Fat Tailed Kelly

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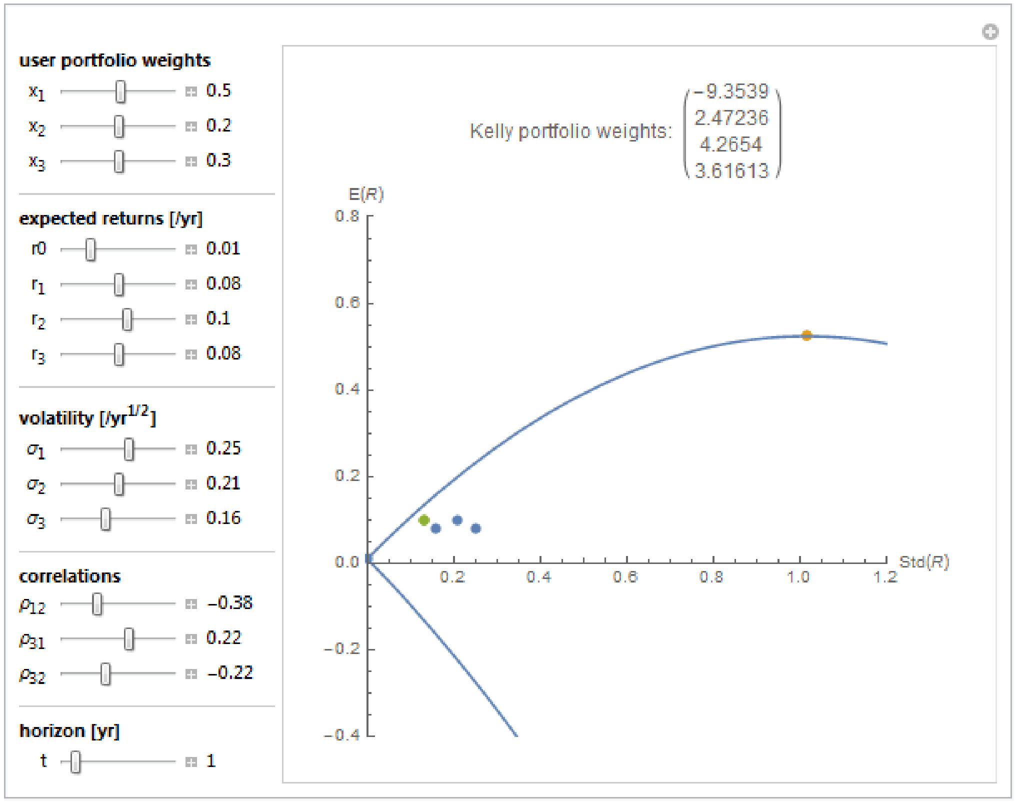
Normal Returns

http://demonstrations.wolfram.com/KellyPortfolioAnalysis



- Assumptions
  - Trade continuously in any size, long or short, with no bid-ask spread
  - Known, normal, distribution
- Results
  - $R_t^{\gamma} \sim N(r_{\gamma} t, \gamma^T \cdot \Sigma \cdot \gamma t)$
  - $r_{\gamma} = r_0 + \gamma^T \cdot (\mu - r_0 1) - \frac{1}{2}\gamma^T \cdot \Sigma \cdot \gamma$
  - $\sigma_{\gamma}^2 = \gamma^T \cdot \Sigma \cdot \gamma$
  - $\mu_i = r_i + \frac{1}{2}\Sigma_{ii}$
  - $\gamma^* = \Sigma^{-1} \cdot (\mu - r_0 1)$

Stable Distribution

Univariate

https://en.wikipedia.org/wiki/Stable_distribution

$X \sim S(\alpha, \beta, \mu, c)$ means its log characteristic function is $\psi(k) = ik\mu - |kc|^{\alpha}[1 - i\beta\, \mathrm{sign}(k) \tan(\alpha\frac{\pi}{2})]$.

Multivariate

https://en.wikipedia.org/wiki/Multivariate_stable_distribution

$X \sim S(\alpha, \mu, \Gamma)$ means

$\psi(k) =$

$ik^T \cdot \mu - \int_{s \in \mathbb{S}} [|k^T \cdot s|^{\alpha} - i\, \mathrm{sign}(k^T \cdot s) \tan(\alpha\frac{\pi}{2})] d\Gamma(s)$.

Further, $\gamma^T \cdot X \sim S(\alpha, \beta_\gamma, \mu_\gamma, c_\gamma)$ where

$c_\gamma^{\alpha} = \int_{s \in \mathbb{S}} |\gamma^T \cdot s|^{\alpha}\, d\Gamma(s)$,

$\beta_\gamma = \int_{s \in \mathbb{S}} |\gamma^T \cdot s|^{\alpha} \mathrm{sign}(\gamma^T \cdot s)\, d\Gamma(s)/c_\gamma^{\alpha}$,

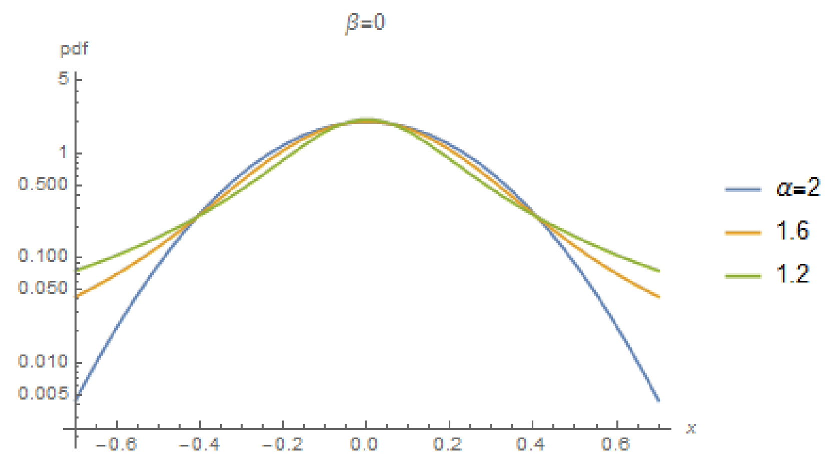
$\mu_\gamma = \gamma^T \cdot \mu$.

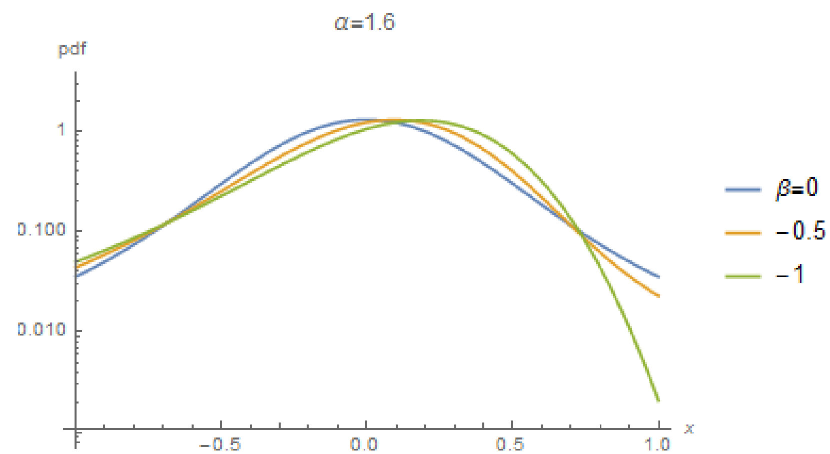
Infinite variance when $\alpha < 2$; infinite mean when $\alpha < 1$.

Generalized Central Limit Theorem

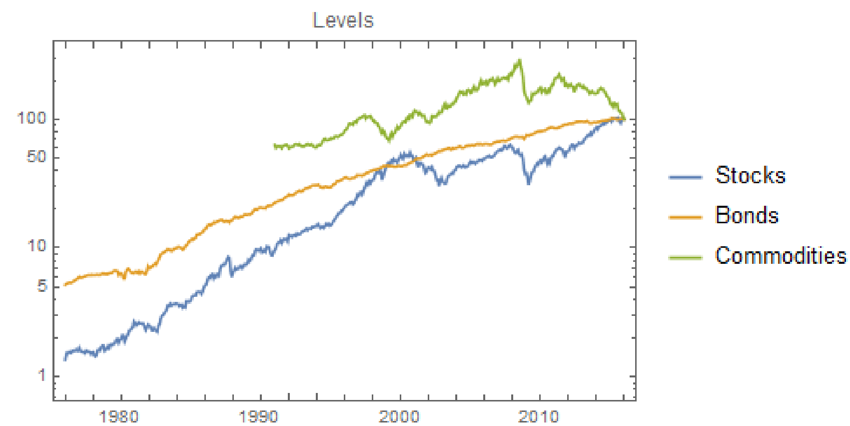
Levy Measure

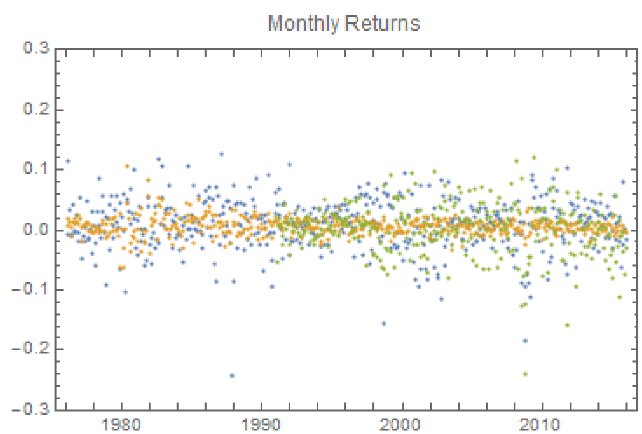
$dF(x) = d\Gamma(s) dr / r^{\alpha+1}$ where $r = |x|$ and $s = x/r$.





Empirically, Returns have Fat Tails

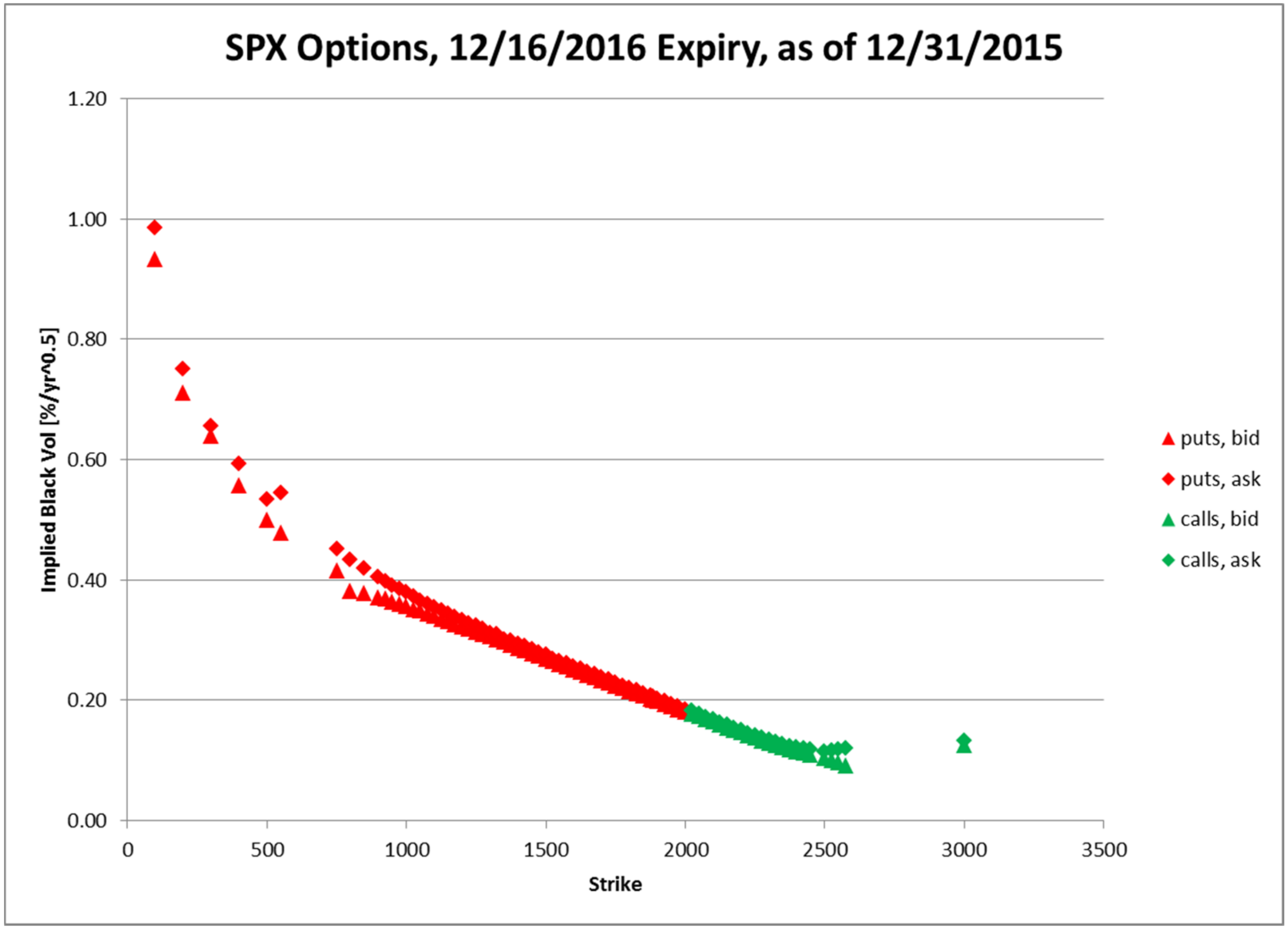




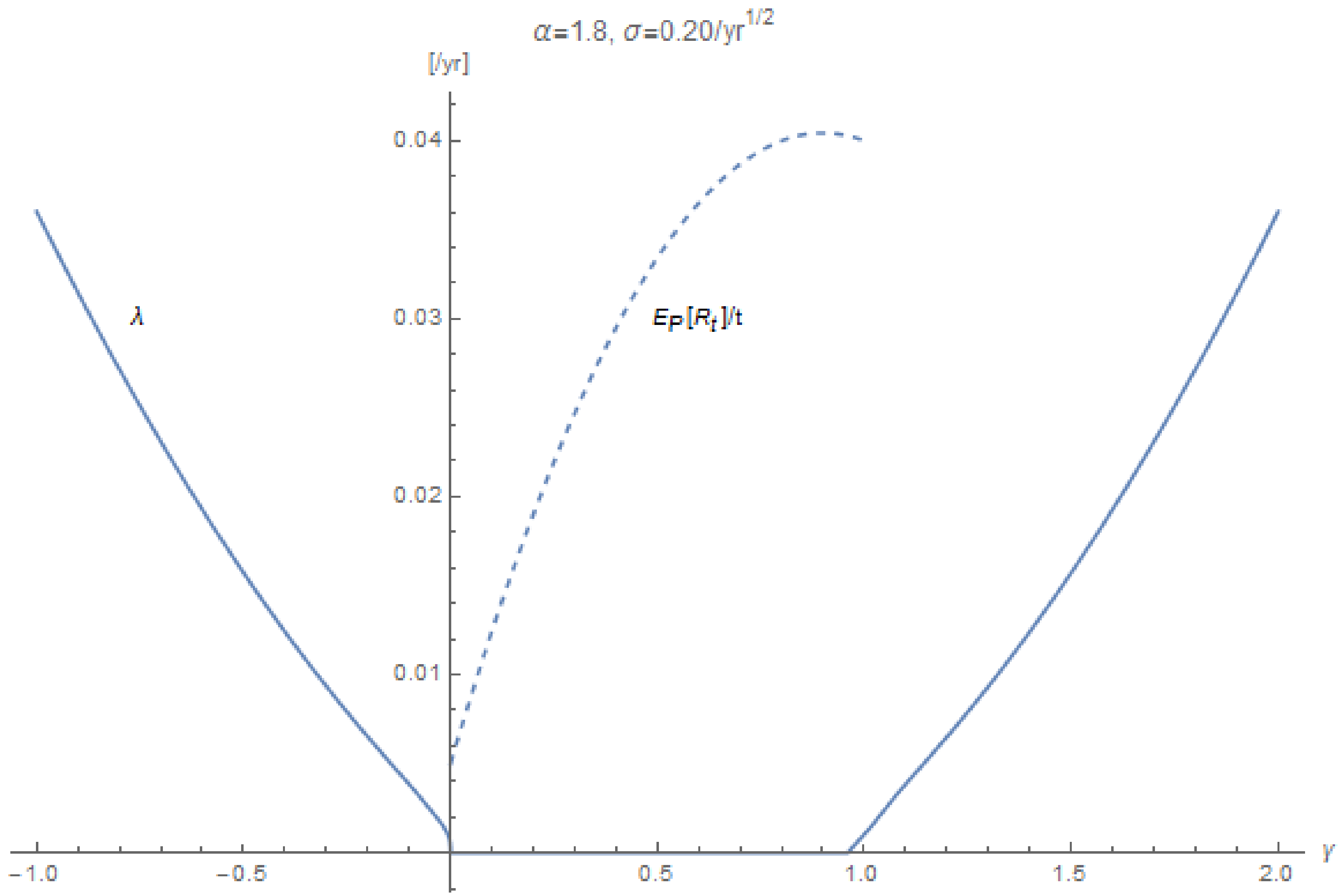
	Tail	Skew	Drift [/yr]	Vol [/yr^0.5]	Begin
Stocks	1.87	-1	10.0%	17.0%	12/31/75
Bonds	1.68	0	7.2%	8.9%	12/31/75
Commodities	1.87	-0.7	1.6%	17.0%	12/31/90

* 12/31/15 end data and monthly frequency for all

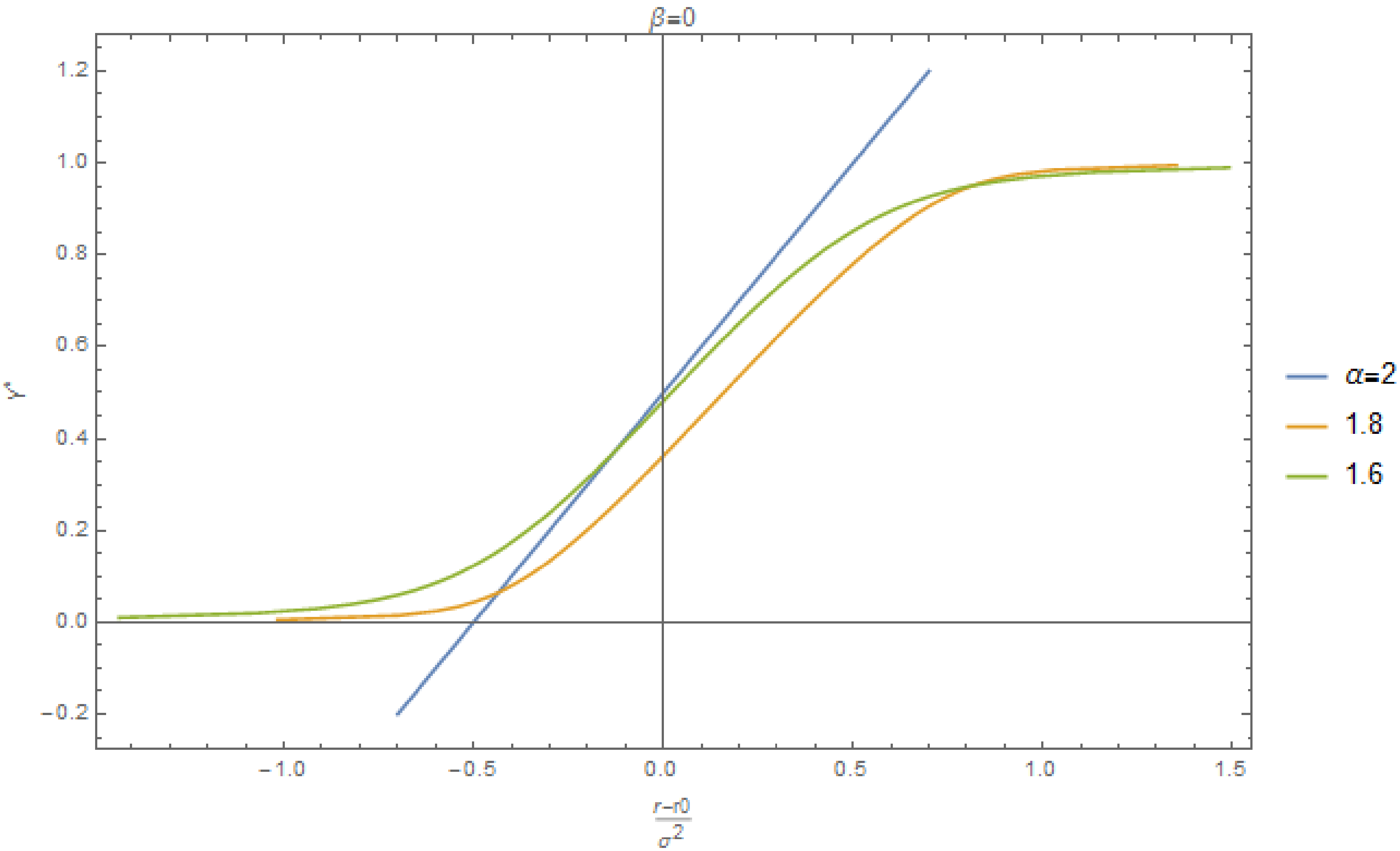
Markets Price Fat Tails



Symmetric Stable Returns (1 Asset)



Symmetric Stable Kelly Criterion (1 Asset)



THE MATH

Levy-Khitchine

- The pdf of a random variable is the Fourier transform of its characteristic function.
- A semi-martingale process is defined by its characteristic triplet (b, c, F) (and a truncation function h).
- The log characteristic function is given by the Levy-Khitchine formula:

$$\psi(k) \equiv ik^T \cdot b - \frac{1}{2}k^T \cdot \mathrm{c} \cdot k + \int (\exp(ik^T \cdot x) - 1 - ik^T \cdot h(x))dF(x)$$

Ito's Lemma

Lemma 5.5 (Itô's formula) *Let U be an open subset of $\mathbb{R}^d$ and X a U-valued semimartingale such that X_- is U-valued as well. Moreover, let $f : U \to \mathbb{R}$ be a function of class C^2. Then $f(X)$ is a semimartingale, and we have*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t Df(X_{s-})^\top dX_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t D^2_{ij} f(X_{s-}) d\langle X^{i,c}, X^{j,c}\rangle_s \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \Big(f(X_{s-}+x) - f(X_{s-}) - Df(X_{s-})^\top x \Big) \mu^X(ds, dx) \end{aligned} \tag{5.7}$$

for any $t \in \mathbb{R}_+$. Here, $Df = (D_1 f, \dots, D_d f)$ and $(D^2_{ij} f)_{i,j=1,\dots,d}$ denote the first and second derivatives of f, respectively.

PROOF. This follows immediately from Jacod (1979), (2.54). Note that $\cup_{n\in\mathbb{N}}[0, R_n] = \mathbb{R}_+$ if X_- is U-valued. □

We now consider the effect of C^2-mappings on the characteristics.

Corollary 5.6 *Let X, U be as in the previous lemma. Moreover, let $f : U \to \mathbb{R}^n$ be a function of class C^2. If (B, C, ν) denote the characteristics of X, then the characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ of the $\mathbb{R}^n$-valued semimartingale $f(X)$ are given by*

$$\begin{aligned} \widetilde{B}^k_t &= \int_0^t Df^k(X_{s-})^\top dB_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t D^2_{ij} f^k(X_{s-}) dC^{ij}_s \\ &\quad + \int_{[0,t]\times\mathbb{R}^d} \Big(h^k_n(f(X_{s-}+x) - f(X_{s-})) - Df^k(X_{s-})^\top h_d(x) \Big) \nu(ds, dx), \end{aligned}$$

$$\widetilde{C}^{kl}_t = \sum_{i,j=1}^d \int_0^t D_i f^k(X_{s-}) D_j f^l(X_{s-}) dC^{ij}_s, \tag{5.8}$$

$$\widetilde{\nu}([0,t]\times G) = \int_{[0,t]\times\mathbb{R}^n} 1_G(f(X_{s-}+x) - f(X_{s-}))\nu(ds, dx) \tag{5.9}$$

for $k, l \in \{1, \dots, d\}$, $t \in \mathbb{R}_+$, $G \in \mathcal{B}^n$.

Stochastic Exponential

Lemma 4.2 (Exponential Lévy processes) *1. Let $\widetilde{L}$ be a real-valued Lévy process with characteristic triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$. Then the process $Z := e^{\widetilde{L}}$ is of the form $\mathscr{E}(L)$ for some Lévy process L whose triplet (b, c, F) is given by*

$$\begin{aligned} b &= \widetilde{b} + \frac{\widetilde{c}}{2} + \int (h(e^x - 1) - h(x))\widetilde{F}(dx), \\ c &= \widetilde{c}, \\ F(G) &= \int 1_G(e^x - 1)\widetilde{F}(dx) \text{ for } G \in \mathcal{B}. \end{aligned}$$

2. Let L be a real-valued Lévy process with characteristic triplet (b, c, F). Suppose that $Z := \mathscr{E}(L)$ is positive. Then $Z = e^{\widetilde{L}}$ for some Lévy process $\widetilde{L}$ whose triplet $(\widetilde{b}, \widetilde{c}, \widetilde{F})$ is given by

$$\begin{aligned} \widetilde{b} &= b - \frac{c}{2} + \int (h(\log(1 + x)) - h(x))F(dx), \\ \widetilde{c} &= c, \\ \widetilde{F}(G) &= \int 1_G(\log(1 + x))F(dx) \text{ for } G \in \mathcal{B}. \end{aligned}$$

Kallsen's Kelly Criterion

Theorem 3.1 (Logarithmic utility) *Let* $u(x) = \log(x)$. *Assume that there exists some* $\gamma \in \mathbb{R}^d$ *such that*

1. $F(\{x \in \mathbb{R}^d : 1 + \gamma^\top x \leq 0\}) = 0$
2. $\int |\frac{x}{1+\gamma^\top x} - h(x)|F(dx) < \infty$
3. $$b - c\gamma + \int \Big(\frac{x}{1+\gamma^\top x} - h(x)\Big) F(dx) = 0.$$

Let

$$\kappa_t := \frac{\varepsilon}{K_T} \mathscr{E}(\gamma^\top L)_t,$$

$$V_t := \kappa_t(K_T - K_t),$$

$$\varphi_t^i := \frac{\gamma^i}{\widehat{S}_{t-}^i} V_{t-} \textit{ for } i = 1, \ldots, d, \quad \varphi_t^0 := \int_0^t \varphi_s^\top d\widehat{S}_s - \sum_{i=1}^d \varphi_t^i \widehat{S}_t^i$$

for $t \in [0, T]$, *where we set* $V_{0-} := 0$. *Then* $(\varphi, \kappa) \in \mathfrak{P}$ *is an optimal portfolio/consumption pair with discounted wealth process* V.

Normal Returns

- Given $(\Omega, \mathcal{F}, \mathbb{F}, P)$, satisfying the usual conditions. P is the physical ("real world") measure.
- $S_t = S_0 \exp(R_t)$ where $R_t \sim N(rt, \Sigma t)$. That is, $\mathbb{T}(R|P) = (r, \Sigma, 0)$.
- $S_t = S_0 \mathcal{E}(\bar{R}_t)$. That is, $\mathbb{T}(\bar{R}|P) = (\mu, \Sigma, 0)$ where $\mu_i = r_i + \frac{1}{2}\Sigma_{ii}$ and $\bar{R}_t \sim N(\mu t, \Sigma t)$.
- An asset's Sharpe ratio is $(r_i - r_0)/\sigma_i$ where $\sigma_i^2 = \Sigma_{ii}$, and its market price of risk is $(\mu_i - r_0)/\sigma_i$. If $\mu_i = r_0$, then the market price of risk is zero and the Sharp ratio is -0.5.
- $V_t = V_0 \mathcal{E}(\bar{R}_t^\gamma)$ where $d\bar{R}_t^\gamma = (1 - \gamma^T \cdot 1) r_0 dt + \gamma^T \cdot d\bar{R}_t$. That is, $\mathbb{T}(\bar{R}^\gamma|P) = (r_0 + \gamma^T \cdot (\mu - r_0 1), \gamma^T \cdot \Sigma \cdot \gamma, 0)$.
- $V_t = V_0 \exp(R_t^\gamma)$. That is, $\mathbb{T}(R^\gamma|P) = (r_\gamma, \sigma_\gamma^2, 0)$ and $R_t^\gamma \sim N(r_\gamma t, \sigma_\gamma^2 t)$ where $r_\gamma = r_0 + \gamma^T \cdot (\mu - r_0 1) - \frac{1}{2}\gamma^T \cdot \Sigma \cdot \gamma$ and $\sigma_\gamma^2 = \gamma^T \cdot \Sigma \cdot \gamma$.
- The Kelly criterion is $\gamma^* = \Sigma^{-1} \cdot (\mu - r_0 1)$.

Alpha Stable Returns (1 Asset)

- Given $(\Omega, \mathcal{F}, \mathbb{F}, P)$, satisfying the usual conditions. P is the physical ("real world") measure.
- $S_t = S_0 \exp(R_t)$ where $R_t \sim S(\alpha, \beta, rt, (\frac{1}{2}\sigma^2 t)^{1/\alpha})$ and $1 < \alpha < 2$. (Note that risk now scales as $t^{1/\alpha}$, not $t^{1/2}$.) That is, $\mathbb{T}(R|P) = (r + r_a, 0, f)$ relative to $h(x)$ where $f(x) = \left[c_- 1_{(-\infty,0)}(x) + c_+ 1_{(0,\infty)}(x)\right]/|x|^{1+\alpha}$, $c_\pm = -\frac{1}{2}(1 \pm \beta)\frac{1}{2}\sigma^2 \Gamma(-\alpha)/\cos(\alpha\frac{\pi}{2})$, and $r_a = \int_{-\infty}^{\infty}[h(x) - x]f(x)dx$. If, for example, $h(x) = x 1_{[-1,1]}(x)$, then $r_a = -(c_+ - c_-)/(\alpha - 1)$.
- $S_t = S_0 \mathcal{E}(\bar{R}_t)$. That is, $\mathbb{T}(\bar{R}\big|P) = (r + r_a + r_b, 0, \bar{f})$ relative to $\bar{h}(x)$ where $\bar{f}(x) = f(\log(1+x))/(1+x)$ for $x > -1$ and $r_b = \int_{-\infty}^{\infty}[\bar{h}(e^x - 1) - h(x)]f(x)dx$. $\bar{R}_t$ has an unnamed distribution.
- $V_t = V_0 \mathcal{E}(\bar{R}_t^\gamma)$ where $d\bar{R}_t^\gamma = (1-\gamma)r_0 dt + \gamma d\bar{R}_t$. Per Ito's lemma, $\mathbb{T}(\bar{R}^\gamma\big|P) = ((1-\gamma)r_0 + \gamma(r + r_a + r_b) + r_c(\gamma), 0, \bar{f}^\gamma)$ relative to $\bar{h}^\gamma(x)$ where $\bar{f}^\gamma(x) = \bar{f}(x/\gamma)/|\gamma|$ for $x/\gamma > -1$ and $r_c(\gamma) = \int_{-\infty}^{\infty}[\bar{h}^\gamma(\gamma(e^x - 1)) - \gamma\bar{h}(e^x - 1)]f(x)dx$.
- The portfolio hazard rate is $\lambda = \int_{-\infty}^{-1}\bar{f}^\gamma(x)dx$. If $\gamma < 0$, then $\lambda = c_+ \log((\gamma - 1)/\gamma)^{-\alpha}/\alpha$; if $\gamma > 1$, then $\lambda = c_- \log(\gamma/(\gamma - 1))^{-\alpha}/\alpha$.
- If $0 \le \gamma \le 1$, then $\lambda = 0$ and $V_t = V_0 \exp(R_t^\gamma)$ where $\mathbb{T}(R^\gamma|P) = ((1-\gamma)r_0 + \gamma(r + r_a + r_b) + r_c(\gamma) + r_d(\gamma), 0, f^\gamma)$ relative to $h^\gamma(x)$, $f^\gamma(x) = \bar{f}^\gamma(e^x - 1)e^x$ for $x > \log(1-\gamma)$, and $r_d = \int_{-\infty}^{\infty}[h^\gamma(\log(1 + \gamma(e^x - 1))) - \bar{h}^\gamma(\gamma(e^x - 1))]f(x)dx$. R_t^γ has an unnamed distribution, but $E_P(R_t^\gamma)/t = (1-\gamma)r_0 + \gamma(r + r_a + r_b) + r_c(\gamma) + r_d(\gamma) + r_e(\gamma)$ where $r_e(\gamma) = \int_{-\infty}^{\infty}[x - h^\gamma(\log(1 + \gamma(e^x - 1)))]f(x)dx$. The portfolio expected return is $E_P(R_t^\gamma)/t = (1-\gamma)r_0 + \gamma r + \int_{-\infty}^{\infty}[\log(1 + \gamma(e^x - 1))]f(x)dx$.
- The Kelly criterion is $I(\gamma^*) = (r - r_0)/\sigma^2$ where $I(\gamma) = \int_{-\infty}^{\infty}\left[(e^x - 1)/\left(1 + \gamma(e^x - 1)\right)\right]f(x)dx/\sigma^2$ for $0 \le \gamma \le 1$.

TBD: Alpha Stable Returns (N Assets)

- Given $(\Omega, \mathcal{F}, \mathbb{F}, P)$, satisfying the usual conditions. P is the physical ("real world") measure.
- $S_t = S_0 \exp(R_t)$ where $R_t \sim S(rt, \Gamma t)$ and $1 < \alpha < 2$. That is, $\mathbb{T}(R|P) = (r + r_a, 0, F)$ relative to $h(x) = x 1_{\{x:|x| \leq 1\}}(x)$ where $dF(x) = d\Gamma(\mathrm{s}) dr / |r|^{1+\alpha}$, $r = |x|$, and $s = x/r$ and
  $r_a = \int_{\mathbb{R}} [h(x) - x] dF(x)$.
- $S_t = S_0 \mathcal{E}(\bar{R}_t)$. That is, $\mathbb{T}(\bar{R}|P) = t(r + r_a + r_b, 0, \bar{F})$ where $r_b =?$ and $d\bar{F}(x) =?$.

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