$



Risk Aversion

- If $\theta > 2.76$ is a bettor's exogenous risk aversion then a bettor considers a bet $p = 10\%$ to be unfavorable to a bet of $p = 0\%$, regardless of their initial wealth W_0 or the number of trials T.



Measuring Risk Aversion

- Barsky et al. (1997) designed a questionnaire given to thousands of individuals in person by Federal interviewers, and about 2/3 of them had relative risk aversion higher than $3.76 = 1 + \theta$.
- Suppose I offer you the chance to play the blackjack $\pi = 0.6$ game 10,000 times instantly on a computer, but if you agree you must use the strategy $p = 10\%$.
- Using the large deviation bound derived above, the long term behavior hinges on

$$\mathbb{P}\left[W_{p,T} \leq W_0 a^T\right] \leq \left(\inf_{\theta > 0} \mathbb{E}\left[\left(\frac{R_{p,1}}{a}\right)^{-\theta}\right]\right)^T$$

Measuring Risk Aversion

- The chances of underperforming an 0.6% benchmark are quite bad:

$$\mathbb{P}\left[W_{10\%,10^4} \leq W_0 1.006^{10^4}\right] \leq \left(\inf_{\theta>0} \mathbb{E}\left[\left(\frac{R_{10\%,1}}{1.006}\right)^{-\theta}\right]\right)^{10^4}$$
$$\leq 0.998^{10^4} < 10^{-8}$$

so it is quite likely you will end up with more than $W_0 \times 10^{24}$, and all you stand to lose is W_0.

Measuring Risk Aversion

- An individual with exogenous risk of $1+\theta = 3.76$ or greater would not want to take this bet because they are principally interested in maximizing

$$\max_{0\le p\le 1} \mathbb{E}\left[-R_{p,1}^{-\theta}\right] = \max_{0\le p\le 1}\left[-0.6(1+p)^{-\theta} - 0.4(1-p)^{-\theta}\right]$$

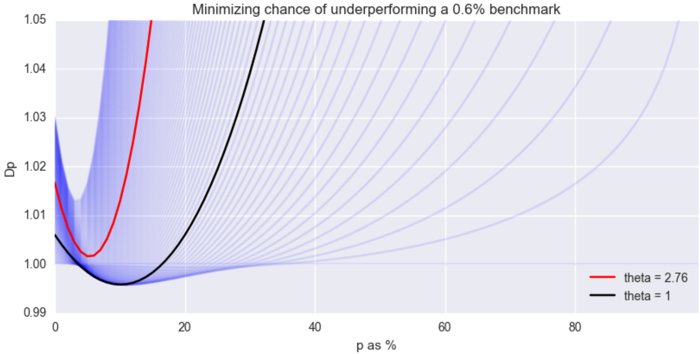
  and the choice $p = 10\%$ is worse (according to their expected utility) than a choice of $p = 0\%$.
- On the other hand, an individual hoping to beat a modest benchmark of 0.6% is hoping to minimize

$$\min_{0\le p\le 1}\ \min_{\theta>0}\left[0.6\left(\frac{1+p}{1.006}\right)^{-\theta} + 0.4\left(\frac{1-p}{1.006}\right)^{-\theta}\right]$$

Measuring Risk Aversion

- Such an individual would be willing to take the bet.

$$D_{p,\theta} = 0.6\left(\frac{1+p}{1.006}\right)^{-\theta} + 0.4\left(\frac{1-p}{1.006}\right)^{-\theta}$$



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