

A note on special Kähler manifolds

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The base space of an algebraically completely integrable Hamiltonian system acquires a rather special differential-geometric structure which plays an important role in modern physical theories such as Seiberg-Witten theory. This structure was formalised by D. Freed [1] as a *Special Kähler manifold*. In that paper he conjectured that there are no compact special Kähler manifolds other than flat ones. In this paper, we prove that there are no nonflat complete special Kähler manifolds, thus verifying the conjecture.

Definition 1. *Let M be a Kähler manifold with Kähler form ω . A special Kähler structure on M is a real flat torsionfree symplectic connection ∇ satisfying*

$$d_{\nabla}I = 0$$

where $I \in \Omega^1(M, TM)$ is the complex structure on M .

The following property is true for any special Kähler manifold.

Theorem 1. *Any special Kähler manifold has nonnegative Ricci curvature. Moreover, if the scalar curvature is identically zero, then the curvature tensor itself vanishes.*

Proof. We take (z^1, \dots, z^n) to be a special coordinate system. Under this coordinate system, the Kähler form can be represented as

$$\omega = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j = \frac{\sqrt{-1}}{2} \operatorname{Im} u_{ij} dz^i \wedge d\bar{z}^j$$

where u is a local holomorphic function [1, page 5] and u_{ij} is defined as $\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}$. Moreover, we define $u_{i_1 \dots i_n}$ to be $\frac{\partial^n u}{\partial z^{i_1} \dots \partial \bar{z}^{i_n}}$.

It is then a straightforward computation that

$$R_{i\bar{j}k\bar{l}} = -\frac{1}{4} h^{m\bar{n}} u_{ikm} \overline{u_{jln}}$$

where our definition for the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - h^{m\bar{n}} \frac{\partial h_{m\bar{j}}}{\partial \bar{z}^l} \frac{\partial h_{i\bar{n}}}{\partial z^k}$$

Let F be the cubic form defined as

$$F = -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)})$$

where $\pi^{(1,0)} \in \Omega^{1,0}(TCM)$ is the projection onto the $(1, 0)$ part of the complexified tangent bundle. F is a global section of the bundle $Sym^3 T^*M$. Locally $F_{ijk} = u_{ijk}$. Thus we have

$$(1) \quad R_{i\bar{j}k\bar{l}} = -\frac{1}{4} h^{m\bar{n}} F_{ikm} \overline{F_{jln}}$$

We remark that the above identity is valid for any local holomorphic coordinate system, because both sides are tensors. From (1) we see that the Ricci curvature is nonnegative. Furthermore, if the scalar curvature ρ

$$(2) \quad \rho = \frac{1}{4} h^{i\bar{i}_1} h^{j\bar{j}_1} h^{k\bar{k}_1} F_{ijk} \overline{F_{i_1 j_1 k_1}}$$

is identically zero, then the curvature tensor vanishes. □

Theorem 2. *If (M, ω) is a complete special Kähler manifold, then the curvature tensor vanishes.*

Proof. Define

$$(3) \quad F_{ijk,l} = \partial_l F_{ijk} - \Gamma_{il}^m F_{mjk} - \Gamma_{jl}^m F_{imk} - \Gamma_{kl}^m F_{ijm}$$

to be the covariant derivative of F_{ijk} . By the Bochner formula

$$\begin{aligned} \Delta \rho &= \frac{1}{4} h^{i\bar{i}_1} h^{j\bar{j}_1} h^{k\bar{k}_1} h^{l\bar{l}_1} F_{ijk,l} \overline{F_{i_1 j_1 k_1 l_1}} \\ &\quad + \frac{3}{4} h^{m_1 \bar{i}_1} h^{i\bar{m}} h^{j\bar{j}_1} h^{k\bar{k}_1} R_{m_1 \bar{m}} F_{ijk} \overline{F_{i_1 j_1 k_1}} \end{aligned}$$

where $R_{m_1 \bar{m}} = -h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta} m_1 \bar{m}}$ is the Ricci tensor. In particular, we have

$$(4) \quad \Delta \rho \geq 3 h^{m_1 \bar{i}_1} h^{i\bar{m}} R_{i\bar{i}_1} R_{m_1 \bar{m}} \geq \frac{3}{n} \rho^2$$

We use the version of the generalized maximal principle in [2, page 582, Lemma 1.1]. The original statement in [2] is quite general. We rewrite it in the following (simplified) way:

Proposition 1. *Let M be a complete Kähler manifold of nonnegative Ricci curvature. Let φ be a nonnegative function satisfying*

$$\Delta\varphi \geq C_1\varphi^\alpha - C_2\varphi - C_3$$

where $\alpha > 1$, $C_1 > 0$, $C_2, C_3 \geq 0$ are constants. Then

$$\sup \varphi \leq \text{Max}\left\{1, \left(\frac{C_2 + C_3}{C_1}\right)^{1/\alpha}\right\}$$

For any positive number a , from (4) we see that $\Delta(a\rho) \geq \frac{3}{na}(a\rho)^2$. By the above proposition we see that $a\rho \leq 1$. Letting $a \rightarrow \infty$ we see that $\rho \equiv 0$. The theorem thus follows from Theorem 1. \square

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