Let The Riemann metric be written as

$$(dr)^2 + h_{ij}(r, \theta)\, d\theta_i d\theta_j$$

What is the asymptotic behavior of $h_{ij}(r.\theta)$ when $r \to 0$.

Note that the Riemannian metric at the reference point is regular. Thus we can define $(\theta_2 \cdots, \theta_n)$ as

$$\theta_j = \frac{x_j}{r} \quad , \quad r = \sqrt{\Sigma x_j^2}$$

Suppose

$$ds^2 = g_{ij}\, dx_i dx_j$$

Let's compare

$$g_{ij}\, dx_i dx_j \sim (dr)^2 + h_{ij}\, d\theta_i d\theta_j$$

We have

$$g_{ij}\, dx_i dx_j = g_{ij}(dr\theta_i + r\, d\theta_i)(dr\theta_j + r\, d\theta_j)$$

$$= g_{ij}\theta_i\theta_j (dr)^2 + 2\, g_{ij}\, x_i\, dr\, d\theta_j + g_{ij}\, r^2 d\theta_i d\theta_j$$

By comparison, we have

$$g_{ij}\, \theta_i\, \theta_j = 1$$

$$\sum_i g_{ij}\, x_i = 0$$

and

$$h_{ij} = r^2\left(g_{ij} - g_{1j}\frac{\theta_i}{\theta_1} - g_{i1}\frac{\theta_j}{\theta_1} + g_{11}\frac{\theta_i\theta_j}{\theta_1^2}\right)$$

Thus

$$\sqrt{\det(h_{ij})} = r^{n-1} \cdot \mu$$

for μ being a regular function (at least for fixed θ_i).

Thus

$$\Delta \rho = \frac{\partial f}{\partial r} \sim \frac{n-1}{r} + \text{small terms.}$$

If we choose (g_{ij}) to be normal, we can actually compute

$$\det\left(\delta_{ij} + \frac{\theta_i \theta_j}{\theta_1^2}\right) = \frac{1}{\theta_1^2}$$

Thus

$$\sqrt{\det h_{ij}}$$

can be extended as a regular function near 0.